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Incomplete markets over an infinite horizon: Long-lived securities and speculative bubbles

Michael Magill [a,*], Martine Quinzii [b]

[a] Department of Economics, University of Southern California, Los Angeles, CA 90089-0253, USA
[b] University of California, Davis, CA 95616, USA

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Abstract

This paper studies sequence economies over an infinite horizon with general security structures. Assumptions are given under which a pseudo-equilibrium exists for all economies and an equilibrium exists for a dense set of (appropriately parameterized) economies. Under these assumptions the indebtedness of the agents in equilibrium can be limited either by an explicit bound on their debts or by a transversality condition limiting the asymptotic growth of their debts. The qualitative properties of equilibrium prices of infinite-lived securities are studied: the prices of infinite-lived securities in zero net supply are shown to permit speculative bubbles and the existence of bubbles can affect the equilibrium allocation. The prices of securities in positive supply (equity contracts) cannot have speculative bubbles: the extent of speculation in this class of model is thus severely limited.

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1. Introduction

Sequence economies over an infinite horizon form the basic framework for modern theoretical macroeconomics. The models are of two kinds: in the first, agents (families) are finitely lived and are succeeded by their children in an

* Corresponding author.

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infinite sequence of overlapping generations (OLG); in the second, there are a finite number of families (dynasties) that are infinitely lived. The latter model, which is the focus of this paper, has mainly been explored in the representative agent case: while this simplifies the analysis it also limits the insights that can be obtained; in a representative agent model of an exchange economy, the equilibrium allocation is given by the initial resources and no change in the structure of the markets or in the monetary environment can have real effects. It is for this reason that the general equilibrium analysis of sequence economies with incomplete markets has been the subject of much recent research – this class of models being referred to more briefly as *general equilibrium with incomplete markets* (GEI). [1] The analysis of the GEI model, which promises to provide a rich framework for the analysis of problems in macroeconomics, has, however, so far been restricted to the somewhat ad hoc setting of a finite horizon.

In a companion paper (Magill and Quinzii, 1994) we showed how the concept of a GEI equilibrium can be extended to an infinite horizon economy. The new element that needs to be incorporated into the definition of an equilibrium is a device to prevent agents from indefinitely postponing the repayment of their debts (so-called Ponzi schemes). We showed how the two traditional approaches for deterministic economies based either on *borrowing constraints* or a *transversality condition* can be extended to a model with incomplete markets. An equilibrium with debt constraints is a simple and intuitively appealing descriptive concept of equilibrium: an equilibrium with a transversality condition is, in principle, more general and is more natural for theoretical (mathematical) analysis; however, some economists may find it hard to accept as a positive concept of equilibrium since it calls for substantial rationality on the part of agents. Under the assumption (B.4) that agents have a uniform lower bound on their degree of impatience at each date-event – defined as the proportion of future consumption that an agent is ready to give up in order to have one more unit of consumption (of the numeraire good) at that date-event – these two types of equilibria were shown to coincide. Thus, equilibria with transversality conditions, which are more convenient for studying existence and qualitative properties, can be used to study the descriptively more satisfactory concept of equilibrium with debt constraints.

The Magill and Quinzii (1994) paper is restricted to the simplest setting of an economy with short-lived securities paying dividends in a numeraire good. The object of the present paper is to extend the analysis to general security structures with the emphasis on the case of infinite-lived securities. The first task is to find conditions under which an equilibrium exists: even in the case of finite horizon economies it is well known since Hart's (1975) paper that discontinuities in

[1] See the Special Issue of the *Journal of Mathematical Economics* on General Equilibrium with Incomplete Markets (vol. 19, no. 1, 1990) and the survey articles of Geanakoplos (1990) and Magill and Shafer (1991).

agents' demands created by changes in the rank of the returns matrices can lead to non-existence of equilibrium. The method used in the recent GEI literature to circumvent this problem consists of introducing the concept of a pseudo-equilibrium: such an equilibrium is shown to exist for all economies and for a generic set of economies every pseudo-equilibrium is shown to be an equilibrium (see Duffie and Shafer, 1985, 1986). To extend this approach to the infinite horizon case we adopt a simpler and more intuitive approach to represent a pseudo-equilibrium of an economy with long-lived securities that amounts to adjoining to the economy a family of potentially equivalent short-lived numeraire assets. This enables us to draw on the arguments in the earlier paper (Magill and Quinzii, 1994) to establish the existence of a pseudo-equilibrium, the proof being obtained by taking limits of pseudo-equilibria of truncated economies. It is here that the techniques introduced by Bewley (1972) for establishing the existence of an Arrow–Debreu equilibrium over an infinite horizon turn out to be most fruitful. For the price of an infinite-lived security (when it is priced at its fundamental value) is the discounted value of its future dividend stream – an expression with an infinite number of terms. The problem of convergence is solved by viewing the present value prices as elements of the norm dual of ℓ_∞ (the space *ba* of bounded finitely additive set functions), the convergence being taken in the weak star topology. Mackey continuity of agents' preference orderings is then used to establish that the limit prices are summable.

The existence of an equilibrium for a dense set of security payoffs is obtained by showing that whenever a pseudo-equilibrium offers a larger subspace of income transfers than the original securities, the commodity payoffs can be perturbed so as to make these subspaces coincide. This result is weaker than the existence result for finite horizon economies, which establishes existence for a generic set of economies. However, we have not found a way to use Smale's version of Sard's theorem for infinite dimensional spaces to establish existence for a generic set of economies: this must be the subject of later research.

Since the equilibria exhibited in the existence proof are obtained by taking limits of equilibria (pseudo-equilibria) of truncated economies, they have the property that all securities are priced at their fundamental values. It is natural to enquire whether this property holds for all possible equilibria: since no terminal condition can automatically be attached to the system of stochastic difference equations that must be satisfied by a security price (the first-order conditions for the portfolio choice of each agent), it is not a priori clear that the price of an infinite-lived security will equal its fundamental value.

The phenomenon of *speculative bubbles* has been the subject of much interest in macroeconomics (see Blanchard and Fischer, 1989, ch. 5). It is sometimes argued that bubbles cannot arise in an economy with a finite number of infinite-lived agents: in Section 6 we show that this statement needs to be qualified. As in Tirole (1982) and Santos and Woodford (1992) we find that there cannot be a speculative bubble on infinite-lived securities in *positive* net supply: since equity contracts constitute a significant segment of the capital market, this places a bound on the extent to which the model predicts the occurrence of speculation. However, the prices of infinite-lived securities in *zero* net supply behave quite differently: they admit substantial amounts of speculation. We show that a speculative bubble can always be added to the price of such a security without affecting the real equilibrium allocation. There is thus a significant nominal indeterminacy in the prices of infinite-lived securities in zero net supply. However, there is a qualitative difference between speculative bubbles that can arise in an equilibrium depending on whether the markets are complete or incomplete. If the markets are complete, then speculative bubbles only introduce a nominal indeterminacy; if the markets are incomplete, then a speculative bubble can have a real effect in the sense that the same equilibrium allocation cannot be supported by a system of prices such that each security is priced at its fundamental value.

2. The infinite horizon economy

We consider an economy with time and uncertainty over an infinite horizon. Let $T=\{0,1,\dots\}$ denote the set of time periods and let S be a set of states of nature. The revelation of information is described by a sequence of partitions of S, $\mathbb{F}=(\mathbb{F}_0,\mathbb{F}_1,\dots,\mathbb{F}_t,\dots)$, where the number of subsets in $\mathbb{F}_t$ is *finite* and $\mathbb{F}_t$ is finer than the partition $\mathbb{F}_{t-1}$ (i.e. $\sigma\in\mathbb{F}_t$, $\sigma'\in\mathbb{F}_{t-1}\Rightarrow\sigma\subset\sigma'$ or $\sigma\cap\sigma'=\emptyset$) for all $t\geqq 1$. At date 0 we assume that there is no information so that $\mathbb{F}_0=S$. The information available at time t (for $t\in T$) is assumed to be the same for all agents in the economy (*symmetric information*) and is described by the subset σ of the partition $\mathbb{F}_t$ in which the state of nature lies. A pair $\xi=(t,\sigma)$ with $t\in T$ and $\sigma\in\mathbb{F}_t$ is called a *date-event* or *node* and $t(\xi)=t$ is the *date* of node ξ. The set D consisting of all date events (or nodes) is called the *event tree* induced by $\mathbb{F}$, $D=\bigcup_{t\in T,\sigma\in\mathbb{F}_t}(t,\sigma)$.

A node $\xi'=(t',\sigma')$ is said to succeed (strictly) a node $\xi=(t,\sigma)$ if $t'\geq t(t'>t)$ and $\sigma'\subset\sigma$; we write $\xi'\geq\xi(\xi'>\xi)$. The set of nodes that succeeds a node $\xi\in D$ is called the subtree $D(\xi)$ and $D^+(\xi)=\{\xi'\in D(\xi)\,|\,\xi'>\xi\}$ is the set of strict successors of ξ. The subset of nodes of $D(\xi)$ at date T is denoted by $D_T(\xi)$ and the subset of nodes between dates $t(\xi)$ and T by $D^T(\xi)$:

$$D_T(\xi)=(\xi'\in D(\xi)\,|\,t(\xi')=T),$$

$$D^T(\xi)=\{\xi'\in D(\xi)\,|\,t(\xi)\leq t(\xi')\leq T\}.$$

When ξ is the initial node (denoted by ξ_0) the notation is simplified to D^+, D_T, D^T.

$\xi^+=\{\xi'\in D(\xi)\,|\,t(\xi')=t(\xi)+1\}$ is the set of immediate successors of ξ. The number of elements of ξ^+ is finite and is called the branching number $b(\xi)$

at $\xi(b(\xi) = \#\xi^+)$. If $\xi = (t, \sigma)$, with $t \geq 1$, then the unique node $\xi^- = (t-1, \sigma')$ with $\sigma \subset \sigma'$ is called the predecessor of ξ.

The economy consists of a finite collection of infinitely-lived consumers (families) $\boldsymbol{I} = \{1, \ldots, I\}$ who purchase commodities on spot markets and trade securities at every node in the event tree $\boldsymbol{D}$ described above. There is a set $\boldsymbol{L} = \{1, \ldots, L\}$ of commodities at each node: the set consisting of all commodities indexed over the event tree is thus:

$$\boldsymbol{D} \times \boldsymbol{L} = \{(\xi, \ell) \mid \xi \in \boldsymbol{D}, \ell \in \boldsymbol{L}\}.$$

Let $\mathbb{R}^{D \times L}$ denote the vector space of all maps $x: \boldsymbol{D} \times \boldsymbol{L} \to \mathbb{R}$ and let $\ell_\infty(\boldsymbol{D} \times \boldsymbol{L})$ denote the subspace of $\mathbb{R}^{D \times L}$ consisting of all *bounded* maps (sequences):

$$\ell_\infty(\boldsymbol{D} \times \boldsymbol{L}) = \left\{ x \in \mathbb{R}^{D \times L} \mid \sup_{(\xi, \ell) \in D \times L} |x(\xi, \ell)| < \infty \right\}.$$

The norm $\|\cdot\|_\infty$ of $\ell_\infty(\boldsymbol{D} \times \boldsymbol{L})$ is defined by $\|x\|_\infty = \sup_{(\xi, \ell) \in D \times L} |x(\xi, \ell)|$. As in Bewley (1972) we take the commodity space to be $\ell_\infty(\boldsymbol{D} \times \boldsymbol{L})$. Each agent $i \in \boldsymbol{I}$ has an *initial endowment* process given by $\omega^i = (\omega^i(\xi, \ell), (\xi, \ell) \in \boldsymbol{D} \times \boldsymbol{L})$ which is assumed to lie in the non-negative orthant $\ell_\infty^+(\boldsymbol{D} \times \boldsymbol{L})$. Let $\omega^i(\xi) = (\omega^i(\xi, \ell), \ell \in \boldsymbol{L}) \in \boldsymbol{R}^L$ denote the agent's endowment of the L goods at node ξ. Agent i chooses a *consumption process* $x^i = (x^i(\xi, \ell), (\xi, \ell) \in \boldsymbol{D} \times \boldsymbol{L})$, which must lie in his consumption set $X^i = \ell_\infty^+(\boldsymbol{D} \times \boldsymbol{L})$; $x^i(\xi) = (x^i(\xi, \ell), \ell \in \boldsymbol{L}) \in \mathbb{R}_+^L$ denotes the agent's consumption at node ξ. Note that this description of the commodity space assumes that each good is perfectly divisible and is perishable (no storable or durable goods) and that the supply of goods does not grow without bound. The agent's preference among consumption processes in X^i is expressed by a *preference ordering* $\succsim_i$.

At each date event there are spot markets on which the L commodities are traded. Let

$$p = (p(\xi, \ell), (\xi, \ell) \in \boldsymbol{D} \times \boldsymbol{L}) \in \mathbb{R}^{D \times L},$$

denote the *spot price process* and let $p(\xi) = (p(\xi, \ell), \ell \in \boldsymbol{L})$ denote the vector of spot prices for the L goods at node ξ. At each node ξ, good 1 plays the role of numeraire good:

$$p(\xi, 1) = 1, \quad \forall \xi \in \boldsymbol{D}, \tag{2.1}$$

so that all payments are denominated in units of good 1.

In this paper we consider the class of real securities: a financial asset is said to be a *real* security if its return at each node ξ (after its node of issue) is the value under the spot prices at node ξ of a specified bundle of the L goods. As is well known, a model with nominal securities can be converted into a family of models with real securities (see Geanakoplos and Mas-Colell, 1989). Let $\boldsymbol{J}$ denote the set of (real) securities. Security $j \in \boldsymbol{J}$ is issued at node $\xi(j) \in \boldsymbol{D}$ and promises to deliver a dividend process $\{p(\xi) A(\xi, j) \in \mathbb{R} \mid \xi \in \boldsymbol{D}^+(\xi(j))\}$ at all nodes strictly succeeding its node of issue $\xi(j)$, which is the value of a bundle $A(\xi, j) = (A(\xi, \ell, j), \ell \in L) \in \mathbb{R}^L$ of the L commodities under the spot prices $p(\xi)$. A maturity node of a security is a node (after its node of issue) beyond which it makes no payment (i.e. $\xi \in \boldsymbol{D}(\xi(j))$ is a maturity node of security j if $A(\xi, j) \neq 0$ and $A(\xi', j) = 0$, $\forall \xi' > \xi$). A security is first traded at its node of issue and is always retraded until the predecessor of a maturity node: a security is usually not retraded after a maturity node is reached, but in some circumstances (which are discussed later) this will be permitted. Security j with node of issue $\xi = \xi(j)$ is said to be *short-lived* if it is traded only at its node of issue and pays dividends only at the immediate successors of this node, namely if $A(\xi', j) = 0$, $\forall \xi' \notin \xi^+(j)$. In all other cases security j is said to be *long-lived*. A security whose commodity payoff is exclusively in good 1 ($A(\xi, \ell, j) = 0$ if $\ell \neq 1$) is called a *numeraire* security. A security that is traded at every node after its node of issue is said to be *infinite-lived*.

The set of *traded securities* at node ξ is denoted by $J(\xi)$: it includes all securities issued at or before node ξ which have not yet reached a maturity node

$$J(\xi) \supset \{ j \in \boldsymbol{J} \mid \xi(j) \leqq \xi \text{ and } \exists \xi' > \xi \text{ with } A(\xi', j) \neq 0 \}. \tag{2.2}$$

As indicated above it is sometimes of interest to include in $J(\xi)$ securities that have already matured. In all circumstances, however, we assume that for each node $\xi \in \boldsymbol{D}$, the *number* of traded securities $\mathcal{J}(\xi) = \# J(\xi)$ is *finite*. It is convenient to extend the definition of the commodity payoff of each security j from its subtree $\boldsymbol{D}^+(\xi(j))$ to the whole event tree $\boldsymbol{D}$ by defining $A(\xi, j) = 0$ for all $\xi \notin \boldsymbol{D}^+(\xi(j))$. Let

$$A(\xi) = (A(\xi, j), j \in \boldsymbol{J}), \qquad A = (A(\xi), \xi \in \boldsymbol{D})$$

denote the commodity payoffs of all securities at node ξ and the *commodity payoff process* of the securities. Note that $A(\xi)$ has at most a finite number of non-zero components.

Definition 2.1. A *security structure* $\mathscr{A} = (\boldsymbol{J}, (\xi(j))_{j \in \boldsymbol{J}}, (J(\xi))_{\xi \in \boldsymbol{D}}, A)$ consists of a set of securities $\boldsymbol{J}$, a node of issue $\xi(j)$ for each security $j \in \boldsymbol{J}$, a set of traded securities $J(\xi) \subset \boldsymbol{J}$ at each node $\xi \in \boldsymbol{D}$ and a commodity payoff process $A \in \mathbb{R}^{D \times L \times J}$ for the securities, which satisfy the following consistency conditions:

(i) $A(\xi, j) = 0$ if $\xi \notin \boldsymbol{D}^+(\xi(j))$;

(ii) $\xi \notin \boldsymbol{D}(\xi(j)) \Rightarrow j \notin J(\xi)$;

(iii) $\xi(j) \leq \xi$ and $j \notin J(\xi) \Rightarrow j \notin J(\xi')$, $\forall \xi' \geqq \xi$;

(iv) (2.2) holds.

Conditions (i) and (ii) assert that a security which has not yet been issued cannot pay a dividend and cannot be traded. (iii) asserts that once a security ceases to be traded, it is never retraded. (iv) states that all 'active' securities which have not yet reached a maturity node are traded.

If $j \in J(\xi)$, then let $q(\xi, j)$ denote the price of one unit of security j after its dividend at node ξ has been paid. It is convenient to define the price process of each security j on the whole event tree D by setting $q(\xi, j) = 0$ if $j \notin J(\xi)$. Let

$$q(\xi) = (q(\xi, j), j \in J), \qquad q = (q(\xi), \xi \in D) \in Q$$

denote the vector of security prices at node ξ and the *security price process*, where $Q = \{q \in \mathbb{R}^{D \times J} \mid q(\xi, j) = 0 \text{ if } j \notin J(\xi)\}$ denotes the *space of security prices*.

Let $z^i(\xi, j) \in \mathbb{R}$ denote the number of units of the jth security purchased (if $z^i(\xi, j) > 0$) or sold (if $z^i(\xi, j) < 0$) by agent i at node ξ: each security is assumed to be perfectly divisible and can be bought and sold in unlimited amounts (no short-sales constraints). If security j is not traded at node ξ, then we adopt the convention $z^i(\xi, j) = 0$. Let

$$z^i(\xi) = (z^i(\xi, j), j \in J), \qquad z^i = (z^i(\xi), \xi \in D) \in \mathscr{Z}$$

denote the agent's portfolio at node ξ and his *portfolio process* (*trading strategy*), where $\mathscr{Z} = \{z^i \in \mathbb{R}^{D \times J} \mid z^i(\xi, j) = 0 \text{ if } j \notin J(\xi)\}$ is the *portfolio space*.

If $\succsim = (\succsim_1, \ldots, \succsim_I)$ and $\omega = (\omega^1, \ldots, \omega^I)$ denote the profiles of preference orderings and endowments of the I agents and $\mathscr{A}$ is the security structure, then $\mathscr{E}_\infty(D, \succsim \omega, \mathscr{A})$ denotes the associated economy over the event tree D. The economy $\mathscr{E}_\infty(D, \succsim \omega, \mathscr{A})$ satisfies the following assumptions.

Assumption B1 (Event-tree). For each node $\xi \in D$ the branching number $b(\xi) = \#\xi^+$ is finite.

Assumption B2 (Endowments). There exist scalars m and m' with $0 < m < m'$ such that $\forall (\xi, \ell) \in D \times L$, $\omega^i(\xi, \ell) > m$, $\forall i \in I$ and $\Sigma_{i \in I} \omega^i(\xi, \ell) < m'$.

Let $\ell_1(D \times L)$ denote the subspace of $\mathbb{R}^{D \times L}$ consisting of all *summable sequences*,

$$\ell_1(D \times L) = \left\{P \in \mathbb{R}^{D \times L} \quad \sum_{(\xi, \ell) \in D \times L} |P(\xi, \ell)| < \infty\right\}.$$

Recall that the *Mackey topology* on $\ell_\infty(D \times L)$ is the strongest locally convex topology such that the dual of $\ell_\infty(D \times L)$ under this topology is $\ell_1(D \times L)$. For a discussion of this topology see Bewley (1972) and Mas-Colell and Zame (1991).

Assumption B3 (Preferences). For $i \in I$, $\succsim_i$ is a transitive, reflexive, complete preference ordering on $X^i = \ell_\infty^+(D \times L)$ which is strictly convex and continuous in the Mackey topology (i.e. for all $\tilde{x}^i \in X^i$, $\{x^i \in X^i \mid x^i \succ_i \tilde{x}^i\}$ is strictly convex and closed in the Mackey topology and $\{x^i \in X^i \mid x^i \succ_i \tilde{x}^i\}$ is open in the Mackey topology). $\succsim_i$ is monotone in the sense that for each $x^i \in X^i$ and for each $y \in \ell_\infty^+(D \times L)$, $x^i + y \succsim_i x^i$.

To express the next assumption on preferences it is convenient to introduce the following notation. Let

$$F = \{y \in \ell_\infty^+(D \times L \mid \| y \|_\infty < m\ell\}$$

be a bounded set which includes as a subset the feasible consumption plans of each agent. For a subset $E \subset D$ of nodes let χ_E denote the characteristic function of E:

$$\chi_E(\xi) = \begin{cases} 1, & \text{if } \xi \in E, \\ 0, & \text{if } \xi \notin E, \end{cases}$$

and for $x \in \ell_\infty(D \times L)$ define $x\chi_E = (x(\xi, l)\chi_E(\xi), (\xi, l) \in D \times L)$. Let $e_\ell^\xi \in \ell_\infty(D \times L)$ denote the process that has all components 0 except for the component of good ℓ at node ξ which is 1:

$$e_\ell^\xi(\xi', \ell') = \begin{cases} 1, & \text{if } (\xi', \ell') = (\xi, \ell), \\ 0, & \text{if } (\xi', \ell') \neq (\xi, \ell). \end{cases}$$

Assumption B4 (Uniform lower bound on impatience). There exists $\beta < 1$ such that for all $i \in I$

$$x^i\chi_{D \setminus D^+(\xi)} + \beta x^i \chi_{D^+(\xi)} + e_1^\xi \succ_i x^i, \quad \forall \xi \in D, \forall x^i \in F.$$

Assumption B5 (Securities). Every security $j \in J$ is a real security with bounded commodity payoff $A(\cdot, j) \in \ell_\infty(D \times L)$ and the number of traded securities $J(\xi)$ is finite at each node $\xi \in D$.

Assumption B6 (Short-lived numeraire bond). At each node $\xi \in D$, a security $j_\xi \in J(\xi)$ is issued which is traded only at this node and has a commodity payoff of one unit of good 1 at each successor:

$$A(\xi', \ell, j_\xi) = \begin{cases} 1, & \text{if } \xi' \in \xi^+ \text{and } \ell = 1, \\ 0, & \text{otherwise}. \end{cases}$$

Assumptions B1–B6 are essentially the same as Assumptions A1–A6 in Magill and Quinzii (1994). The main difference is that A3 in Magill and Quinzii (1994) only requires convexity of agents' preferred sets, while Assumption B3 requires that these sets be strictly convex. The reason for this strengthening of the convexity requirement is as follows. The proof of existence of an equilibrium for the infinite horizon economy is based on taking limits of equilibria of truncated finite horizon economies. In the case of short-lived numeraire securities, the existence of an equilibrium in a finite horizon economy only requires the use of a standard Kakutani fixed-point argument, which can be applied to an economy in which agents' demands are expressed by correspondences. For a finite horizon economy with a general security structure, the existence of a (pseudo-) equilibrium (see Section 4) requires the use of degree arguments and the currently available results (Duffie and Shafer, 1985, 1986; Husseini et al., 1990; Hirsch et al., 1990) use degree theory for functions. The proofs thus require that agents have well-defined demand functions – a property ensured by the strict convexity assumption in B3. It seems likely that the existence of a pseudo-equilibrium for a finite horizon economy can be extended to the case of demand correspondences, but we will not enter into such refinements here. B3 invokes only the assumption of weak monotonicity, but recall that strict convexity and weak monotonicity imply strict monotonicity with respect to every good at every node.

The assumption that there is a uniform lower bound on the impatience of each agent (Assumption B4) means that at each node ξ, each agent is prepared to give up at least the fraction $1-\beta>0$ of his future consumption plan after node ξ in exchange for an additional unit of commodity 1 at that node, $1-\beta$ being uniform across all nodes of the event tree and across all feasible consumption plans. B4 plays an important role in the analysis and is essentially the only new assumption added to those made by Bewley (1972). Assumptions B3 and B4 are satisfied by a preference ordering $\underset{i}{\succsim}$ represented by an additively separable utility function,

$$u^i(x^i)=\sum_{\xi\in D}\rho(\xi)\delta_i^{t(\xi)}v_i(x^i(\xi)), \tag{2.3}$$

where $\rho(\xi)$ is the probability of ξ (induced by a probability measure ρ on the measurable subsets of S), $\delta_i\in(0,1)$ is a discount factor and $v_i:\mathbb{R}_+^L\to\boldsymbol{R}$ is a continuous, increasing concave function with $v_i(0)=0$.

3. Definition of an equilibrium

The problem of defining an appropriate concept of an equilibrium for an infinite horizon sequence economy with incomplete markets has been discussed in Magill and Quinzii (1994). The simplest concept is that of an equilibrium with debt constraints: to show that such an equilibrium exists we introduced the more abstract concept of an equilibrium with a transversality condition. As we shall see below, this more abstract concept provides a particularly powerful tool to prove the existence of an equilibrium for an economy with infinite lived securities.

In an exchange economy, equilibrium concepts differ only by the specification of the agents' budget sets. For agent i, faced with the price process $(p,q)\in\mathbb{R}_+^{D\times L}\times Q$, let us consider the following budget sets:

(i) The budget set with an *explicit debt constraint M*

$$\mathscr{B}_\infty^M(p,q,\omega^i,\mathscr{A})=\left\{x^i\in X^i\;\middle|\;\begin{array}{l}\exists z^i\in\mathscr{Z},\text{ such that }\forall\xi\in D\\ q(\xi)z^i(\xi)\geqq -M\\ p(\xi)(x^i(\xi)-\omega^i(\xi))=V(\xi)z^i(\xi^-)-q(\xi)z^i(\xi)\end{array}\right\},$$

where $V(\xi)=p(\xi)A(\xi)+q(\xi)$ gives the vector of returns at the successor ξ of the $J(\xi^-)$ securities traded at ξ^- (ignoring the remaining zero components).

(ii) The budget set with an *implicit debt constraint*

$$\mathscr{B}_\infty^{\mathrm{DC}}(p,q,\omega^i,\mathscr{A}) = \left\{ x^i \in X^i \;\middle|\; \begin{array}{l} \exists z^i \in \mathscr{Z}, \text{ with } (qz^i)\in \ell_\infty(\boldsymbol{D}) \\ \text{such that } \forall \xi \in \boldsymbol{D} \\ p(\xi)(x^i(\xi)-\omega^i(\xi)) = V(\xi)z^i(\xi^-)-q(\xi)z^i(\xi) \end{array} \right\},$$

where $(qz^i)=(q(\xi)z^i(\xi),\ \xi\in\boldsymbol{D})$.

(iii) The budget set with a *transversality condition*

$$\mathscr{B}_\infty^{\mathrm{TC}}(p,q,\pi^i,\omega^i,\mathscr{A}) = \left\{ x^i \in X^i \;\middle|\; \begin{array}{l} \exists z^i \in \mathscr{Z}, \text{ such that } \forall \xi \in \boldsymbol{D} \\ \lim\limits_{T\to\infty} \sum\limits_{\xi' \in \boldsymbol{D}_T(\xi)} \pi^i(\xi')q(\xi')z^i(\xi') = 0 \\ p(\xi)(x^i(\xi)-\omega^i(\xi)) = V(\xi)z^i(\xi^-)-q(\xi)z^i(\xi) \end{array} \right\},$$

where $\pi^i=(\pi^i(\xi),\ \xi\in\boldsymbol{D})$ is a process of present value prices.

The first budget set, $\mathscr{B}_\infty^M$, leads to the simplest concept of an equilibrium with explicit debt constraint M. Since the object of the debt constraint is to eliminate Ponzi schemes (the indefinite postponement of debt) and not to introduce a new imperfection over and above the incompleteness of the markets, we do not want the debt constraints to be binding in an equilibrium. It is clear that if M is chosen independently of the characteristics of the economy, there will always be some economies for which M is too small and is binding in equilibrium. The budget set (ii) avoids the bad choice of a bound M by leaving the bound unspecified: proving that an equilibrium with budget set (ii) exists is thus a convenient way of proving that there is an appropriate bound M for a given economy such that an equilibrium with budget set (i) exists in which the bound M is never binding. The requirement that the constraints that prevent Ponzi schemes be non-binding is also emphasized in the approach of Levine and Zame (1996).

If the present value process π^i is summable ($\pi^i\in\ell_1(\boldsymbol{D})$), then condition (iii), that the asymptotic present value of debt tends to zero on any subtree, is a weaker restriction than a debt constraint. By showing that an equilibrium with budget set (iii) exists, where π^i is the present value process of agent i at equilibrium, we will show that an equilibrium with budget set (ii) exists.

If (x^i,z^i) is a consumption-portfolio plan for agent i satisfying the constraints in the budget set $\mathscr{B}_\infty^*$ (where an asterisk takes the place of the superscript indicating the type of constraint involved in the budget set), then z^i is said to *finance* x^i and we write $(x^i;z^i)\in\mathscr{B}_\infty^*(p,q,\omega^i,\mathscr{A})$. $(\bar{x}^i;\bar{z}^i)$ is said to be $\succsim_i$ *maximal* in $\mathscr{B}_\infty^*(p,q,\omega^i,\mathscr{A})$ if $\bar{z}^i$ finances $\bar{x}^i$ and $\bar{x}^i\succsim_i x^i$ for all $(x^i;z^i)\in\mathscr{B}_\infty^*(p,q,\omega^i,\mathscr{A})$.

The debt constrained budget sets $\mathscr{B}_\infty^M$ and $\mathscr{B}_\infty^{DC}$ lead to the following equilibrium concepts.

Definition 3.1. An *equilibrium with implicit debt constraint* (resp. *with explicit debt constraint M*) of the economy $\mathscr{E}_\infty(\boldsymbol{D} \succsim\ \omega, \mathscr{A})$ is a pair

$$((\bar{x}, \bar{z}), (\bar{p}, \bar{q})) \in \ell_\infty^+(\boldsymbol{D} \times \boldsymbol{L} \times \boldsymbol{I}) \times Z^I \times \mathbb{R}^{D \times L} \times Q,$$

where $(\bar{x}, \bar{z}) = (\bar{x}^1, \ldots, \bar{x}^I, \bar{z}^1, \ldots, \bar{z}^I)$, such that

(i) $(\bar{x}^i; \bar{z}^i)$ is $\succsim_i$ maximal in $\mathscr{B}_\infty^{DC}(\bar{p}, \bar{q}, \omega^i, \mathscr{A})$ (resp. in $\mathscr{B}_\infty^M(\bar{p}, \bar{q}, \omega^i, \mathscr{A})$), for each $i \in \boldsymbol{I}$;

(ii) $\sum_{i \in I}(\bar{x}^i - \omega^i) = 0$;

(iii) $\sum_{i \in I} \bar{z}^i = 0$.

The transversality conditions in the budget set $\mathscr{B}_\infty^{TC}$ require that on every subtree the present value of an agent's debt be asymptotically zero. However, when markets are incomplete there is no objective present value vector (which can be deduced from the market prices $(\bar{p}, \bar{q})$) for evaluating an agent's indebtedness. For this reason we use agent i's present value vector $\bar{\pi}^i$ to evaluate the asymptotic value of his debt. The implicit prices $(\bar{\pi}^i)_{i \in I}$ must thus be added to the objective market prices $(\bar{p}, \bar{q})$ to define an equilibrium with the transversality condition.

Definition 3.2. An *equilibrium with the transversality condition* (a *TC equilibrium*) of the economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \mathscr{A})$ is a pair

$$((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$$
$$\in \ell_\infty^+(\boldsymbol{D} \times \boldsymbol{L} \times \boldsymbol{I}) \times Z^I \times \mathbb{R}^{D \times L} \times Q \times \ell_1^+(\boldsymbol{D} \times \boldsymbol{I})$$

such that

(i) $(\bar{x}^i; \bar{z}^i)$ is $\succsim_i$ maximal in $\mathscr{B}_\infty^{TC}(\bar{p}, \bar{q}, \bar{\pi}^i, \omega^i, \mathscr{A})$, for each $i \in \boldsymbol{I}$;

(ii) for each $i \in \boldsymbol{I}$

(a) $\bar{\pi}^i(\xi) > 0$, $\forall \xi \in \boldsymbol{D}$, and $\bar{P}^i \in \ell_1^+(\boldsymbol{D} \times \boldsymbol{L})$, where $\bar{P}^i = (\bar{P}^i(\xi), \xi \in \boldsymbol{D}) = (\bar{\pi}^i(\xi)\bar{p}(\xi), \xi \in \boldsymbol{D})$,

(b) $\bar{x}^i$ is $\succsim_i$ maximal in $\mathscr{B}_\infty(\bar{P}^i, \omega^i) = \{x^i \in X^i \mid \bar{P}^i(x^i - \omega^i) \leqq 0\}$,

(c) $\bar{\pi}^i(\xi)\bar{q}(\xi, j) = \sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi')(\bar{p}(\xi')A(\xi', j) + \bar{q}(\xi', j))$, $\forall j \in J(\xi)$, $\forall \xi \in \boldsymbol{D}$;

(iii) $\sum_{i \in I}(\bar{x}^i - \omega^i) = 0$;

(iv) $\sum_{i \in I} \bar{z}^i = 0$.

Remark. Condition (ii) characterizes the equilibrium present value vector $\bar{\pi}^i$ of agent i. (b) and (c) express the fact that if $\bar{\pi}^i(\xi)$ is the multiplier associated with the budget equation for node ξ (for all $\xi \in \boldsymbol{D}$), then the first-order conditions with respect to x^i and z^i are satisfied. Mackey continuity of the agent's preference ordering implies that $\bar{P}^i$ lies in $\ell_1(\boldsymbol{D} \times \boldsymbol{L})$, the condition required in (a).

4. Existence of a pseudo-equilibrium

Equilibria with the debt constraint and transversality conditions are two ways of extending the notion of an equilibrium with incomplete markets for a finite horizon economy (often called a GEI equilibrium) to an infinite horizon. As is well known, even for a finite horizon economy a GEI equilibrium may not exist. The problem arises from the fact that at a node ξ the dimension of the subspace of $\mathbb{R}^{b(\xi)}$ spanned by the columns of the $b(\xi)\times j(\xi)$ returns matrix,

$$V(\xi^+) = [V(\xi', j)]_{\substack{\xi'\in\xi^+ \\ j\in J(\xi)}} = [p(\xi')A(\xi', j) + q(\xi', j)]_{\substack{\xi'\in\xi^+ \\ j\in J(\xi)}}, \quad (4.1)$$

can change when prices $(p(\xi'), q(\xi'))_{\xi'\in\xi^+}$ vary. This creates discontinuities in agents' demands which may lead to the non-existence of a GEI equilibrium. The way to circumvent this problem, adopted in the recent literature, is to introduce the concept of a pseudo-equilibrium: our objective is to show how this approach can be extended to the infinite horizon case. In this section we show that a pseudo-equilibrium exists for all infinite horizon economies; in the next section it will be shown that 'for most' economies a pseudo-equilibrium is a TC equilibrium. The statement 'for most' economies will mean for a dense rather than a generic set of economies, so that the existence result is weaker than in the finite horizon case.

The idea of a pseudo-equilibrium is as follows: typically if $j(\xi)$ securities are traded at a node ξ which has $b(\xi)$ successors, then the rank of the returns matrix $V(\xi^+)$ in (4.1) is

$$a(\xi) = \min(b(\xi), j(\xi)). \quad (4.2)$$

However, for some prices (p, q) the rank may fall. This statement presupposes that the securities traded at node ξ have non-trivial dividends and/or capital values at the immediate successors of ξ. In this section we consider only equilibria in which securities with zero dividends have a zero capital value (securities are priced at their fundamental values). There is thus no loss of generality in assuming that once a security has matured, it is no longer traded. This amounts to modifying (2.2) to

$$J(\xi) = \{j\in J \mid \xi(j) \leq \xi \text{ and } \exists \xi' > \xi \text{ with } A(\xi', j) \neq 0\}. \quad (4.3)$$

A pseudo-equilibrium is an equilibrium of an economy in which agents are given an *artificial subspace* of income transfers of dimension $a(\xi)$ at node ξ which contains the subspace of transfers achievable with the existing securities – but which is larger when the matrix $V(\xi^+)$ has rank less than $a(\xi)$.

In the analysis of finite horizon economies, a pseudo-equilibrium is defined using a vector of discounted date 0 prices and an abstract subspace at each node. The subspaces are parameterized in a way that is convenient to prove existence. Here we adopt an alternative representation which is more convenient for the passage to the limit from the finite to the infinite case: this representation consists in defining the artificial subspace at each node by an orthogonal basis which may be interpreted as the returns of $a(\xi)$ short-lived numeraire securities issued at node ξ.

Definition 4.1. $\mathscr{K} = (K, (\xi(k))_{k \in K}, (K(\xi))_{\xi \in D}, \Gamma)$ is an *artificial short-lived numeraire security structure* of subspace dimensions $(a(\xi))_{\xi \in D}$ if

(i) $K = \bigcup_{\xi \in D} K(\xi)$, where $K(\xi)$ consists of $a(\xi)$ short-lived numeraire securities issued at node ξ:

$$k \in K(\xi) \Rightarrow \xi(k) = \xi \quad \text{and} \quad \Gamma(\xi', \ell, k) = 0, \quad \text{if } \xi' \notin \xi^+ \text{or } \ell \neq 1;$$

(ii) at each node ξ the returns of the securities issued at node ξ are pairwise orthogonal:

$$\sum_{\xi' \in \xi^+} \Gamma(\xi', 1, k)\Gamma(\xi', 1, k') = 0, \quad \forall k, k' \in K(\xi), k \neq k';$$

(iii) the payoff on each security is non-zero and is normalized so that

$$\max_{\xi' \in \xi^+} |\Gamma(\xi', 1, k)| = 1, \quad \forall k \in K(\xi), \forall \xi \in D.$$

Using a short-lived security structure $\mathscr{K}$ to describe the artificial subspaces of income transfers available to agents leads to the following definition of a pseudo-equilibrium. Note that if V is an $n \times k$ matrix, then $\langle V \rangle$ will denote the subspace of $\mathbb{R}^n$ spanned by the k columns of V.

Definition 4.2. $((\bar{x}, \bar{y}), (\bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i \in I}))$ is a *pseudo-equilibrium* of the economy $\mathscr{E}_\infty(D, \succsim \omega, \mathscr{A})$ if there is an artificial short-lived numeraire security structure $\mathscr{K}$ with subspace dimensions $(a(\xi))_{\xi \in D}$ given by (4.2) such that

(i) $((\bar{x}, \bar{y}), (\bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i \in I}))$ is a TC equilibrium for $\mathscr{E}_\infty(D, \succsim, \omega, \mathscr{K})$;

(ii) there exists $\bar{q} \in \mathbb{R}^{D \times J}$ such that

$$\bar{q}(\xi, j) = \frac{1}{\bar{\pi}^i(\xi)} \sum_{\xi' \in D^+(\xi)} \bar{\pi}^i(\xi')\bar{p}(\xi')A(\xi', j),$$
$$\forall j \in J(\xi), \forall \xi \in D, \forall i \in I; \tag{4.4}$$

(iii)

$$\left\langle [\bar{p}(\xi')A(\xi', j) + \bar{q}(\xi', j)]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} \right\rangle \subset \left\langle [\Gamma(\xi', 1, k)]_{\substack{\xi' \in \xi^+ \\ k \in K(\xi)}} \right\rangle,$$
$$\forall \xi \in D. \tag{4.5}$$

If, in a pseudoequilibrium, the returns matrix of the original security structure $\mathscr{A}$ has maximal rank $a(\xi)$ at each node $\xi \in D$, then the inclusion in (4.5) is an equality. In this case trading in the artificial securities gives each agent access to the same opportunity set as trading in the original securities. Thus when the rank condition in (4.6) below is satisfied, up to converting the portfolios and security prices from the artificial to the original securities, a pseudo-equilibrium is a TC equilibrium.

Proposition 4.3. If $((\bar{x}, \bar{\gamma}), (\bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i \in I}))$ *is a pseudo-equilibrium of* $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \mathscr{A})$ *that satisfies*

$$\operatorname{rank} \left[\bar{p}(\xi') A(\xi', j) + \bar{q}(\xi', j) \right]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} = a(\xi), \quad \forall \xi \in \boldsymbol{D}, \tag{4.6}$$

where $\bar{q}$ *is defined by* (4.4), *then there exists a vector of portfolios for the agents* $\bar{z} = (\bar{z}^1, \dots, \bar{z}^I)$ *such that* $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$ *is a TC equilibrium of* $\mathscr{E}_\infty(\boldsymbol{D}, \succsim \omega, \mathscr{A})$.

Proof. By (4.4), $\forall j \in J(\xi), \forall \xi \in \boldsymbol{D}, \forall i \in \boldsymbol{I}$:

$$\begin{aligned} &\bar{\pi}^i(\xi) \bar{q}(\xi, j) \\ &= \sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi') \left(\bar{p}(\xi') A(\xi', j) + \frac{1}{\bar{\pi}^i(\xi')} \sum_{\xi'' \in \boldsymbol{D}^+(\xi')} \bar{p}(\xi'') A(\xi'', j) \right) \\ &= \sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi') V(\xi', j), \end{aligned}$$

which can be written as $\bar{\pi}^i(\xi)\bar{q}(\xi) = \bar{\pi}^i(\xi^+)V(\xi^+)$, where $\bar{\pi}^i(\xi^+) = (\bar{\pi}^i(\xi'), \xi' \in \xi^+)$, denotes agent i's $b(\xi)$ row vector of present value prices at the successors ξ^+. Let $\Gamma(\xi^+) = [\Gamma(\xi', 1, k)]_{\xi' \in \xi^+_{k \in K(\xi)}}$, then by (4.5) and (4.6):

$$\langle V(\xi^+) \rangle = \langle \Gamma(\xi^+) \rangle, \quad \forall \xi \in \boldsymbol{D}.$$

This implies that the $\mathscr{A}$-budget set $\mathscr{B}_\infty^{\mathrm{TC}}(\bar{p}, \bar{q}, \bar{\pi}^i, \omega^i, \mathscr{A})$ and the $\mathscr{K}$-budget set $\mathscr{B}_\infty^{\mathrm{TC}}(\bar{p}, \bar{\rho}, \bar{\pi}^i, \omega^i, \mathscr{K})$ are the same. Note that if $z^i(\xi)$ and $\gamma^i(\xi)$ are portfolios generating the same income transfer

$$V(\xi^+) z^i(\xi) = \Gamma(\xi^+) \gamma^i(\xi), \tag{4.7}$$

then

$$\begin{aligned} \bar{\pi}^i(\xi) \bar{q}(\xi) z^i(\xi) &= \bar{\pi}^i(\xi^+) V(\xi^+) z^i(\xi) = \bar{\pi}^i(\xi^+) \Gamma(\xi^+) \gamma^i(\xi) \\ &= \bar{\pi}^i(\xi) \bar{\rho}(\xi) \gamma^i(\xi) \end{aligned}$$

so that they have the same cost. Thus, since every trading strategy γ^i in the $\mathscr{K}$-budget set has an equivalent trading strategy z^i in the $\mathscr{A}$-budget and conversely, the two budget sets coincide. Let $\bar{z}^i$ be a trading strategy in the $\mathscr{A}$-budget set corresponding to $\bar{\gamma}^i$ in the $\mathscr{K}$-budget set for $i = 2, \dots, I$ and let $\bar{z}^1 = \sum_{i=2}^I \bar{z}^i$. It is easy to check that $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$ is a TC equilibrium of $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \mathscr{A})$. □

We now prove the main theorem of this section.

Theorem 4.4. Each economy $\mathscr{E}_\infty(D, \succsim, \omega, \mathscr{A})$ that satisfies Assumptions B1–B6 has a pseudo-equilibrium.

Proof. The theorem will be proved by taking limits of pseudo-equilibria of truncated economies in which trade stops at some finite date.

Let $\mathscr{E}_\infty(D, \succsim, \omega, \mathscr{A}))$ be an infinite horizon economy: the associated T-truncated economy $\mathscr{E}_T(D, \succsim, \omega, \mathscr{A}))$ is the economy with the same characteristics as $\mathscr{E}_\infty$ in which agents are constrained to stop trading at date T. If $(p_T, q_T) \in \mathbb{R}^{D\times L} \times Q$ is a commodity and security price process, then the budget set of agent i in the truncated economy $\mathscr{E}_T$ is defined by

$$\mathscr{B}_T\big(p_T, q_T, \omega^i, \mathscr{A}\big) = \left\{ x^i \in X^i \;\middle|\; \begin{array}{l} \exists z^i \in Z,\ z^i(\xi) = 0, \ \text{if } t(\xi) \geqq T \\ p_T(\xi)(x^i(\xi) - \omega^i(\xi)) \\ \quad = (p_T(\xi)A(\xi) + q_T(\xi))z^i(\xi^-) \\ \qquad - q_T(\xi)z^i(\xi), \ \text{if } t(\xi) \leqq T \\ x^i(\xi) = \omega^i(\xi), \ \text{if } t(\xi) > T \end{array} \right\}.$$

Even though the consumption-portfolio process of an agent is defined over the whole event tree, a T-truncated economy is essentially a finite horizon economy with $T + 1$ periods since an agent's consumption-portfolio process is fixed after date T.

Definition 4.6. A GEI *equilibrium* of the truncated economy $\mathscr{E}_T(D, \succsim, \omega, \mathscr{A})$ is a pair

$$((\bar{x}_T, \bar{z}_T), (\bar{p}_T, \bar{q}_T)) \in \ell_\infty^+(D \times L \times I) \times Z^I \times \mathbb{R}^{D\times L} \times Q$$

such that

(i) $(\bar{x}_T^i; \bar{z}_T^i)$ is $\succsim_i$ maximal in $\mathscr{B}_T(\bar{p}_T, \bar{q}_T, \omega^i, \mathscr{A})$, $\forall i \in I$;
(ii) $\sum_{i\in I}(\bar{x}_T^i - \omega^i) = 0$;
(iii) $\sum_{i\in I} \bar{z}_T^i = 0$;
(iv) $\bar{p}_T(\xi) = 0$, if $t(\xi) > T$, and $\bar{q}_T(\xi) = 0$, if $t(\xi) \geqq T$.

Since only the prices of the commodities and securities that are traded in $\mathscr{E}_T$ are well determined, (iv) is a natural way to extend these prices to the whole event tree. Since in an equilibrium of the truncated economy the terminal condition $z_T^i(\xi) = 0$ for all ξ, with $t(\xi) \geqq T$ replacing the transversality condition of the budget set $\mathscr{B}_\infty^{TC}$, the present value vectors of the agents do not appear explicitly in Definition 4.6. However each agent has a well-defined present value vector in an equilibrium of $\mathscr{E}_T$ defined by the first-order conditions of the agent's maximization problem. When an agent's preference ordering is represented by a differentiable utility function, then the present value vector is simply the vector of Lagrange multipliers given by the Kuhn–Tucker Theorem. When the preference ordering is assumed not to be differentiable (as is the case here), the existence of a present value vector can be derived by a direct separation argument. In Magill and Quinzii (1994, lemma 5.6) it is shown that under Assumption B3 if $((\bar{x}_T, \bar{z}_T), (\bar{p}_T, \bar{q}_T))$ is a GEI equilibrium of $\mathscr{E}_T(D, \succsim, \omega, \mathscr{A})$, then each agent $i \in I$ has a present value vector $\bar{\pi}_T^i \in \mathbb{R}^D$ satisfying

(a) $\bar{\pi}_T^i(\xi) > 0$, if $t(\xi) \leqq T$, and $\bar{\pi}_T^i(\xi) = 0$, if $t(\xi) > T$;

(b) $\bar{x}_T^i$ is $\underset{i}{\succsim}$ maximal in

$$B_T(\bar{P}_T^i, \omega^i) = \left\{ x^i \in X^i \middle| \begin{array}{l} \bar{P}_T^i(x^i - \omega^i) \leqq 0 \\ \\ x^i(\xi) = \omega^i(\xi), \quad \text{if } t(\xi) > T \end{array} \right\},$$

where $\bar{P}_T^i = (\bar{P}_T^i(\xi), \xi \in D) = (\bar{\pi}_T^i(\xi)\bar{p}_T(\xi), \xi \in D)$

(c) $\bar{\pi}_T^i(\xi)\bar{q}_T(\xi, j) = \sum_{\xi' \in \xi^+} \bar{\pi}_T^i(\xi')(\bar{p}_T(\xi')A(\xi', j) + \bar{q}_T(\xi', j))$, $\forall j \in J(\xi)$, $t(\xi) \leqq T - 1$.

The following definition is the finite horizon analog of Definition 4.2.

Definition 4.7. $((\bar{x}_T, \bar{\gamma}_T), (\bar{p}_T, \bar{\rho}_T))$ is a *pseudo-equilibrium* of the truncated economy $\mathscr{E}_T(D, \succsim, \omega, \mathscr{A})$ if there is an artificial short-lived numeraire security structure $\mathscr{K}_T$ with subspace dimensions $(a(\xi))_{\xi \in D}$, where $a(\xi) = \min(b(\xi), J(\xi))$ for $\xi \in D^{T-1}$, such that

(i) $((\bar{x}_T, \bar{\gamma}_T), (\bar{p}_T, \bar{\rho}_T))$ is a GEI equilibrium for $\mathscr{E}_T(D, \succsim, \omega, \mathscr{K}_T)$;

(ii) if π_T is any positive vector of node prices satisfying

$$\pi_T(\xi)\bar{\rho}(\xi) = \pi_T(\xi^+)\Gamma_T(\xi^+), \quad t(\xi) \leqq T - 1, \tag{4.8}$$

and if $\bar{q}_T$ is defined by

$$\bar{q}_T(\xi, j) = \begin{cases} \dfrac{1}{\pi_T(\xi)} \displaystyle\sum_{\xi' \in D^+(\xi)} \pi_T(\xi')\bar{p}_T(\xi')A(\xi', j), \\ \quad \text{if } j \in J(\xi), \ \xi \in D^{T-1}, \\ 0, \quad \text{otherwise}, \end{cases} \tag{4.9}$$

then

$$\left\langle [\bar{p}_T(\xi')A(\xi', j) + \bar{q}_T(\xi', j)]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} \right\rangle \subset \left\langle [\Gamma_T(\xi', 1, k)]_{\substack{\xi' \in \xi^+ \\ k \in K(\xi)}} \right\rangle,$$

$$\forall \xi \in D^{T-1}. \tag{4.10}$$

Remark. When (4.10) holds, the definition of $\bar{q}_T$ in (4.9) is independent of the choice of the vector of node prices π_T satisfying (4.8). This can be seen as follows:

$$\bar{q}_T(\xi, j) = \frac{1}{\pi_T(\xi)} \sum_{\xi' \in \xi^+} \pi_T(\xi')(\bar{p}_T(\xi')A(\xi', j) + \bar{q}_T(\xi', j))$$
$$= \frac{1}{\pi_T(\xi)} \sum_{\xi' \in \xi^+} \pi_T(\xi') \sum_{k \in K(\xi)} \alpha_k^j \Gamma_T(\xi', 1, k)$$
$$= \sum_{k \in K(\xi)} \alpha_k^j \bar{p}(\xi, k),$$

where $(\alpha_k^j, k \in K(\xi))$ are the coordinates of the vector $(\bar{p}_T(\xi')A(\xi', j) + \bar{q}_T(\xi', j))_{\xi' \in \xi^+}$ on the basis $(\Gamma_T(\cdot, 1, k), k \in K(\xi))$. The reverse calculation with any other vector of nodes prices $(\tilde{\pi}_T(\xi), \xi \in D^T)$ satisfying (4.8) leads to (4.9) using the terminal condition $\bar{q}_T(\xi, j) = 0$ if $t(\xi) = T$.

It follows from the extension of the existence result of Hirsch et al. (1990) to the multiperiod case [2] (see Magill and Shafer, 1991, theorem 16) that a pseudo-equilibrium $((\bar{x}_T, \bar{y}_T), (\bar{p}_T, \bar{\rho}_T))$ of the truncated economy $\mathscr{E}_T(\boldsymbol{D}, \succsim, \omega, \mathscr{A})$ exists for every finite T. Let $\bar{\mathscr{K}}_T$ denote the associated artificial short-lived numeraire security structure and let $\bar{\Gamma}_T$ denote its (numeraire) commodity payoff process. Since $((\bar{x}_T, \bar{y}_T), (\bar{p}_T, \bar{\rho}_T))$ is a GEI equilibrium of $\mathscr{E}_T(\boldsymbol{D}, \succsim, \omega, \bar{\mathscr{K}}_T)$, it has associated with it present value vectors and discounted prices for the agents $(\bar{\pi}_T^i, \bar{P}_T^i)_{i \in I}$ satisfying (a) and (b) above, (c) being replaced by (4.8). $\bar{\pi}_T^i$, and hence $\bar{P}_T^i$, can be normalized by setting

$$\bar{P}_T^i \mathbb{1} = \sum_{(\xi, \ell) \in D^T \times L} \bar{P}_T^i(\xi, \ell) = 1, \quad \forall i \in \boldsymbol{I}, \forall T \in \boldsymbol{T}, \tag{4.11}$$

where $\mathbb{1} = (1, \ldots, 1, \ldots) \in \ell_\infty(\boldsymbol{D} \times \boldsymbol{L})$ denotes the vector all the components of which are equal to 1. Let $\bar{q}_T$ denote the prices of the original securities defined by (4.9). In view of the remark following Definition 4.7 for each $i \in \boldsymbol{I}$:

$$\bar{q}_T(\xi, j) = \frac{1}{\bar{\pi}_T^i(\xi)} \sum_{\xi' \in D^+(\xi)} \bar{\pi}_T^i(\xi')\bar{p}_T(\xi')A(\xi', j),$$
$$\forall j \in J(\xi), \forall \xi \in \boldsymbol{D}^{T-1},$$

which can be written as

$$\bar{q}_T(\xi, j) = \frac{1}{\bar{\pi}_T^i(\xi)} \bar{P}_T^i A(\cdot, j) \chi_{D^+(\xi)}, \quad \forall j \in J(\xi), \forall \xi \in \boldsymbol{D}^{T-1}. \tag{4.12}$$

[2] This result cannot be derived from the existence result of Duffie and Shafer (1986) for multiperiod economies which depends on the differentiability of agents' demand functions – a property which is not required here.

Let $e_T = (\bar{x}_T, \bar{y}_T, \bar{p}_T, \bar{\rho}_T, (\bar{\pi}_T^i)_{i \in I}, \bar{\Gamma}_T)$ denote the actions, prices and payoff process characterizing the T-period pseudo-equilibrium. Since $\bar{x}_T \in \mathbb{R}^{D \times L \times I}$, $\bar{q}_T \in \mathbb{R}^{D \times J \times I}$, etc. e_T (and hence the sequence $(e_T)_{T \in T}$) lies in a space Y which is a product of Euclidean spaces. The discounted prices $(\bar{P}_T^i, i \in I)_{T \in T}$ can be viewed as elements of $ba(D \times L) = \ell_\infty^*(D \times L)$, i.e. the norm dual of $\ell_\infty(D \times L)$, consisting of bounded finitely additive set functions on $D \times L$. Let $\| \cdot \|_{ba}$ denote the norm of $ba(D \times L)$ and let $\sigma(ba, \ell_\infty)$ denote the weak * topology of $ba(D \times L)$.

The idea is to take limits of e_T in the product topology and of $(P_T^i, i \in I)$ in the $\sigma(ba, \ell_\infty)$ topology. In Magill and Quinzii (1994, step 1 in the proof of theorem 5.1) it is shown that the actions and prices $(\bar{x}_T, \bar{y}_T, \bar{p}_T, \bar{\rho}_T, (\bar{\pi}_T^i)_{i \in I})$ can be bounded independently of T and that for each $\xi \in D$ there exists $c_\xi > 0$ such that $\bar{\pi}_T^i(\xi) \geqq c_\xi$, $\forall i \in I$, $\forall T \in T$. By property (iii) of Definition 4.1 the commodity payoffs $\bar{\Gamma}_T$ are bounded independently of T, $|\bar{\Gamma}_T(\xi, 1, k)| \leqq 1$, $\forall k \in K$, $\forall \xi \in D$. It follows from Tychonov's Theorem that $(e_T)_{T \in T}$ lies in a subset of Y which is compact in the product topology. Since (4.11) implies $\| P_T^i \|_{ba} = 1$, $\forall i \in I$, $\forall T \in T$, these prices belong to the unit sphere in $ba(D \times L)$ which, by Alaoglu's theorem, is compact in the $\sigma(ba, \ell_\infty)$ topology. Thus there exists a directed net $(\Lambda, \geqq)$ and a subnet $\{(\bar{P}_{T_\lambda}^i, i \in I, \lambda \in (\Lambda, \geqq)\}$ such that $\bar{P}_{T_\lambda}^i$ converges to $\bar{P}^i$ in the $\sigma(ba, \ell_\infty)$ topology, $\forall i \in I$. By extracting an appropriate subnet, e_{T_λ} converges to $e = (\bar{x}, \bar{y}, \bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i \in I}, \bar{\Gamma})$ in the product topology.

It is clear that $\bar{\Gamma}$ satisfies (ii) and (iii) in Definition 4.2 and has $a(\xi)$ linearly independent payoffs at each node ($\xi \in D$: let $\mathscr{K}(\xi)$ denote these securities and let $K = \bigcup_{\xi \in D} K(\xi)$. Then $\bar{\mathscr{K}} = (K, \xi(k)_{k \in K}, (K(\xi))_{\xi \in D}, \bar{\Gamma})$ is an artificial short-lived numeraire security structure with subspace dimensions $(a(\xi))_{\xi \in D}$. It follows from Magill and Quinzii (1994, step 3 in the proof of theorem 5.1) that $((\bar{x}, \bar{y}), (\bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i \in I}))$ is a TC equilibrium of $\mathscr{E}_\infty(D, \succsim, \omega, \bar{\mathscr{K}})$: in particular it is shown that for each $i \in I$, the limit price $\bar{P}^i$ lies in $\ell_1(D \times L)$ and that $\bar{P}^i(\xi, \ell) = \bar{\pi}^i(\xi)\bar{p}(\xi, \ell)$, $\forall (\xi, \ell) \in D \times L$. Since $A(\cdot, j)\chi_{D^+(\xi)} \in \ell_\infty(D \times L)$ and since $\bar{P}_{T_\lambda}^i$ converges to $\bar{P}^i$ in the $\sigma(ba, \ell_\infty)$ topology,

$$\bar{P}_{T_\lambda}^i A(\cdot, j)\chi_{D^+(\xi)} \to \bar{P}^i A(\cdot, j)\chi_{D^+(\xi)}.$$

Thus (4.12) implies that $\forall i \in I$, $\forall j \in J(\xi)$, $\forall \xi \in D$:

$$\bar{q}_{T_\lambda}(\xi, j) \to \frac{1}{\bar{\pi}^i(\xi)} \bar{P}^i A(\cdot, j)\chi_{D^+(\xi)}$$

$$= \frac{1}{\bar{\pi}^i(\xi)} \sum_{\xi' \in D^+(\xi)} \bar{\pi}^i(\xi')\bar{p}(\xi')A(\xi', j).$$

Let $\bar{q}(\xi, j)$ denote this limit. Then $\bar{q} = (\bar{q}(\xi, j), \xi \in D, j \in J(\xi))$ satisfies (4.4). Passing to the limit in (4.10) in the obvious way gives (4.5), and the proof is complete. □

Remark. Note that it is the fact that, for each agent $i \in \boldsymbol{I}$, the personalized prices $\bar{P}^i_{T_\lambda}$ converge to $\bar{P}^i$ in the $\sigma(ba, \ell_\infty)$ topology and that the limit price satisfies $\bar{P}^i \in \ell_1(\boldsymbol{D} \times \boldsymbol{L})$ and $\bar{P}^i(\xi, \ell) = \bar{\pi}^i(\xi)\bar{p}(\xi, \ell)$, $\forall (\xi, \ell) \in \boldsymbol{D} \times \boldsymbol{L}$, which enables us to show that each security price $\bar{q}_{T_\lambda}(\cdot, j)$ converges to the discounted sum of its future dividends for each agent. This result seems difficult to obtain using convergence of the personalized prices in the product topology.

5. Existence of an equilibrium

In this section we show that for an economy with given uncertainty and agent characteristics $(\boldsymbol{D}, \succsim, \omega)$ and for given nodes of issue and maturity for the securities $(\boldsymbol{J}, (\xi(j))_{j \in \boldsymbol{J}}, (J(\xi))_{\xi \in \boldsymbol{D}})$ there is a dense set of security payoffs $\boldsymbol{A}^*$ such that for all $A \in \boldsymbol{A}^*$ the economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \mathscr{A})$, with $\mathscr{A} = (\boldsymbol{J}, (\xi(j))_{j \in \boldsymbol{J}}, (J(\xi))_{\xi \in \boldsymbol{D}}, A)$, has a pseudo-equilibrium which is a TC equilibrium. To establish such a result we first define a set of admissible security payoffs compatible with a given set of nodes of issue and maturity. More precisely, let $(\boldsymbol{J}, (\xi(j))_{j \in \boldsymbol{J}}, (J(\xi))_{\xi \in \boldsymbol{D}})$ be a collection of securities, nodes of issue and traded securities such that $J(\xi)$ is finite for all $\xi \in \boldsymbol{D}$ and such that the consistency conditions (ii) and (iii) in Definition 2.1 are satisfied. The set $\boldsymbol{A}$ of *admissible payoffs* for the securities is defined by

$$\boldsymbol{A} = \left\{ A \in \ell_\infty(\boldsymbol{D} \times \boldsymbol{L} \times \boldsymbol{J}) \;\middle|\; \begin{array}{l} A(\xi, j) = 0, \quad \text{if } \xi \notin \boldsymbol{D}^+(\xi(j)) \\ \xi \in \boldsymbol{D}(\xi(j)) \text{ and} \\ \quad j \notin J(\xi) \Rightarrow A(\xi', j) = 0, \quad \forall \xi' > \xi \\ A(\xi', \ell, j_\xi) = \begin{cases} 1, & \text{if } \xi' \in \xi^+, \ell = 1 \\ 0, & \text{otherwise} \end{cases} \end{array} \right\}.$$

The definition of $\boldsymbol{A}$ ensures that each payoff process $A \in \boldsymbol{A}$ is compatible with $(\boldsymbol{J}, \xi(j)_{j \in \boldsymbol{J}}, (J(\xi)_{\xi \in \boldsymbol{D}})$ – in the sense that (i) and (iv) in Definition 2.1 are satisfied – and that Assumptions B5 and B6 hold.

Since $\boldsymbol{A} \subset \ell_\infty(\boldsymbol{D} \times \boldsymbol{L} \times \boldsymbol{J})$, it is natural to endow $\boldsymbol{A}$ with the norm topology, the norm of a payoff process A being defined by

$$\| A \|_\infty = \sup_{(\xi, \ell, j) \in \boldsymbol{D} \times \boldsymbol{L} \times \boldsymbol{J}} | A(\xi, \ell, j) |.$$

Note that $\boldsymbol{A}$ is a closed subset of $\ell_\infty(\boldsymbol{D} \times \boldsymbol{L} \times \boldsymbol{J})$.

Theorem 5.1. Under Assumptions B1–B6 there exists a dense subset $\boldsymbol{A}^* \subset \boldsymbol{A}$ such that if $A \in \boldsymbol{A}^*$, then the infinite horizon economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \mathscr{A})$ has an equilibrium with the transversality condition.

Proof. See the appendix.

The idea of the proof is to show that if we pick a payoff process $\bar{A} \in \boldsymbol{A}$ for which the rank condition

$$\operatorname{rank}\left[\bar{p}(\xi')\bar{A}(\xi', j) + \bar{q}(\xi', j)\right]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} = a(\xi), \forall \xi \in \boldsymbol{D}$$

is not satisfied in a pseudo-equilibrium $((\bar{x}, \bar{y}), (\bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i \in I})$ of $\mathscr{E}_x(\boldsymbol{D}, \succsim, \omega, \bar{\mathscr{A}})$ then for all $\epsilon > 0$, there exists a payoff process $A \in \boldsymbol{A}$ in the ball of radius ϵ around $\bar{A}$ such that $((\bar{x}, \bar{y}), (\bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i \in I})$ is a pseudo-equilibrium for $\mathscr{E}_x(\boldsymbol{D}, \succsim, \omega, \mathscr{A}))$ in which the rank condition is satisfied for A.

5.1. Equilibrium with a debt constraint and the transversality condition

For a deterministic sequence economy over an infinite horizon in which there is a market for short-term borrowing and lending at each date, it has long been recognized that a concept of equilibrium based on a transversality condition on the indebtedness of each agent – requiring that an agent's debt grows asymptotically slower than the rate of interest – permits the link to be made between the equilibria of the sequence economy and the Arrow–Debreu equilibria (see, for example, Kehoe, 1989). The concept of an equilibrium with the transversality condition in Definition 3.2 provides an extension of this concept to the case of stochastic economies with incomplete markets. When markets are incomplete there is no unique present value vector (representing implicit prices of income in the future) which links the prices of the securities to their future dividends, i.e. which satisfies

$$\pi(\xi)\bar{q}(\xi) = \sum_{\xi'\xi^+} \pi(\xi')(\bar{p}(\xi')A(\xi') + \bar{q}(\xi')), \quad \forall \xi \in \boldsymbol{D}.$$

For this reason the indebtedness of each agent is evaluated using his own present value vector $\bar{\pi}^i$ at the equilibrium. Some economists may argue that this makes the concept of a TC equilibrium somewhat questionable as a descriptive concept of an equilibrium: since agents are assumed to assess their capacity to postpone the repayment of their debts (i.e. their capacity to borrow from the market) using their own present value vector, the concept of an equilibrium is based on self-monitoring rather than some objective market-based scheme for controlling the indebtedness of agents.

Even for deterministic economies, macroeconomists have typically been more comfortable with a concept of equilibrium based on explicit debt constraints. It is thus of some interest that *under Assumption B4 the equilibria of an economy with implicit debt constraints can be shown to coincide with the equilibria with the transversality condition*. This, in turn, implies that the equilibria can be obtained by imposing an explicit bound M on the indebtedness of the agents at each node, where M can be chosen so that it is never binding. Proposition 5.3 below permits the advantages of each concept of equilibrium to be retained, since for a theoretical (mathematical) analysis a TC equilibrium is more natural, while an equilibrium with a debt constraint is more plausible as a descriptive concept of equilibrium.

Proposition 5.2. Under Assumption B4, if $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i\in I}))$ *is a TC equilibrium of* $\mathscr{E}_\infty(D, \succsim, \omega, \mathscr{A})$, *then* $(\bar{q}\bar{z}^i) \in \ell_\infty(D)$ *for all* $i \in I$.

Proof. Let $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i\in I}))$ be a TC equilibrium. Pick any node $\bar{\xi} \in D$ and consider an agent who is a net lender at $\bar{\xi}$, i.e. $\bar{q}(\bar{\xi})\bar{z}^i(\bar{\xi}) \geqq 0$. Agent i can consider scaling down his portfolio from node $\bar{\xi}$ onwards, i.e. to change $\bar{z}^i$ to $\tilde{z}^i$ defined by

$$\tilde{z}^i(\xi) = \begin{cases} \bar{z}^i(\xi), & \text{if } \xi \notin D(\bar{\xi}), \\ \beta\bar{z}^i(\xi), & \text{if } \xi \in D(\bar{\xi}), \end{cases}$$

where $\beta < 1$ is the factor defined by Assumption B4. It is easy to check that this new portfolio satisfies the transversality conditions

$$\lim_{T\to\infty} \sum_{\xi' \in D_T(\xi)} \bar{\pi}^i(\xi')\bar{q}(\xi')\tilde{z}^i(\xi') = 0$$

for all $\xi \in D$. With the trading strategy $\tilde{z}^i$, agent i can consume $\bar{x}^i(\xi)$ if $\xi \in D(\bar{\xi})$, at least $\beta\bar{x}^i(\xi)$ if $\xi \in D^+(\bar{\xi})$ since

$$\begin{aligned} &p(\xi)\omega^i(\xi) + \beta V(\xi)\bar{z}^i(\xi^-) - \beta\bar{q}(\xi)\bar{z}^i(\xi) \\ &\quad = \beta\bar{p}(\xi)\bar{x}^i(\xi) + (1-\beta)\bar{p}(\xi)\omega^i(\xi) \geqq \beta\bar{p}(\xi)\bar{x}^i(\xi), \end{aligned}$$

and can increase his consumption of good 1 at node $\bar{\xi}$ by $(1-\beta)\bar{q}(\bar{\xi})\bar{z}^i(\bar{\xi})$ since

$$\begin{aligned} &\bar{p}(\xi)\left(\bar{x}^i(\bar{\xi}) + (1-\beta)\bar{q}(\bar{\xi})\bar{z}^i(\bar{\xi})e_1^{\bar{\xi}}\right) \\ &\quad = \bar{p}(\bar{\xi})\omega^i(\bar{\xi}) + V(\bar{\xi})\bar{z}^i(\bar{\xi}^-) - \beta\bar{q}(\bar{\xi})\bar{z}^i(\bar{\xi}). \end{aligned}$$

By Assumption B4 the increment $(1-\beta)\bar{q}(\bar{\xi})\bar{z}^i(\bar{\xi})$ to his consumption of good 1 must be less than 1 so that

$$\bar{q}(\bar{\xi})\bar{z}^i(\bar{\xi}) \geqq 0 \Rightarrow \bar{q}(\bar{\xi})\bar{z}^i(\bar{\xi}) \leqq \frac{1}{1-\beta}.$$

Since $\sum_{i\in I}\bar{q}(\bar{\xi})\bar{z}^i(\bar{\xi}) = 0$, agents who are net borrowers must find net lenders. Thus

$$\bar{q}(\bar{\xi})\bar{z}^i(\bar{\xi}) \leqq 0 \Rightarrow -\left(\frac{I-1}{1-\beta}\right) \leqq \bar{q}(\bar{\xi})\bar{z}^i(\bar{\xi})$$

so that

$$-\left(\frac{I-1}{1-\beta}\right) \leqq \bar{q}(\bar{\xi})\bar{z}^i(\bar{\xi}) \leqq \frac{1}{1-\beta}, \quad \forall i \in \boldsymbol{I}.$$

Thus for each agent $i \in \boldsymbol{I}$, $(\bar{q}\bar{z}^i) \in \ell_\infty(\boldsymbol{D})$. □

Proposition 5.3. Under Assumptions B1–B4, $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}))$ is an equilibrium with an implicit debt constraint of $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \mathscr{A})$ if and only if there exist present value vectors $(\bar{\pi}^i)_{i \in I}$ such that $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$ is a TC equilibrium.

Proof. ($\Leftarrow$) The fact that a TC equilibrium $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$ is also an equilibrium with implicit debt constraints follows from Proposition 5.2. Since for all $i \in \boldsymbol{I}$, $(\bar{x}^i; \bar{z}^i) \in \mathscr{B}_\infty^{DC}(\bar{p}, \bar{q}, \omega^i, \mathscr{A}) \subset \mathscr{B}_\infty^{TC}(\bar{p}, \bar{q}, \bar{\pi}^i, \omega^i, \mathscr{A})$ and since $(\bar{x}^i; \bar{z}^i)$ is $\succsim_i$ maximal in the larger budget set $\mathscr{B}_\infty^{TC}$, it is $\succsim_i$ maximal in $\mathscr{B}_\infty^{DC}$. Thus $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}))$ is an equilibrium with an implicit debt constraint.

($\Rightarrow$) The main ideas of the proof are as follows: for a complete proof see Magill and Quinzii (1993, theorem 5.2). Let $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}))$ be an equilibrium with an implicit debt constraint. Since $\bar{x}^i$ is $\succsim_i$ maximal in $\mathscr{B}_\infty^{DC}(\bar{p}, \bar{q}, \omega^i, \mathscr{A})$, a separation argument gives the existence of a price $P^i \in ba(\boldsymbol{D} \times \boldsymbol{L})$ which separates the preferred set $\mathscr{U}^i = \{x^i \in \ell_\infty^+(\boldsymbol{D} \times \boldsymbol{L}) \mid x^i \succsim_i \bar{x}^i\}$, which has a non-empty interior in the norm topology, from $\mathscr{B}_\infty^{DC}$. Mackey continuity of $\succsim_i$ implies that $P^i \in \ell_1(\boldsymbol{D} \times \boldsymbol{L})$. Since $\mathscr{B}_\infty^{DC}$ is defined by linear inequalities, Farkas' Lemma applied to a sequence of truncated budget sets leads to the existence of a present value vector $\bar{\pi}^i \in l_1(\boldsymbol{D})$ which is strictly positive by the monotonicity of $\succsim_i$ and satisfies (a)–(c) in Definition 3.2(ii). Since for each $i \in \boldsymbol{I}$, $\bar{x}^i \in \mathscr{B}_\infty^{DC} \subset \mathscr{B}_\infty^{TC} \subset B(P^i, \omega^i)$ and since by the separation argument $\bar{x}^i$ is $\succsim_i$ maximal in the larger set $B(P^i, \omega^i)$, it is also $\succsim_i$ maximal in $\mathscr{B}_\infty^{TC}$. Thus $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$ is a TC equilibrium. □

Corollary 5.4. Under Assumption B1–B6 there exists a dense subset $A^* \subset A$ such that if $A \in A^*$, then the infinite horizon economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \mathscr{A})$ has an equilibrium with an explicit debt constraint M which is never binding.

Proof. Let A^* be the dense subset in Theorem 5.1. If the commodity payoff process A lies in A^*, then the economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \mathscr{A})$ has a TC equilibrium $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$, which by Proposition 5.3 is an equilibrium with an implicit debt constraint. Any number $M > \bar{M}$, where $\bar{M} = \max_{i \in I} \sup_{\xi \in D} |\bar{q}(\xi)\bar{z}^i(\xi)|$, provides a debt constraint that is never binding. □

5.2. Generalization to securities in positive supply

Although the model introduced in Section 2 is restricted to an exchange economy in which securities are in zero net supply it can be generalized to include securities in positive supply (such as equity contracts) that arise naturally in a production economy.

Let $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \delta, \mathscr{A})$ denote an economy which is identical in all respects to that considered in Section 2 except that a subset $J_0 \subset J(\xi_0)$ of the securities issued at date 0 can have *positive initial supply* $\delta = (\delta_j, j \in J_0)$, where $\delta_j = \sum_{i \in I} \delta_j^i$ and δ_j^i is agent i's initial holding of security j. (Securities issued after date 0 could also be permitted to be in positive initial supply, but this complicates the notation unnecessarily.) A TC equilibrium of the economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \delta, \mathscr{A})$ is given by Definition 3.2 with the following modifications: the new budget set $\mathscr{B}_\infty^{\mathrm{TC}}(p, q, \pi^i, \omega^i, \delta^i, \mathscr{A})$ of agent i is identical to that defined in Section 3 except for the equation at the initial node which becomes

$$p(\xi_0)\big(x^i(\xi_0) - \omega^i(\xi_0)\big) = q(\xi_0)\big(\delta^i - z^i(\xi_0)\big), \tag{5.1}$$

and the market-clearing condition (iv) becomes

$$\sum_{i \in \boldsymbol{I}} \bar{z}^i(\xi, j) = \delta_j, \quad \forall \xi \in \boldsymbol{D}, \forall j \in \boldsymbol{J}, \tag{5.2}$$

where $\delta_j = 0$ if $j \notin J_0$.

Securities in positive supply model ownership rights to the income (dividend) streams created by productive assets such as firms, land or other durable capital goods. An economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \delta, \mathscr{A})$ in which there are securities in positive (initial) supply thus serves as a model of a production economy in which all decisions regarding the use of the productive assets (i.e. the production plans) are fixed: for such an economy we make the following additional assumption.

Assumption B7. If $\delta_j > 0$, then $A(\cdot, j) \in \ell_\infty^+(\boldsymbol{D} \times \boldsymbol{L})$ and $\delta_j^i \geqq 0$ for all $i \in \boldsymbol{I}$. If $\delta_j = 0$, then $\delta_j^i = 0$ for all $i \in \boldsymbol{I}$.

Thus securities in positive supply represent productive assets with non-negative payoffs and agents only inherit non-negative initial shares of such assets. For securities in zero supply agents do not inherit any initial debt (or credit).

Let us show how the above existence results can be extended to such economies. To this end let [3]

$$\Omega = \left\{\omega = (\omega^1, \ldots, \omega^I) \in \ell_\infty^+(\boldsymbol{D} \times \boldsymbol{L} \times \boldsymbol{I}) \mid \omega^i \gg m\boldsymbol{1}, i \in \boldsymbol{I}\right\}$$

[3] For $x, y \in \ell_\infty(\boldsymbol{D} \times \boldsymbol{L})$, $x \gg y$ means $x(\xi, \ell) > y(\xi, \ell)$ for all $(\xi, \ell) \in \boldsymbol{D} \times \boldsymbol{L}$.

denote the space of initial endowments of the agents satisfying Assumption B2. Theorem 5.1 leads to the following result.

Theorem 5.5. Under Assumptions B1–B7 there exists a dense subset $\Delta \subset \Omega \times A$ such that if $(\omega, A) \in \Delta$, then the infinite horizon economy $\mathscr{E}(\boldsymbol{D}, \succsim, \omega, \delta, \mathscr{A})$ has a TC equilibrium.

Proof. With an economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \delta, \mathscr{A})$ in which some securities are in positive initial supply we may associate an economy $\tilde{\mathscr{E}}_\infty(\boldsymbol{D}, \succsim, \omega, \mathscr{A})$ in which all securities are in zero initial supply and agents have the modified endowments

$$\underset{\sim}{\omega}^i(\xi) = \omega^i(\xi) + \sum_{j \in J_0} \delta^i_j A(\xi, j), \quad \forall \xi \in \boldsymbol{D}.$$

If $\tilde{\mathscr{E}}_\infty(\boldsymbol{D}, \succsim, \omega, \mathscr{A})$ does not have a TC equilibrium, then by (the construction of the equilibrium in the proof of) Theorem 5.1 there exists, for each $\epsilon > 0$, $A' \subset A^*$ such that $\| A' - A \|_\infty < \epsilon$ and $\tilde{\mathscr{E}}_\infty(\boldsymbol{D}, \succsim, \omega, \mathscr{A}')$ has a TC equilibrium $((\bar{x}, \bar{\bar{z}}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$, where $\bar{q}$ satisfies the pricing equations (4.4). Let us show that if for each $i \in \boldsymbol{I}$

$$\bar{z}^i(\cdot, j) = \bar{\bar{z}}^i(\cdot, j) + \delta^i_j, \quad \forall j \in \boldsymbol{J},$$

and ω' is such that for all $i \in \boldsymbol{I}$

$$\omega'^i(\xi) + \sum_{j \in J_0} \delta^i_j A'(\xi, j) = \underset{\sim}{\omega}^i(\xi) = \omega^i(\xi) + \sum_{j \in J_0} \delta^i_j A(\xi, j), \quad \forall \xi \in \boldsymbol{D}, \tag{5.3}$$

then $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$ is a TC equilibrium of $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega', \delta, \mathscr{A}')$. To see this it suffices to check that if $z^i(\cdot, j) = \bar{\bar{z}}^i(\cdot, j) + \delta^i_j$ for all $j \in \boldsymbol{J}$, then

$$(x^i; \bar{\bar{z}}^i) \in \mathscr{B}^{\mathrm{TC}}_\infty(\bar{p}, \bar{q}, \bar{\pi}^i, \underset{\sim}{\omega}^i, \mathscr{A}')$$
$$\Leftrightarrow (x^i; z^i) \in \mathscr{B}^{\mathrm{TC}}_\infty(\bar{p}, \bar{q}, \bar{\pi}^i, \omega'^i, \delta^i, \mathscr{A}').$$

The budget equations at each node are clearly the same, and if the security prices satisfy (4.4), then for all $j \in \boldsymbol{J}$, $i \in \boldsymbol{I}$ and $\xi \in \boldsymbol{D}$:

$$\lim_{T \to \infty} \sum_{\xi' \in \boldsymbol{D}_T(\xi)} \bar{\pi}^i(\xi') \bar{q}^i(\xi')$$
$$= \lim_{T \to \infty} \sum_{\xi' \in \boldsymbol{D}(\xi) \setminus \boldsymbol{D}^T(\xi)} \bar{\pi}^i(\xi') \bar{p}(\xi') A'(\xi', j) = 0,$$

since $\bar{P}^i \in \ell_1(\boldsymbol{D} \times \boldsymbol{L})$ and $A'(\cdot, j) \in \ell_\infty(\boldsymbol{D} \times \boldsymbol{L})$. Thus the transversality conditions for the two budget sets are equivalent since for all $\xi \in \boldsymbol{D}$:

$$\lim_{T \to \infty} \sum_{\xi' \in \boldsymbol{D}_T(\xi)} \bar{\pi}^i(\xi') \bar{q}(\xi') z^i(\xi')$$
$$= \lim_{T \to \infty} \sum_{\xi' \in \boldsymbol{D}_T(\xi)} \bar{\pi}^i(\xi') \bar{q}(\xi') (\bar{\bar{z}}^i(\xi') + \delta^i)$$
$$= \lim_{T \to \infty} \sum_{\xi' \in \boldsymbol{D}_T(\xi)} \pi^i(\xi') \bar{q}(\xi') \bar{\bar{z}}^i(\xi').$$

Since (5.3) implies $\| \omega - \omega' \|_\infty \leq (\Sigma_{j \in J_0} \delta_j) \| A - A' \|_\infty$, (ω', A') can be chosen arbitrarily close to (ω, A). Thus the set Δ of parameters (ω, A), such that $\mathscr{E}_\infty(D, \succsim,$
$\omega, \delta, \mathscr{A})$ has a TC equilibrium, is dense in $\Omega \times A$. □

Since Propositions 5.2 and 5.3 can clearly be applied to an economy $\mathscr{E}_\infty(D, \succsim, \omega, \delta, \mathscr{A})$, it follows that if $(\omega, A) \in \Delta$, then any TC equilibrium of the economy $\mathscr{E}_\infty(D, \succsim, \omega, \delta, \mathscr{A})$ can be obtained by imposing an explicit debt constraint M that is never binding.

6. Equilibrium prices of infinite lived securities

In any equilibrium the prices of the securities $\bar{q}$ and each agent's present value vector $\bar{\pi}^i$ must satisfy the equations

$$\bar{\pi}^i(\xi)\bar{q}(\xi, j) = \sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi')(\bar{p}(\xi')A(\xi', j) + \bar{q}(\xi', j)),$$
$$\forall \xi \in D, \forall i \in I, \tag{6.1}$$

which express the first-order conditions for the agent's portfolio. In a finite horizon economy, or as here in a T-truncated economy $\mathscr{E}_T$, 'integrating' these equations (by successive substitution) and using the terminal condition

$$\bar{q}_T(\xi) = 0, \quad \forall \xi \in D_T, \tag{6.2}$$

gives

$$\bar{q}_T(\xi, j) = \frac{1}{\bar{\pi}^i_T(\xi)} \sum_{\xi' \in D^+(\xi)} \bar{\pi}^i_T(\xi')\bar{p}_T(\xi')A(\xi', j), \quad \forall \xi \in D, \forall i \in I, \tag{6.3}$$

so that the equilibrium price of each security is equal to the present value of its future income stream for each agent. The expression on the right-hand side of (6.3) is called the *fundamental value for agent i* of security j (at node ξ).

It is evident that in an infinite horizon economy (6.3) holds for any finitely-lived security. However, for an infinite-lived security there is no terminal condition (6.2) that can be added to Eq. (6.1) that would force the equilibrium price of the security to equal its fundamental value for each agent. This leads to the following definition.

Definition 6.1. Let $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$ be a TC equilibrium of the economy $\mathscr{E}_\infty(D, \succsim, \omega, \mathscr{A})$. Security $j \in J$ is said to be priced at its *fundamental value* if, for all agents $i \in I$,

$$\bar{q}(\xi, j) = \frac{1}{\bar{\pi}^i(\xi)} \sum_{\xi' \in D^+(\xi)} \bar{\pi}^i(\xi')\bar{p}(\xi')A(\xi', j), \quad \forall \xi \in D(\xi(j)) \tag{6.4}$$

Security j is said to have a *speculative bubble* if (6.4) is not satisfied for some agent $i \in \boldsymbol{I}$.

It is a natural consequence of the method used in Sections 4 and 5 to construct equilibria of an infinite horizon economy $\mathscr{E}_\infty$ that in these equilibria all securities are priced at their fundamental values: since in the truncated economies $\mathscr{E}_T$ (6.3) is satisfied, this property is transmitted to the equilibrium prices of the securities in the limit. To what extent are such equilibria typical? Do there exist equilibria of $\mathscr{E}_\infty$ in which some of the securities have speculative bubbles?

The answers depend crucially on the type of infinite-lived securities available in the economy: securities in positive supply can never have speculative bubbles, while infinite-lived securities in zero net supply always admit the possibility of speculative bubbles.

Proposition 6.2. Under Assumptions B4 and B7, if $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$ *is a TC equilibrium of* $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \delta, \mathscr{A})$*, then the price of every security in positive supply* $(\delta_j > 0)$ *is equal to its fundamental value.*

Proof. This result could be deduced from Santos and Woodford (1992), but a particularly simple proof can be given in the present context. Let $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$ be a TC equilibrium. For all $\xi \in \boldsymbol{D}$, $\sum_{i \in I} \bar{q}(\xi)\bar{z}^i(\xi) = \sum_{j \in J_0} \bar{q}(\xi, j)\delta_j$. Since Assumption B4 holds, by Proposition 5.2, $(\overline{qz}^i) \in \ell_\infty(\boldsymbol{D})$ for all $i \in \boldsymbol{I}$. Thus $\sum_{j \in J_0} \bar{q}(\cdot, j)\delta_j \in \ell_\infty(\boldsymbol{D})$. By Assumption B7, $\delta_j > 0$ implies $\bar{q}(\xi, j) \geq 0$, $\forall \xi \in \boldsymbol{D}$. Thus every security in positive supply $(j \in J_0)$ satisfies $\bar{q}(\cdot, j) \in \ell_\infty(\boldsymbol{D})$. Integrating the first-order condition (6.1) of agent i up to date T gives

$$\bar{\pi}^i(\xi)\bar{q}(\xi, j) = \sum_{\substack{\xi' \in D^T(\xi) \\ \xi' > \xi}} \bar{\pi}^i(\xi')\bar{p}(\xi')A(\xi', j) + \sum_{\xi' \in D_T(\xi)} \bar{\pi}^i(\xi')\bar{q}(\xi').$$

Since $\bar{\pi}^i \in \ell_1(\boldsymbol{D})$ and $\bar{q}(\cdot, j) \in \ell_\infty(\boldsymbol{D})$, $\sum_{\xi' \in D_T(\xi)} \bar{\pi}^i(\xi')\bar{q}(\xi') \to 0$ as $T \to \infty$, $\forall i \in \boldsymbol{I}$, so that (6.4) holds for each agent. □

We now show that the equilibrium prices of infinite-lived securities in zero net supply are not tied to their fundamental values. Proposition 6.3(i) below shows that it is always possible to add a bubble component to the equilibrium price of such a infinite-lived security so that the resulting price remains an equilibrium price. However, there is a difference between speculative bubbles with complete and incomplete markets. If, in an equilibrium, the financial markets are complete even without securities with speculative bubbles, then the same equilibrium allocation can be supported by pricing every security at its fundamental value;

removing the bubble component in the price of any infinite-lived security does not alter the span of the markets and hence does not affect the real equilibrium allocation (Proposition 6.3(ii)). When markets are incomplete there exist equilibria in which infinite-lived securities have speculative bubbles that are *non-trivial* in the sense that the same equilibrium allocation cannot be obtained if securities are priced at their fundamental values: the bubble components in the prices of the infinitely-lived securities affect the span of the markets in such a way that they cannot be removed without altering the real equilibrium allocation (Proposition 6.3(iii)).

The following preliminary remark will be helpful before stating Proposition 6.3. Suppose $((\bar{x}, \bar{z}), (\bar{p}, \tilde{q}, (\bar{\pi}^i)_{i\in I}))$ is a TC equilibrium of an economy $\mathscr{E}_\infty(D, \succeq, \omega, \delta, \mathscr{A})$ in which the price $\tilde{q}(\cdot, j)$ of security j has a bubble. If $\bar{q}(\cdot, j)$ is the fundamental value of the security defined by (6.4), then both $\tilde{q}(\cdot, j)$ and $\bar{q}(\cdot, j)$ satisfy the first-order condition (6.1). Thus the difference $\rho(\cdot, j)$ between the two prices,

$$\rho(\xi, j) = \tilde{q}(\xi, j) - \bar{q}(\xi, j), \quad \forall \xi \in D, \tag{6.5}$$

called the *bubble component* [4] of $\tilde{q}(\cdot, j)$, satisfies the following homogeneous equation:

$$\bar{\pi}^i(\xi)\rho(\xi) = \sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi')\rho(\xi'), \quad \forall i \in I. \tag{6.6}$$

In a deterministic economy ($b(\xi) = 1, \forall \xi \in D$) in which there is short-term borrowing and lending at each date, markets are complete ($\bar{\pi}^i = \bar{\pi}, \forall i \in I$) and the equilibrium short-term interest rate satisfies

$$\frac{1}{1 + \bar{r}_t} = \frac{\bar{\pi}_{t+1}}{\bar{\pi}_t}, \quad \forall t \in T.$$

In this case (6.6) becomes $\rho_t = [1/(1 + \bar{r}_t)]\rho_{t+1}, \forall t \in T$, which implies $\rho_{t+1} = \alpha \prod_{\theta=0}^{t}(1 + \bar{r}_\theta)$ and leads to the well-known conclusion that in a deterministic economy a bubble must grow at the rate of interest.

This property generalizes to stochastic economies in which markets are not necessarily complete, as follows. The equilibrium short-term interest rate $\bar{r}(\xi)$ at node ξ satisfies

$$\frac{1}{1 + \bar{r}(\xi)} = \frac{\sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi')}{\bar{\pi}^i(\xi)}, \quad \forall \xi \in D,$$

[4] Since a bubble assigns a non-zero value to a zero dividend stream, when the price of a security has a bubble component, the price ceases to be a linear functional on the space of income streams $\ell_\infty(D)$. Thus the bubbles that arise in the model of this paper are not the same as the bubbles studied by Gilles and LeRoy (1992) which come from a pure charge component of a continuous linear functional on $\ell_\infty(D)$.

and the product $\bar{p}(\xi)$ of the random interest rates on the path $[\xi_0, \xi^-]$ from the initial node ξ_0 to the predecessor ξ^- of ξ,

$$\bar{p}(\xi) = \prod_{\xi' \in [\xi_0, \xi^-]} (1 + \bar{r}(\xi')), \quad \forall \xi \in \boldsymbol{D}, \tag{6.7}$$

satisfies homogeneous equation (6.6) since, for all $\xi \in \boldsymbol{D}$ and for all $i \in \boldsymbol{I}$,

$$\begin{aligned}\sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi')\bar{p}(\xi') &= \Big(\sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi')\Big) \prod_{\xi'' \in [\xi_0, \xi]} (1 + \bar{r}(\xi'')) \\ &= \frac{\bar{\pi}^i(\xi)}{1 + \bar{r}(\xi)} \prod_{\xi'' \in [\xi_0, \xi]} (1 + \bar{r}(\xi'')) \\ &= \bar{\pi}^i(\xi) \prod_{\xi'' \in [\xi_0, \xi^-]} (1 + \bar{r}(\xi'')) = \bar{\pi}^i(\xi)\bar{p}(\xi).\end{aligned}$$

Proposition 6.3. Let $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \delta, \mathscr{A})$ be an economy satisfying Assumption B6, with at least one infinite-lived security ($j \in J(\xi_0)$) in zero net supply ($\delta_j = 0$, $\delta_j^i = 0, \forall i \in \boldsymbol{I}$).

(i) *If $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$ is a TC equilibrium of the economy and if $\tilde{q}$ is defined by*

$$\tilde{q}(\cdot, j) = \bar{q}(\cdot, j) + \bar{p}(\cdot), \qquad \tilde{q}(\cdot, j') = \bar{q}(\cdot, j'), \quad j' \neq j,$$

where $\bar{p} = (\bar{p}(\xi), \xi \in \boldsymbol{D})$ is the bubble component given by (6.7), then $\tilde{q}$ is also an equilibrium security price vector; i.e. there exist portfolios $\tilde{z} = (\tilde{z}^1, \ldots, \tilde{z}^I)$ such that $((\bar{x}, \tilde{z}), (\bar{p}, \tilde{q}, (\bar{\pi}^i)_{i \in I}))$ is a TC equilibrium of the economy.

(ii) *Conversely, if $((\bar{x}, \tilde{z}), (\bar{p}, \tilde{q}, (\bar{\pi}^i)_{i \in I}))$ is a TC equilibrium of $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \delta, \mathscr{A})$ and if the financial markets are complete even without the infinite-lived securities whose prices exhibit a bubble, then there exists a vector of portfolios $\bar{z} = (\bar{z}^1, \ldots, \bar{z}^I)$ and a vector of security prices $\bar{q}$, under which every security is priced at its fundamental value, such that $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$ is a TC equilibrium.*

(iii) *If the hypothesis in* (ii) *is not satisfied, then the existence of bubble components in the security prices can have real effects, i.e. there exist equilibria in which the price of some infinite-lived security has a speculative bubble, such that the same real allocation cannot be supported by a vector of security prices under which every security is priced at its fundamental value.*

Proof. (i) For a vector of security prices q, let $V(q, \xi^+) = [\bar{p}(\xi')A(\xi', j) + q(\xi')]_{\xi' \in \xi, j \in J(\xi)}$ denote the $b(\xi) \times J(\xi)$ matrix of returns at the successors of ξ. Since $\bar{p}(\xi')$ has the same value for each successor $\xi' \in \xi^+$, the vector $(\bar{p}(\xi'), \xi' \in \xi^+)$ is collinear to $(1, \ldots, 1)$. Since by Assumption B6, $(1, \ldots, 1)^{\mathrm{T}} \in \langle V(\bar{q}, \xi^+) \rangle$, adding the bubble component to the price of security j does not change the subspace of returns at the successors of ξ, $\langle V(\tilde{q}, \xi^+)\rangle = \langle V(\bar{q}, \xi^+)\rangle$. Since $\bar{q}$ and $\tilde{q}$ satisfy (6.1),

$$\left\langle\begin{bmatrix} -\tilde{q}(\xi) \\ V(\tilde{q}, \xi^+)\end{bmatrix}\right\rangle = \left\langle\begin{bmatrix} -\bar{q}(\xi) \\ V(\bar{q}, \xi^+)\end{bmatrix}\right\rangle, \quad \forall \xi \in \boldsymbol{D}, \tag{6.8}$$

so that the subspace of income transfers is unchanged at each node. Since no agent inherits debt at date 0 in the infinite-lived security j, $\bar{q}(\xi_0)\delta^i = \tilde{q}(\xi_0)\delta^i$, so that the agents' wealth at date 0 is unchanged. Thus $\mathscr{B}(\bar{p}, \bar{q}, \bar{\pi}^i, \omega^i, \delta^i, A) = \mathscr{B}(\bar{p}, \tilde{q}, \bar{\pi}^i, \omega^i, \delta^i, A)$. For all agents $i \in \boldsymbol{I}$, $i \neq 1$, we define the new portfolio $\tilde{z}^i$ by

$$\begin{bmatrix} -\tilde{q}(\xi) \\ V(\tilde{q}, \xi^+)\end{bmatrix}\tilde{z}^i(\xi) = \begin{bmatrix} -\bar{q}(\xi) \\ V(\bar{q}, \xi^+)\end{bmatrix}\bar{z}^i(\xi), \quad \forall \xi \in \boldsymbol{D}. \tag{6.9}$$

Let $\tilde{z}^1(\xi) = -\sum_{i=2}^{I}\tilde{z}^i(\xi)$, $\forall \xi \in \boldsymbol{D}$. The spot market-clearing equations imply that $\tilde{z}^1(\cdot)$ satisfies (6.9). Thus $(\bar{x}^i; \tilde{z}^i)$ is $\succsim_i$ maximal in $\mathscr{B}(\bar{p}, \tilde{z}, \bar{\pi}^i, \omega^i, \delta^i, A)$, $\forall i \in \boldsymbol{I}$. Since $\sum_{i \in I}\tilde{z}^i = 0$ and $\sum_{i \in I}(\bar{x}^i - \omega^i) = 0$ it follows that $((\bar{x}, \tilde{z})(\bar{p}, \tilde{q}, (\bar{\pi}^i)_{i \in I}))$ is a TC equilibrium.

(ii) Let $\bar{\pi}^i = \bar{\pi}$, $\forall i \in \boldsymbol{I}$, denote the common present value vector of the agents and let $\bar{q}$ denote the vector of security prices defined by (6.4). Then $\bar{q}(\cdot, j) = \tilde{q}(\cdot, j)$ except possibly for some infinite-lived assets in zero net supply. By assumption, $\langle V(\tilde{q}, \xi^+)\rangle = \langle V(\bar{q}, \xi^+)\rangle = \mathbb{R}^{b(\xi)}$. Since both $\bar{q}$ and $\tilde{q}$ satisfy (6.1), it follows that (6.8) holds. Since $\delta_j^i = 0$ for securities in zero net supply, $\mathscr{B}(\bar{p}, \tilde{q}, \bar{\pi}^i, \omega^i, \delta^i, A) = \mathscr{B}(\bar{p}, \bar{q}, \bar{\pi}^i, \omega^i, \delta^i, A)$. By the same argument as in (i) there exists $\bar{z}$ such that $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$ is a TC equilibrium.

(iii) When the subspaces $\langle V(\tilde{q}, \xi^+)\rangle$ depend on the bubble components of infinite-lived securities – in the sense that replacing prices with speculative bubbles $\tilde{q}$ by the fundamental values $\bar{q}$ of the securities leads to different subspaces $\langle V(\bar{q}, \xi^+)\rangle$ – then the equilibrium $((\bar{x}, \tilde{z}), (\bar{p}, \tilde{q}, (\bar{\pi}^i)_{i \in I})$ cannot be achieved with prices $\bar{q}$ and a change in portfolios. This will be shown by two examples. The first has the merit of simplicity, but does not satisfy Assumption B6: it illustrates how a speculative bubble on an intrinsically worthless security can enable agents to carry out mutually advantageous borrowing and lending in the absence of a bond. Constructing an example satisfying Assumption B6 is necessarily more complicated since, as shown in (i), when the short-term bond is traded, the simplest bubble given by (6.7) cannot have real effects. Thus Example 2, which satisfies Assumption B6, exhibits a more sophisticated bubble which has real effects.

Example 1. Suppose that the event tree $\boldsymbol{D}$ is such that the only uncertainty is at date 1, the future after date 1 being infinite but certain. Let (ξ_1, ξ_1') denote the two nodes at date 1, the node following ξ_1 (resp. ξ_1') at date t being denoted by ξ_t (resp. ξ_t') for $t \geq 1$ (see Fig. 1). We assume that the two nodes at date 1 are

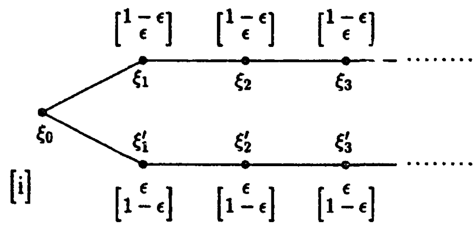

Fig. 1. Event tree and agents' endowments for Example 1.

equally probable $(\rho(\xi_1)) = (\rho(\xi'_1) = 1/2)$. Suppose the economy has an equal number of agents of two types and one good (income). Every agent has the same additively separable utility function (2.3) with

$$u^i(x) = \sum_{\xi \in D} \rho(\xi)\,\delta^{t(\xi)}[x(\xi)]^{1/2}$$
$$= \frac{1}{2}\sum_{t=0}^{\infty} \delta^t[x(\xi_t)]^{1/2} + \frac{1}{2}\sum_{t=0}^{\infty} \delta^t[x(\xi'_t)]^{1/2}.$$

The endowments of the two types of agents are shown in Fig. 1: both types have one unit of income at node ξ_0, and if nature chooses node ξ_1, then type 1 agents have $1-\epsilon$ for ever and type 2 agents have ϵ for ever; if nature chooses node ξ'_1, then the incomes of the two types are reversed. The financial structure consists of two securities issued at node ξ_0. Security 1 (2) pays one unit of income at node ξ_1 (ξ'_1) and 0 elsewhere and is retraded at all nodes on the upper (lower) branch $(\xi_t(\xi'_t),\ t \geqq 1)$.

If the securities are priced at their fundamental values, then the equilibrium prices are

$$\bar{q}_1(\xi_0) = \bar{q}_2(\xi_0) = \frac{\delta}{2^{1/2}}, \qquad \bar{q}_1(\xi_t) = \bar{q}_2(\xi'_t) = 0, \quad t \geqq 1,$$

and the equilibrium allocation is

$$\bar{x}^i(\xi_0) = 1, \quad \bar{x}^i(\xi_1) = \bar{x}^i(\xi'_1) = \tfrac{1}{2}, \quad \bar{x}^1(\xi_t) = \bar{x}^2(\xi'_t) = 1-\epsilon,$$
$$\bar{x}^1(\xi'_t) = \bar{x}^2(\xi_t) = \epsilon, \quad t \geqq 2.$$

Trading the securities only permits income transfers at date 1, the agents remaining at their initial endowments thereafter.

If there is an equilibrium in which both securities have speculative bubbles, then the financial markets are complete; the allocation must correspond to the Arrow–Debreu equilibrium allocation

$$\bar{x}^i(\xi_0) = 1, \qquad \bar{x}^i(\xi_t) = \bar{x}^i(\xi'_t) = \tfrac{1}{2}, \quad t \geqq 1,\ i = 1, 2, \tag{6.10}$$

and the agents' present value vectors (normalized by $\bar{\pi}^i(\xi_0) = 1$) coincide with the Arrow–Debreu prices $\bar{\pi}$ given by

$$\bar{\pi}(\xi_0) = 1, \qquad \bar{\pi}(\xi_t) = \bar{\pi}(\xi'_t) = \frac{\delta^t}{2^{1/2}}, \quad t \geqq 1.$$

Since the prices of the securities $q=(q_1, q_2)$ must satisfy (6.1), we must have

$$q_1(\xi_0)=\frac{\delta}{2^{1/2}}(1+q_1(\xi_1)), \qquad \frac{\delta^t}{2^{1/2}}q_1(\xi_t)=\frac{\delta^{t+1}}{2^{1/2}}q_1(\xi_{t+1}), \quad t \geqq 1,$$

$$q_2(\xi_0)=\frac{\delta}{2^{1/2}}(1+q_2(\xi_1')), \qquad \frac{\delta^t}{2^{1/2}}q_2(\xi_t')=\frac{\delta^{t+1}}{2^{1/2}}q_2(\xi_{t+1}'), \quad t \geqq 1,$$

in which the initial prices of the securities $(q_1(\xi_0), q_2(\xi_0))$ are arbitrary. [5] Choosing

$$\tilde{q}_1(\xi_0)=\tilde{q}_2(\xi_0)=\frac{1+\delta}{2^{1/2}} \text{ gives } \tilde{q}_1(\xi_t)=\tilde{q}_2(\xi_t')=\frac{1}{\delta^t}, \quad t \geqq 1. \tag{6.11}$$

The portfolios (z^1, z^2) that finance the allocation (6.10) must satisfy $z^2=-z^1$, where $z^1=(z_1^1, z_2^1)$ is a solution of

$$1=1+\tilde{q}_1(\xi_0)z_1^1(\xi_0)+\tilde{q}_2(\xi_0)z_2^1(\xi_0),$$

$$\tfrac{1}{2}=1-\epsilon+(1+\tilde{q}_1(\xi_1))z_1^1(\xi_0)-\tilde{q}_1(\xi_1)z_1^1(\xi_1),$$

$$\tfrac{1}{2}=\epsilon+(1+\tilde{q}_2(\xi_1'))z_2^1(\xi_0)-\tilde{q}_2(\xi_1')z_2^1(\xi_1'),$$

and for $t \geqq 2$:

$$\tfrac{1}{2}=1-\epsilon+\tilde{q}_1(\xi_t)z_1^1(\xi_{t-1})-\tilde{q}_1(\xi_t)z_1^1(\xi_t),$$

$$\tfrac{1}{2}=\epsilon+\tilde{q}_2(\xi_t')z_2^1(\xi_{t-1}')-\tilde{q}_2(\xi_t')z_2^1(\xi_t'),$$

so that

$$z_1^1(\xi_t)=(1+\delta)z_1^1(\xi_0)+\delta(\tfrac{1}{2}-\epsilon)\frac{1-\delta^t}{1-\delta},$$

$$z_2^1(\xi_t')=(1+\delta)z_2^1(\xi_0)-\delta(\tfrac{1}{2}-\epsilon)\frac{1-\delta^t}{1-\delta}.$$

Since $\bar{\pi}(\xi_t)\tilde{q}_1(\xi_t)=\bar{\pi}(\xi_t')\tilde{q}_2(\xi_t')=1/\sqrt{2}$, $t \geqq 1$, the transversality conditions are satisfied if and only if $\lim_{t\to\infty} z_1^1(\xi_T)=\lim_{t\to\infty} z_2^1(\xi_T')=0$. The unique initial portfolio for which the transversality conditions are satisfied is thus given by

$$\bar{z}_1^1(\xi_0)=-\bar{z}_2^1(\xi_0)=-\frac{\delta}{1-\delta^2}\left(\frac{1}{2}-\epsilon\right) \tag{6.12}$$

so that the equilibrium portfolio of agent 1 is given by (6.12) for $t=0$ and for $t \geqq 1$ by

$$\bar{z}_1^1(\xi_t)=-\frac{\delta^{t+1}}{1-\delta}\left(\frac{1}{2}-\epsilon\right)=-\bar{z}_2^1(\xi_t').$$

[5] This is a *nominal* indeterminacy which has no real effects, since a change in $q_j(\xi_0)$ can always be compensated for by rescaling the portfolios z_j^i, $i=1,2$.

If ϵ is small ($\epsilon < \frac{1}{2}$), then agents of type 1 are rich on the upper and poor on the lower branch of the event tree, with the converse for agents of type 2. To finance a transfer of income from the upper to the lower branch at date zero, type 1 agents take a long position in security 2 covered by a short position in security 1: they are thus debtors on the upper and creditors on the lower branch. The situation is reversed when ϵ is large ($\epsilon > \frac{1}{2}$), and when $\epsilon = \frac{1}{2}$ there is no trade since the initial endowment is Pareto optimal. The securities enable the agents to achieve risk-sharing only if they have a positive value after date 1: this in turn arises only if each agent believes that other agents believe that they have value. The equilibrium thus depends on the beliefs of the agents, not on the fundamentals of the securities.

Note that the transversality conditions determine the only value of $z^i(\xi_0)$ for which the asymptotic present value of the debt is zero. If $\bar{z}_1^1(\xi_0) > -[\delta/(1-\delta^2)](\frac{1}{2}-\epsilon)$, then agents of type 1 pay off their debts on the upper branch too fast and become creditors at infinity – which cannot be optimal; similarly, if $\bar{z}_1^1(\xi_0) < -[\delta/(1-\delta^2)](\frac{1}{2}-\epsilon)$, then agents of type 2 become lenders at infinity (on the upper branch).

Example 2. Suppose the event tree D is such that at each node there are two possible events, $\{u, d\}$ ('up' and 'down'), which can follow, each with probability $\frac{1}{2}$. The initial node ξ_0 is event d. There are two (types of) agents ($I = 2$) and one good ($L = 1$) at each node. The preferences of agent i are represented by a utility function

$$u^i(x^i) = \sum_{\xi \in D} \left(\tfrac{1}{2}\delta\right)^{t(\xi)} v_i(x^i(\xi)), \quad i = 1, 2,$$

where $v_i : \mathbb{R}_+ \to \mathbb{R}$, $v_i' > 0$, $v_i'' < 0$, $v_i(0) = 0$, and $0 < \delta < 1$. The endowment of agent 1 is given by

$$\omega^1(\xi) = \begin{cases} \alpha^1 - \alpha(\xi), & \text{if } \xi' = d, \forall \xi' \in [\xi_0, \xi], \\ \alpha^1 - \beta(\xi), & \text{if } \xi = u \text{ and } \xi' = d, \forall \xi' \in [\xi_0, \xi^-], \\ \alpha^1, & \text{otherwise}, \end{cases}$$

where $[\xi_0, \xi]$ denotes the path from the initial node ξ_0 to node ξ and where $\alpha(\cdot)$ and $\beta(\cdot)$ are given by

$$\alpha(\xi_0) = -\frac{1}{1-\delta},$$

$$\alpha(\xi) = \left(1 + \frac{\delta}{1-\delta} + \left(\frac{2}{\delta}\right)^{t(\xi)}\right)\left(\frac{\delta}{2}\right)^{t(\xi)-1} - \left(\frac{\delta}{1-\delta} + \left(\frac{2}{\delta}\right)^{t(\xi)}\right)\left(\frac{\delta}{2}\right)^{t(\xi)},$$

$$\beta(\xi) = \left(1 + \frac{\delta}{1-\delta}\right)\left(\frac{\delta}{2}\right)^{t(\xi)-1},$$

the endowment of agent 2 being given by $\omega^2(\xi) = \alpha^2 + \alpha^1 - \omega^1(\xi)$, $\forall \xi \in \boldsymbol{D}$. α^1 and α^2 are chosen so that $\omega^1(\cdot)$ and $\omega^2(\cdot)$ are uniformly positive.

If there is a single infinite-lived security (a consol) with payoff $A(\xi, 1) = 1$, $\forall \xi \in \boldsymbol{D}^+$, then it is easy to check that $((\bar{x}, \bar{z}), (\bar{p}, \tilde{q}, (\bar{\pi}^i)_{i \in I}))$ is a TC equilibrium where

$$\bar{x}^1(\xi) = \alpha^1 \quad \text{and} \quad \bar{x}^2(\xi) = \alpha^2, \qquad \bar{p}(\xi) = 1, \quad \forall \xi \in \boldsymbol{D},$$

$$\bar{z}^1(\xi) = \begin{cases} \left(\dfrac{\delta}{2}\right)^{t(\xi)}, & \text{if } \xi' = d, \forall \xi' \in [\xi_0, \xi], \\ 0, & \text{otherwise,} \end{cases}$$

$$\tilde{q}(\xi) = \begin{cases} \dfrac{\delta}{1-\delta} + \left(\dfrac{2}{\delta}\right)^{t(\xi)}, & \text{if } \xi' = d, \forall \xi' \in [\xi_0, \xi], \\ \dfrac{\delta}{1-\delta}, & \text{otherwise,} \end{cases}$$

$$\bar{z}^2(\xi) = -\bar{z}^1(\xi) \quad \text{and} \quad \bar{\pi}^1(\xi) = \bar{\pi}^2(\xi) = \left(\frac{\delta}{2}\right)^{t(\xi)}, \quad \forall \xi \in \boldsymbol{D}.$$

The structure of the event tree and the price of the consol are taken from Santos and Woodford (1993, example 6), the price $\tilde{q}$ being a solution of the stochastic difference equation

$$q_t = \delta E(1 + q_{t+1} | \mathcal{F}_t),$$

where $\mathcal{F}_t$ denotes the information available at date t, which can also be written in event tree notation:

$$q(\xi_t) = \tfrac{1}{2}\delta(1 + q(\xi_{t+1}^u)) + \tfrac{1}{2}\delta(1 + q(\xi_{t+1}^d)) \tag{6.13}$$

$(\xi_{t+1}^u, \xi_{t+1}^d)$ denoting the successors of ξ_t. The price $\tilde{q}$ is the sum of two components: the first is the *fundamental value* $\bar{q}(\xi) = \delta/(1-\delta)$, $\forall \xi \in \boldsymbol{D}$, and the second is a *stochastic bubble* that grows at the rate $(2/\delta)$ on the lower branch $[d, d, \ldots]$ and bursts the first time that event u occurs. To compensate for the probability of bursting, the return on the lower branch $[d, d, \ldots]$ exceeds the implicit riskless return $1/\delta$.

Note that since the allocation $(\bar{x}^1, \bar{x}^2)$ is Pareto optimal, the equilibrium would not be affected if the security structure were modified to include the short-lived riskless bond at each node (Assumption B6): the same equilibrium would be obtained with zero trade in the riskless bond. However, the equilibrium allocation $(\bar{x}^1, \bar{x}^2)$ cannot be obtained if the consol is priced at its fundamental value $\bar{q}$: the only portfolio $\bar{z} = (\bar{z}^1, \bar{z}^2)$ that would permit the agents to finance the allocation $(\bar{x}^1, \bar{x}^2)$ under price $\bar{q}$ can be computed by forward induction and it is easy to check that the transversality conditions (for example, on the subtree that begins with event u at date 1) do not hold for $\bar{z}$.

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Appendix

Proof of Theorem 5.1 Let $(\boldsymbol{D}, \succsim, \omega, (J, (\xi(j))_{j\in J}, (J(\xi))_{\xi\in \boldsymbol{D}})$ be the fixed characteristics of the economy and let $\boldsymbol{A}^* \subset \boldsymbol{A}$ be the set of commodity payoff processes for which the economy has a TC equilibrium. For $\xi \in \boldsymbol{D}$ let $j(\xi) = \#J(\xi)$ and let $a(\xi) = \min(j(\xi), b(\xi))$. Let us consider a payoff process $\bar{A} \in \boldsymbol{A}$ and let $((\bar{x}, \bar{\gamma}), (\bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i\in I}))$ be a pseudo-equilibrium of $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \mathscr{A}))$ generated by an artificial security structure $(\mathscr{K} = (\boldsymbol{K}, (\xi(k))_{k\in K}, (\mathscr{K}(\xi))_{\xi\in D}, \Gamma)$, with

$$\operatorname{rank}[\Gamma(\xi', 1, k)]_{\substack{\xi'\in\xi^+ \\ k\in K(\xi)}} = a(\xi), \quad \forall \xi \in \boldsymbol{D}.$$

Let $\bar{q}$ be the vector of prices of the original securities defined by (4.4). There exists a vector of portfolios $\bar{z} \in Z$ such that $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i\in I}))$ is a TC equilibrium of $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \mathscr{A})$ if and only if

$$\operatorname{rank}\left[\bar{p}(\xi')\bar{A}(\xi', j) + \bar{q}(\xi', j)\right]_{\substack{\xi'\in\xi^+ \\ j\in J(\xi)}} = a(\xi), \quad \forall \xi \in \boldsymbol{D}. \tag{A1}$$

To prove the theorem we show that if (A1) is not satisfied for $\bar{A}$, then for every $\epsilon > 0$ there exists a commodity payoff process $A \in \boldsymbol{A}$ with $\| A - \bar{A} \|_\infty < \epsilon$ such that (A1) is satisfied for A, so that $A \in \boldsymbol{A}^*$. We show that this can be done without changing the underlying pseudoequilibrium $((\bar{x}, \bar{\gamma}), (\bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i\in I})$.

Let $\boldsymbol{H} \subset \boldsymbol{A}$ denote the subset of commodity payoff processes A such that $((\bar{x}, \bar{\gamma}), (\bar{p}, \bar{\rho}, (\bar{\pi}^i)_{i\in I}))$ is a pseudo-equilibrium of $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, \mathscr{A}))$. The payoffs in $\boldsymbol{H}$ must satisfy (4.4), i.e. for all $\xi \in \boldsymbol{D}$,

$$\frac{1}{\bar{\pi}^i(\xi)} \sum_{\xi'\in D^+(\xi)} \bar{\pi}^i(\xi')\bar{p}(\xi')A(\xi')$$

must be independent of i, and if $q(\xi)$ denotes this common value, then (4.5) must be satisfied with $\bar{q}$ replaced by q. Thus $A \in \boldsymbol{H}$ if $A \in \boldsymbol{A}$ and if for all $\xi \in \boldsymbol{D}$,

$$\sum_{\xi'\in D^+(\xi)} \frac{\bar{\pi}^i(\xi')\bar{p}(\xi')A(\xi')}{\bar{\pi}^i(\xi)} = \sum_{\xi'\in D^+(\xi)} \frac{\bar{\pi}^1(\xi')\bar{p}(\xi')A(\xi')}{\bar{\pi}^1(\xi)}, \quad \forall i \in \boldsymbol{I},$$

$$\left\langle \left[\bar{p}(\xi')A(\xi', j) + \sum_{\xi'' \in D^+(\xi')} \frac{\bar{\pi}^1(\xi'')\bar{p}(\xi'')A(\xi'')}{\pi^1(\xi')} \right]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} \right\rangle$$

$$\subset \left\langle [\Gamma(\xi', 1, k)]_{\substack{\xi' \in \xi^+ \\ k \in K(\xi)}} \right\rangle.$$

These two equations can be written with date 0 discounted prices as: for all $\xi \in \boldsymbol{D}$,

$$\sum_{\xi' \in D^+(\xi)} \frac{\bar{P}^i(\xi')A(\xi')}{\bar{P}^i(\xi, 1)} = \sum_{\xi' \in D^+(\xi)} \frac{\bar{P}^1(\xi')A(\xi')}{\bar{P}^1(\xi', 1)}, \quad \forall i \in \boldsymbol{I},$$

$$\left\langle \left[\sum_{\xi'' \in D(\xi')} \bar{P}^1(\xi'')A(\xi'', j) \right]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} \right\rangle \subset \left\langle \left[\bar{P}^1(\xi')\Gamma(\xi', k) \right]_{\substack{\xi' \in \xi^+ \\ k \in K(\xi)}} \right\rangle.$$

To simplify notation, if $x \in \ell_\infty(\boldsymbol{D} \times \boldsymbol{L})$ and $P \in \ell_1(\boldsymbol{D} \times \boldsymbol{L})$, then we define

$$P \cdot_\xi x = \sum_{\xi' \in D(\xi)} P(\xi')x(\xi')$$

and let

$$P \,\square_{\xi^+} x = \left(P \cdot_{\xi'} x\right)_{\xi' \in \xi^+}$$

denote the $b(\xi)$ vector of date 0 discounted values of x from all the successors of ξ onwards. For $A \in \boldsymbol{A}$ we define

$$P \cdot_\xi A = \left(P \cdot_\xi A(\cdot, j),\ j \in J(\xi)\right), \qquad P \,\square_{\xi^+} A = \left[P \cdot_{\xi'} A(\cdot, j)\right]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}}.$$

Then $\boldsymbol{H}$ can be written as

$$\boldsymbol{H} = \left\{ A \in \boldsymbol{A} \;\middle|\; \begin{array}{l} \dfrac{1}{\bar{P}^i(\xi, 1)} \displaystyle\sum_{\xi' \in \xi^+} \bar{P}^i \cdot_{\xi'} A = \dfrac{1}{\bar{P}^1(\xi, 1)} \displaystyle\sum_{\xi' \in \xi^+} \bar{P}^1 \cdot_{\xi'} A, \\ \forall \xi \in \boldsymbol{D}, \forall i \in \boldsymbol{I} \\ \langle \bar{P}^1 \square_{\xi^+} A \rangle \subset \bar{\mathscr{S}}(\xi), \quad \forall \xi \in \boldsymbol{D} \end{array} \right\},$$

where

$$\bar{\mathscr{S}}(\xi) = \left\langle \left[\bar{P}^1(\xi')\Gamma(\xi', k) \right]_{\substack{\xi' \in \xi^+ \\ k \in K(\xi)}} \right\rangle.$$

Since $\bar{A} \in \boldsymbol{H}$, $\boldsymbol{H}$ is not empty. Since for each $\xi \in \boldsymbol{D}$ the second condition $\langle \cdot \rangle \subset \bar{\mathscr{S}}(\xi)$ can be expressed by a system of linear equations. $\boldsymbol{H}$ is the intersection of a countable collection of closed hyperplanes. Thus $\boldsymbol{H}$ is a closed subset of $\boldsymbol{A}$. Since $\boldsymbol{A}$ is a closed subset of the Banach space $\ell_\infty(\boldsymbol{D} \times \boldsymbol{L} \times \boldsymbol{J})$, it follows that $\boldsymbol{H}$ is a Baire space (Rudin, 1973, p. 42, theorem 2.2).

For each node ξ let $\hat{J}(\xi)$ be a subset of $J(\xi)$ consisting of $a(\xi)$ securities including the short-lived numeraire bond. For a payoff process $A \in \boldsymbol{A}$ let $\hat{A}$ denote the payoffs of the securities in $\hat{J}(\xi)$. Let $\hat{\xi}^+$ be a subset of $a(\xi)$ nodes of ξ^+ (to be chosen below). Consider the subset of $\boldsymbol{H}$:

$$\boldsymbol{H}_{\xi} = \left\{ A \in \boldsymbol{H} \mid \det\left[\bar{P}^1 \square_{\hat{\xi}^+} \hat{A} \right] = 0 \right\}.$$

We show that $\boldsymbol{H}_{\xi}$ has an empty interior in $\boldsymbol{H}$. It suffices to show that we can perturb any $A \in \boldsymbol{H}_{\xi}$, $A \to A + \Delta A$, with $A + \Delta A \in \boldsymbol{H}$ in such a way that

$$\det\left[\bar{P}^1 \square_{\hat{\xi}^+} (\hat{A} + \Delta \hat{A}) \right] \neq 0. \tag{A2}$$

Consider changes ΔA in the commodity payoff process consisting solely of changes in the amounts of commodity 1 which can be decomposed as follows:

$$\Delta A(\cdot, j) = \begin{cases} \sum\limits_{\xi' \in \hat{\xi}^+} \alpha^j_{\xi'} e^{\xi'}_1 + \sum\limits_{\xi' \in \xi^+ \setminus \hat{\xi}^+} \beta^j_{\xi'} e^{\xi'}_1 + \gamma^j e^{\xi}_1, & \text{if } j \in \hat{J}'(\xi), \\ 0, & \text{if } j \notin \hat{J}'(\xi). \end{cases} \tag{A3}$$

where $\hat{J}'(\xi)$ is the subset of $\hat{J}(\xi)$ which excludes the short-lived numeraire bond. Note that the commodity payoffs are perturbed only at node ξ and its immediate successors ξ^+ and only securities in $\hat{J}'(\xi)$ have their payoffs perturbed. For security $j \in \hat{J}'(\xi)$, its payoff in good 1 is perturbed by $\alpha^j_{\xi'}$ at node $\xi' \in \hat{\xi}^+$, by $\beta^j_{\xi'}$ at node $\xi' \in \xi^+ \setminus \hat{\xi}^+$ and by γ^j at node ξ. For brevity we write

$$\alpha = \left(\alpha^j_{\xi'},\ j \in \hat{J}'(\xi),\ \ \xi' \in \hat{\xi}^+ \right) \in \mathbb{R}^{(a(\xi)-1)a(\xi)},$$

$$\beta = \left(\beta^j_{\xi'},\ j \in \hat{J}'(\xi),\ \ \xi' \in \xi^+ \setminus \hat{\xi}^+ \right) \in \mathbb{R}^{(a(\xi)-1)(b(\xi)-a(\xi))}$$

$$\gamma = \left(\gamma^j,\ j \in \hat{J}'(\xi) \right) \in \mathbb{R}^{a(\xi)-1}.$$

α can be chosen so that (A2) is satisfied. To see this, let $g : \mathbb{R}^{(a(\xi)-1)a(\xi)} \to \mathbb{R}$ be defined by

$$g(\alpha) = \det\left[\bar{P}^1 \square_{\hat{\xi}^+} (\hat{A} + \Delta \hat{A}) \right], \tag{A4}$$

where ΔA is defined by (A3). While ΔA is a function of (α, β, γ), the determinant in (A4) depends only on α. A straightforward but tedious calculation shows that g has partial derivatives of order $a(\xi) - 1$ evaluated at $\alpha = 0$, which are not zero since $\bar{P}^1(\xi', 1) > 0$, $\forall \xi' \in \hat{\xi}^+$. Thus g is not locally constant in a neighborhood of $\alpha = 0$. It follows that there exists α arbitrarily small such that $g(\alpha) \neq 0$.

If $a(\xi) < b(\xi)$, then β is chosen to ensure that

$$\langle \bar{P}^1 \square_{\xi^+} (A + \Delta A) \rangle \subset \bar{\mathscr{L}}(\xi). \tag{A5}$$

Since $\bar{\mathscr{S}}(\xi)$ is a non-trivial subspace of $\mathbb{R}^{b(\xi)}$, there is a choice of $a(\xi)$ nodes $\hat{\xi}^+ \subset \xi^+$ and an appropriate ordering of the nodes such that $\bar{\mathscr{S}}(\xi)$ can be represented by a system of $b(\xi) - a(\xi)$ equations of the form

$$\bar{\mathscr{S}}(\xi) = \{v \in \mathbb{R}^{b(\xi)} \mid [I \mid E]v = 0\}.$$

The nodes are ordered so that the subset $\xi^+ \setminus \hat{\xi}^+$ constitutes the first $b(\xi) - a(\xi)$ nodes. I is the $(b(\xi) - a(\xi)) \times (b(\xi) - a(\xi))$ identity matrix and E is a $(b(\xi) - a(\xi)) \times a(\xi)$ matrix (see Magill and Shafer, 1991, p. 1544). For each $j \in \hat{J}'(\xi)$, once α has been chosen, there is a unique vector $\beta^j = (\beta^j_{\xi'}, \xi' \in \xi^+ \setminus \hat{\xi}^+) \in \mathbb{R}^{b(\xi) - a(\xi)}$ such that the vector

$$v^j = \left(\left(\bar{P}^1(\xi', 1)\beta^j_{\xi'}, \xi' \in \xi^+ \setminus \hat{\xi}^+\right), \left(\bar{P}^1(\xi', 1)\alpha^j_{\xi'}, \xi' \in \hat{\xi}^+\right)\right)$$

satisfies the equation $[I \mid E]v^j = 0$.

If $a(\xi) = b(\xi)$, $\bar{\mathscr{S}}(\xi) = \mathbb{R}^{b(\xi)}$ so that (A5) is automatically satisfied.

We have to ensure that $\langle \bar{P}^1 \square_{\tilde{\xi}^+}(A + \Delta A)\rangle \subset \bar{\mathscr{S}}(\tilde{\xi})$ is satisfied for all nodes $\tilde{\xi} \neq \xi$. By construction, this inclusion holds for all nodes $\tilde{\xi}$ which are not on the path from ξ_0 to ξ. If γ^j is chosen so that

$$\bar{P}^1(\xi, 1)\gamma^j + \sum_{\xi' \in \xi^+ \setminus \hat{\xi}^+} \bar{P}^1(\xi', 1)\beta^j_{\xi'} + \sum_{\xi' \in \hat{\xi}^+} \bar{P}^1(\xi', 1)\alpha^j_{\xi'} = 0,$$

for $j \in \hat{J}'(\xi) \cap J(\xi^-)$, and if $\gamma^j = 0$ for $j \notin \hat{J}'(\xi) \cap J(\xi^-)$, then (A5) is satisfied for all nodes $\tilde{\xi} \in \boldsymbol{D}$. Thus $A + \Delta A$ satisfies the subspace requirements (the second condition) in the definition of $\boldsymbol{H}$. To show that the first condition in the definition of $\boldsymbol{H}$ is satisfied it only remains to show that if Δq is defined by

$$\Delta q(\tilde{\xi}, j) = \frac{1}{\bar{\pi}^1(\tilde{\xi})} \sum_{\xi' \in \boldsymbol{D}^+(\xi')} \bar{\pi}^1(\xi')\bar{p}(\xi')\Delta A(\xi', j),$$

then for all $i \in \boldsymbol{I}$,

$$\frac{1}{\bar{\pi}^i(\tilde{\xi})} \sum_{\xi' \in \boldsymbol{D}^+(\tilde{\xi})} \bar{\pi}^i(\xi')\bar{p}(\xi')\Delta A(\xi', j) = \Delta q(\tilde{\xi}, j),$$

$$\forall j \in J(\tilde{\xi}), \forall \tilde{\xi} \in \boldsymbol{D},$$

so that all agents agree on the induced changes in the security prices. Since ΔA has only a finite number of non-zero terms this follows from $\langle \bar{P}^1 \square_{\tilde{\xi}^+} \Delta A\rangle \subset \bar{\mathscr{S}}(\tilde{\xi}), \forall \tilde{\xi} \in \boldsymbol{D}$.

Since α can be chosen to be arbitrarily small and since (β, γ) are deduced from α by linear relations with bounded coefficients, the perturbation ΔA can be made arbitrarily small. Thus for all $\xi \in \boldsymbol{D}$, $\boldsymbol{H}_\xi$ is closed and has an empty interior in $\boldsymbol{H}$. Since $\boldsymbol{H}$ is a Baire space, the countable union $\bigcup_{\xi \in \boldsymbol{D}} \boldsymbol{H}_\xi$ has an empty interior in $\boldsymbol{H}$. Thus for all $\epsilon > 0$ there exists an ϵ-perturbation A of $\bar{A}$ such that $A \in \boldsymbol{H}$ and

$$\langle \bar{P} \square_{\xi^+} A \rangle = \bar{\mathscr{S}}(\xi), \quad \forall \xi \in \boldsymbol{D},$$

which completes the proof. □

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