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Infinite horizon CAPM equilibrium*

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Summary. This paper derives the equilibrium of an infinite-horizon discrete-time CAPM economy in which agents have discounted expected quadratic utility functions. We show that there is an income stream obtainable by trading on the financial markets which best approximates perfect consumption smoothing (called the *least variable income stream* or LVI) such that the equilibrium consumption of each agent is some multiple of the LVI and some share of aggregate output. The welfare of agents is a decreasing function of the lack of consumption smoothing achievable, measured by the distance of the LVI from the perpetuity of one unit of income for ever. If in addition the economy has a Markov structure, the LVI, and hence the equilibrium, can be calculated by dynamic programming. When the model is calibrated to US data a striking prediction emerges: the quasi-irrelevance of the bond market. Infinitely-lived agents achieve almost all their desired consumption smoothing by applying carryover strategies to equity, the proportion of agents' portfolios in bonds rarely exceeding 3%.

Keywords and Phrases: CAPM equilibrium, Consumption smoothing, Incomplete markets, Markov economy

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1 Introduction

The Capital Asset Pricing Model (CAPM) provides insight into the two basic components of a financial market equilibrium, the *prices* of the securities and the *consumption streams* of the agents. The well-known CAPM *formula* explains the logic underlying the equilibrium pricing of securities by asserting that the risk premium on a security depends on the covariance of its payoff with aggregate output, while the *two-fund separation theorem* exhibits the form of agents' equilibrium consumption streams and portfolios, asserting that the consumption stream of each agent can expressed as some share of aggregate output and some multiple of the payoff of the riskless bond. Most of the research on extending CAPM to a multiperiod and continuous-time framework (Rubinstein [30], Breeden [2], Duffie-Zame [9]) has focused on generalizing the pricing formula, showing that it can be extended to a sequence of per-period formulae linking the per-period risk premium of a security to the covariance of its returns with increments of aggregate consumption, the so-called CCAPM *formula*. Much less attention has been devoted to the structure of agents' intertemporal consumption streams and portfolios: one of the objectives of this paper is to provide the appropriate generalization of the two-fund separation theorem for a discrete-time economy over an infinite horizon.

The key hypothesis of the CAPM model is that the first and second moments of the random variables are all that matters to agents in an equilibrium. As is well known, there are two ways of achieving such a result: the first is to restrict preference orderings to mean-variance preferences, with no restriction on the stochastic nature of agents' endowments or the security returns; the second is to restrict endowments and security returns to be normally distributed or, in the continuous-time case, to follow an Ito process (Duffie-Zame[1] [9]) with no (or only mild) restrictions on agents' preferences. In this paper we adopt the first approach, assuming that preference orderings can be represented by expected discounted quadratic utility functions. The other assumption required to obtain the CAPM result in the two-period (or static) model is that the endowments of agents lie in the span of the market (see Duffie [8] or Magill-Quinzii [26]): this is the second hypothesis that we also adopt in the intertemporal model. This assumption, while strong, is a natural assumption to make when the objective is to study the equity and bond markets – the traditional focus of the CAPM model – for it asserts that the individual risks of the agents are those that arise from their ownership shares of firms and the equity markets permit these risks to be shared among agents. This assumption is less restrictive than the assumption that markets are complete, since it does not guarantee that a risk-averse agent can obtain a constant consumption stream by trading on the markets. As we shall see, this inability to acheive perfect consumption smoothing will play an important role in the analysis that follows.

[1] Their framework requires the additional hypothesis that the financial markets are complete and that agents have expected utility preferences.

The technique that we use builds on the Hilbert space approach that we used in Magill-Quinzii [26], [28] for analyzing two-period financial market equilibria. Hilbert space techniques were introduced by Chamberlain [3], [4] to study arbitrage pricing theory (see also Gilles-LeRoy [12]) and were used by Duffie [7], [8] to obtain short proofs of the properties of the two-period CAPM model. In this paper we exploit the fact that expected discounted quadratic utilities induce a natural metric on the space of income streams, in which each node is weighted by its discounted probability. When endowed with this metric, the space of income streams is a Hilbert space and an agent's utility can be expressed as a loss function from an ideal constant consumption stream.

The key step in finding an equilibrium of the infinite-horizon model lies in finding the income stream obtained by trading on the financial markets which best approximates *perfect consumption smoothing*: this is the stream which lies closest in the above metric to the constant annuity of one unit of income for ever, and we call it the *least variable income stream* (LVI) in the marketed subspace. We show that the structure of equilibrium is such that at the equilibrium agents have diversified their individual risks and only have to resolve a trade-off between consumption smoothing (achievable by holding the LVI) and holding some share of aggregate output (which defines the aggregate risk of the economy). Risk-averse agents choose a high component on the LVI and hold only a small share of aggregate output: such agents favor a high degree of intertemporal consumption smoothing. Risk-tolerant agents, by contrast, choose leveraged portfolios and carry a larger share of the aggregate risks: their consumption is higher on average but also more variable. The incompleteness of the market is measured by the lack of consumption smoothing achievable on the financial markets – namely the distance of the LVI from the constant annuity of one unit of income for ever – and the utility of agents at equilibrium can be expressed in terms of this distance: the greater the incompleteness of the markets, the smaller the equilibrium welfare of the agents.

To complete the characterization of equilibrium, the LVI has to be found. Explicitly calculating the dynamic portfolio strategy which minimizes the infinite sum of deviations from the constant stream 1 for ever is a tractable problem only if the economy has a recursive structure. Thus in Section 3 we add the assumption that the uncertainty which drives the basic endowment processes and the payoffs of the securities has a Markovian structure. Using dynamic programming arguments we show that the rule which underlies the porfolio strategy for the LVI is a state-dependent autoregressive process: while this rule is rather complex when there are many securities, the basic underlying principle is to combine *carryover* strategies – namely depleting the holding of a security in "bad " states and augmenting the stock in "good" states – and *hedging* strategies – holding offsetting long and short positions in securities whose payoffs are positively correlated. While the use of hedging strategies (and diversification) is the typical way of achieving consumption smoothing in a two-period setting, carryover strategies constitute the new instrument that can be used when there is a long horizon:

these are powerful strategies which permit significant consumption smoothing with very few securities when an agent is faced with stationary risks.

Since we obtain an explicit closed-form solution for the equilibrium, it is natural to study the performance of the model when calibrated to postwar data for the US economy[2]. We examine the equity-bond composition of agents' portfolios generated by the model. A striking prediction emerges which may perhaps best be called the quasi-irrelevance of the bond market: for in an economy with growth and inflation in which agents are infinitely lived and face stationary risks, most of the desired consumption smoothing can be achieved using carryover strategies on equity, with trading in bonds providing only minor improvements in welfare. The general form of the trading strategy used in the model by a risk-averse investor consists in holding a portfolio composed mainly of equity, compensating for the low dividends in bad times by making marginal sales, while taking advantage of the high dividends in good times to replenish the holdings of equity. Bonds are used in offsettting strategies to provide additional consumption smoothing: with a single bond, the bond is purchased when equity is sold and conversely, and with two or more bonds hedging strategies are employed with some bonds being sold short to exploit the positive correlation of the bonds payoffs and prices. Although bonds are used as buffers, and do indeed provide additional consumption smoothing, the remarkable feature is that the percentage of the value of the LVI portfolio invested in bonds rarely exceeds 3%. Thus while the model gives firm support to one aspect of popular investment lore, that agents seeking to participate in the long-run growth of the economy should invest in the stock market, it firmly rejects the other popular maxim that risk-averse agents should have a significant proportion of their portfolio invested in bonds.

Section 2 lays out the basic characteristics of the agents and the securities which are traded, and derives the equilibrium of the associated infinite-horizon CAPM economy. Section 3 introduces the assumption of Markov endowments and security payoffs and derives the value function which is minimized by the LVI: the associated first-order conditions define the portfolio strategies which yield maximum income smoothing. Section 4 concludes with a study of the calibrated version of the model.

2 Equilibrium of an infinite horizon CAPM economy

Consider a one-good endowment economy with a finite number I of infinitely-lived agents each having a stochastic endowment stream. At each date the econ-

[2] There are two differences between the calibration that we undertake in Section 4 and that undertaken in the recent macroeconomic literature on calibration of models with incomplete markets (Huggett [17], Telmer [33], Lucas [22], Heaton-Lucas [15], Den Haan [5], [6]). For the models considered in these papers have constant relative risk aversion utility functions and individual risks, and for such models no closed form solutions are currently known, making the computation of equilibrium a difficult problem: for our model computation of equilibrium is straightforward once the form of the equilibrium is analytically derived. While their objective is to study how uninsurable individual risks affect the interest rate or the equity premium, our objective is to study the structure of agents' equilibrium portfolios on the equity and bond markets.

omy experiences one of a finite number of shocks, $s_t \in \boldsymbol{S} = \{1, \ldots, S\}$, the shock s_0 being exogenously given and known at date 0. Let $\sigma_t = (s_0, \ldots, s_t)$ denote the history of the shocks up to date t: let $\boldsymbol{\Sigma}_t = \boldsymbol{S} \times \ldots \times \boldsymbol{S}$ denote the set of all such histories to date t and let $\boldsymbol{\Sigma} = \bigcup_{t=0}^{\infty} \boldsymbol{\Sigma}_t$ denote the collection of all such histories for all dates. $\boldsymbol{\Sigma}$ defines an event-tree: for any node $\sigma_t \in \boldsymbol{\Sigma}$, define the *predecessor* σ_t^- of σ_t and the set σ_t^+ of S *immediate successors* of σ_t by

$$\sigma_t^- = (s_0, \ldots, s_{t-1}), \quad \sigma_t^+ = \{(s_0, \ldots, s_t, s_{t+1}) \mid s_{t+1} \in \boldsymbol{S}\}$$

Let $\rho(\sigma_t) > 0$ denote the probability of node σ_t, where $\rho : \boldsymbol{\Sigma} \longrightarrow [0, 1]$ is a map satisfying $\rho(\sigma_0) = 1$, $\sum_{\sigma_t \in \boldsymbol{\Sigma}_t} \rho(\sigma_t) = 1$ and $\rho(\sigma_t) = \sum_{\sigma' \in \sigma_t^+} \rho(\sigma')$. Let $\boldsymbol{\omega}^i = \left(\omega^i(\sigma_t), \sigma_t \in \boldsymbol{\Sigma}\right)$ and $\boldsymbol{x}^i = \left(x^i(\sigma_t), \sigma_t \in \boldsymbol{\Sigma}\right)$ denote agent i's stochastic endowment and consumption streams, $\omega^i(\sigma_t)$ denoting the endowment at date t if the history is σ_t: ω^i is non-negative and bounded i.e. there exists $M_i > 0$ such that $0 \leq \omega^i(\sigma_t) \leq M_i$ for all $\sigma_t \in \boldsymbol{\Sigma}$. Each agent is assumed to have a preference ordering defined by an additively separable quadratic utility function

$$u^i(\boldsymbol{x}^i) = -\frac{1}{2} \sum_{\sigma_t \in \boldsymbol{\Sigma}} \delta^t \rho(\sigma_t) \left(\alpha^i - x^i(\sigma_t)\right)^2 \quad i = 1, \ldots, I \tag{1}$$

Thus agents have common probability beliefs $\rho(\sigma_t)$ and discount the future at the same rate δ with $0 < \delta < 1$. Quadratic utility functions induce a natural geometry on the space of income streams. To see this, consider a numbering (a bijective map) $\widetilde{n} : \boldsymbol{\Sigma} \longrightarrow \mathbb{N}$ of the nodes of the event-tree to the integers, which respects the dates of the nodes i.e. which is such that $\widetilde{n}(\sigma_0) = 1$, and if two nodes σ_t and σ_τ are such that $t > \tau$, then $\widetilde{n}(\sigma_t) > \widetilde{n}(\sigma_\tau)$. Let $\boldsymbol{\Delta}$ denote the infinite diagonal matrix whose n^{th} diagonal element is $\Delta_n = \delta^t \rho(\sigma_t)$ for the node σ_t such that $\widetilde{n}(\sigma_t) = n$ and let $\mathbb{R}^{\boldsymbol{\Sigma}}$ denote the vector space of all maps $\boldsymbol{x} : \boldsymbol{\Sigma} \longrightarrow \mathbb{R}$ (i.e. the income streams defined on $\boldsymbol{\Sigma}$). Define the space

$$\ell_2^{\Delta}(\boldsymbol{\Sigma}) = \left\{ \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{\Sigma}} \mid \boldsymbol{x}^\top \boldsymbol{\Delta} \boldsymbol{x} = \sum_{\sigma_t \in \boldsymbol{\Sigma}} \delta^t \rho(\sigma_t) \mid x(\sigma_t) \mid^2 < \infty \right\}$$

of all income streams which are square summable with respect to the metric $\boldsymbol{\Delta}$. Note that the expression $\boldsymbol{x}^\top \boldsymbol{\Delta} \boldsymbol{x}$ is written with the assumption that all vectors $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{\Sigma}}$ are expressed as sequences whose components are ordered by $\widetilde{n}$. If for all $\boldsymbol{x}, \boldsymbol{y} \in \ell_2^{\Delta}(\boldsymbol{\Sigma})$ we define the inner product

$$[\![\boldsymbol{x}, \boldsymbol{y}]\!]_{\Delta} = \sum_{\sigma_t \in \boldsymbol{\Sigma}} \delta^t \rho(\sigma_t) x(\sigma_t) y(\sigma_t)$$

with associated norm[3] $\| \boldsymbol{x} \|_{\Delta} = \sqrt{\boldsymbol{x}^\top \boldsymbol{\Delta} \boldsymbol{x}}$, then $\left(\ell_2^{\Delta}(\boldsymbol{\Sigma}), [\![\, ,]\!]_{\Delta}\right)$ is a Hilbert space. To permit projection techniques of $\ell_2^{\Delta}(\boldsymbol{\Sigma})$ to be used to explicitly compute the

[3] In the notation of stochastic processes used in finance and macroeconommics, $[\![\boldsymbol{x}, \boldsymbol{y}]\!]_{\Delta} = E(\sum_{t=0}^{\infty} \delta^t x_t y_t)$ and $\| \boldsymbol{x} \|_{\Delta} = (E\left(\sum_{t=0}^{\infty} \delta^t (x_t)^2\right))^{1/2}$. For simplicity in the analysis that follows we omit the subscript Δ on the inner product and the norm.

equilibrium, we assume that the consumption space is all of $\ell_2^{\Delta}(\boldsymbol{\Sigma})$: later we show how restrictions can be placed on the parameters of the model to ensure that each agent i' s consumption stream is non-negative and does not exceed the ideal consumption level α^i.

To smooth their consumption streams agents can trade J securities at each date. A security can be short-lived, paying a dividend only in the next period (e.g. a one-period bond), long-lived with a finite maturity (e.g. a k-period bond) or infinite-lived (e.g. an equity contract). To accommodate these cases while always retaining the same fixed number J of tradeable securities, we adopt the following approach: let $h : \{1, \ldots, J\} \longrightarrow \{\emptyset, 1, \ldots, J\}$ be a map which indicates that security j at date t becomes security $h(j)$ at date $t + 1$, with $h(j) = \emptyset$ if the security is short lived, and $h(j) = j$ if the security is infinite lived. For example if securities $1, \ldots, k$ are bonds with maturities $1, \ldots, k$, then $h(1) = \emptyset,\ h(2) = 1, \ldots,\ h(k) = k - 1$. In a standard model one unit of a long-lived security j delivers one unit of security $h(j)$ at date $t + 1$. However, if the model is the reduced form of a monetary model with the possibility of inflation and growth (as in Section 4) and if the characteristics of the securities are to remain time invariant, then one unit of security j at date t may yield less than one unit of security $h(j)$ at date $t+1$. Let $f^j(\sigma_{t+1})$, with $0 \leq f^j(\sigma_{t+1}) \leq 1$, denote the amount of security $h(j)$ that security j traded at date-event σ_t delivers at node $\sigma_{t+1} \in \sigma_t^+$: then security j purchased for the price $q^j(\sigma_t)$ at date-event σ_t has the payoff

$$R^j(\sigma_{t+1}) \ = \ d^j(\sigma_{t+1}) \ + \ f^j(\sigma_{t+1}) q^{h(j)}(\sigma_{t+1}) \tag{2}$$

at the immediate successors $\sigma_{t+1} \in \sigma_t^+$. We assume the sequence of dividends $\boldsymbol{d^j} = (d^j(\sigma_t), \sigma_t \in \boldsymbol{\Sigma})$ lies in $\ell_2^{\Delta}(\boldsymbol{\Sigma})$.

Let $z_j^i(\sigma_t)$ denote the number of units of security j purchased (if $z_j^i(\sigma_t) > 0$) or sold (if $z_j^i(\sigma_t) < 0$) by agent i at node σ_t. For any node which is not the initial node $t \neq 0$, the budget equation of agent i is given by

$$x^i(\sigma_t) - \omega^i(\sigma_t) = \sum_{j=1}^{J} R^j(\sigma_t) z_j^i(\sigma_t^-) - \sum_{j=1}^{J} q^j(\sigma_t) z_j^i(\sigma_t) \tag{3}$$

At date 0 we assume $z_j^i(\sigma_0^-) = 0$, so that for $t = 0$ the first term in (3) is absent. Since the model is written under the assumption that there are no transactions costs, we use the standard convention that at each date-event σ_t and for each security j, the position $z_j^i(\sigma_t^-)$ taken at the predecessor is closed out (sold) and a new amount $z_j^i(\sigma_t)$ is purchased.

In order to write the budget equations in a more condensed form, consider the infinite matrix[4] $\boldsymbol{W}$ constructed in the following way. The columns of $\boldsymbol{W}$ give, for each node σ_t and for each security j, the stream of income over the event-tree $\boldsymbol{\Sigma}$ generated by purchasing one unit of security j at node σ_t and selling it at each of the immediate successors $\sigma_{t+1} \in \sigma_t^+$. Thus each of the columns of the matrix

[4] This matrix was introduced in Magill-Quinzii ([26, Section 20]) for a T-period economy and leads to substantial simplifications in the analysis of the multiperiod finance model.

correspond to each of the elements of $\boldsymbol{\Sigma} \times \boldsymbol{J}$, while each row corresponds to a node of $\boldsymbol{\Sigma}$, and the ordering of the columns and rows respects the ordering of the nodes given by $\widetilde{n}$. The column corresponding to security j traded at node σ_t has the entry $-q^j(\sigma_t)$ in row $\widetilde{n}(\sigma_t)$ and the entries $R^j(\sigma')$ in rows $\widetilde{n}(\sigma')$ with $\sigma' \in \sigma_t^+$: in every other row of this column the entry is zero. The first J columns of the matrix correspond to the J securities traded at the initial node σ_0 (with $\widetilde{n}(\sigma_0) = 1$), then comes the block of J columns corresponding to the J securities traded at the node with number 2 (this is an immediate successor of σ_0) and so on. Each column (σ_t, j) of $\boldsymbol{W}$ is a vector in $\ell_2^{\Delta}(\boldsymbol{\Sigma})$ which is orthogonal to all other columns (σ', j') except those corresponding to the node σ_t itself, its immediate predecessor and its immediate successors i.e. columns (σ', j') with $\sigma' = \sigma_t$ or $\sigma' = \sigma_t^-$ or $\sigma' \in \sigma_t^+$. Let

$$\boldsymbol{q} = \left((q^1(\sigma_t), \ldots, q^J(\sigma_t)), \sigma_t \in \boldsymbol{\Sigma}\right) \in \mathbb{R}^{\boldsymbol{\Sigma} \times \boldsymbol{J}}$$

denote the vector of security prices: the matrix constructed above depends on $\boldsymbol{q}$ – when we want to stress this dependence, we write $\boldsymbol{W}(\boldsymbol{q})$, and when it is not essential to do so, we use the simpler notation $\boldsymbol{W}$.

Let $\boldsymbol{z}^i = \left((z_1^i(\sigma_t), \ldots, z_J^i(\sigma_t)), \sigma_t \in \boldsymbol{\Sigma}\right)$ denote agent i's portfolio strategy for the J securities across the event-tree $\boldsymbol{\Sigma}$. With this notation the budget equations of agent i can be written as $\boldsymbol{x}^i - \boldsymbol{\omega}^i = \boldsymbol{W}(\boldsymbol{q})\boldsymbol{z}^i$. Some restriction must be placed on agent i's portfolio strategy $\boldsymbol{z}^i$, for otherwise, regardless of the prices of the securities, agent i could always borrow so as to consume the ideal consumption α^i at each node, postponing indefinitely the repayment of the debt. As recent papers (Magill-Quinzii [25], [27], Levine-Zame [20], Hernandez-Santos [16]) have shown, in an infinite-horizon GEI economy a no-Ponzi scheme condition is one of the key ingredients required to ensure existence of an equilibrium. To eliminate Ponzi schemes it is sufficient to insist that the strategies followed by agents be limits of strategies in which they repay their debt in finite time. Let $\langle \boldsymbol{W}(\boldsymbol{q}) \rangle$ denote the smallest closed subspace of $\ell_2^{\Delta}(\boldsymbol{\Sigma})$ containing all *finite* linear combinations of the columns of $\boldsymbol{W}(\boldsymbol{q})$. A trading strategy $\boldsymbol{z}^i$ over the infinite horizon gives rise to an income stream $\boldsymbol{\tau}^i$ in $\langle \boldsymbol{W}(\boldsymbol{q}) \rangle$ if $\boldsymbol{\tau}^i$ is the limit in $\ell_2^{\Delta}(\boldsymbol{\Sigma})$ of the income streams obtained by following truncated strategies $\boldsymbol{z}_T^i$ such that $z_T^i(\sigma_t) = z(\sigma_t)$ if $t < T$ and $z_T^i(\sigma_t) = 0$ if $t \geq T$. This leads to the following *transversality condition*

$$\lim_{T \longrightarrow \infty} \sum_{t=0}^{T-1} \delta^t \rho(\sigma_t) \left| R(\sigma_t) z^i(\sigma_t^-) - q(\sigma_t) z^i(\sigma_t) \right|^2 + \sum_{\sigma \in \boldsymbol{\Sigma}_T} \delta^T \rho(\sigma) \left| R(\sigma) z^i(\sigma^-) \right|^2 < \infty \tag{TR}$$

where $\boldsymbol{\Sigma}_T$ denotes the set of all nodes at date T. The budget set of agent i over the infinite horizon is thus defined by

$$\mathbb{B}(\boldsymbol{q}, \boldsymbol{\omega}^i) = \left\{ \boldsymbol{x}^i \in \ell_2^{\Delta}(\boldsymbol{\Sigma}) \,\middle|\, \begin{array}{l} \boldsymbol{x}^i - \boldsymbol{\omega}^i = \boldsymbol{W}(\boldsymbol{q})\boldsymbol{z}^i, \\ \boldsymbol{z}^i \in \mathbb{R}^{\boldsymbol{\Sigma} \times \boldsymbol{J}}\ \boldsymbol{z}^i \text{ satisfies } (TR) \end{array} \right\}$$

Let $\boldsymbol{u} = (u^1, \ldots, u^I)$ and $\boldsymbol{\omega} = (\boldsymbol{\omega}^1, \ldots, \boldsymbol{\omega}^I)$ denote the agents' characteristics and let $\boldsymbol{D} = (\boldsymbol{d}^1, \ldots, \boldsymbol{d}^J, \boldsymbol{f}^1, \ldots, \boldsymbol{f}^J, h)$ denote the basic data on security payoffs, then $\mathscr{E}(\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{D})$ will denote the associated economy.

Definition. A *financial market equilibrium* of the economy $\mathscr{E}(\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{D})$ is a pair of actions and prices

$$((\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}), \bar{\boldsymbol{q}}) = \left((\bar{\boldsymbol{x}}^1, \ldots, \bar{\boldsymbol{x}}^I, \bar{\boldsymbol{z}}^1, \ldots, \bar{\boldsymbol{z}}^I), \bar{\boldsymbol{q}}\right)$$

such that

(i) $\bar{\boldsymbol{x}}^i = \arg\max \left\{u^i(\boldsymbol{x}^i) \,\middle|\, \boldsymbol{x}^i \in \mathbb{B}(\bar{\boldsymbol{q}}, \boldsymbol{\omega}^i)\right\}$ and $\bar{\boldsymbol{x}}^i - \boldsymbol{\omega}^i = \boldsymbol{W}(\bar{\boldsymbol{q}})\bar{\boldsymbol{z}}^i$, $i = 1, \ldots, I$

(ii) $\sum_{i=1}^I \bar{\boldsymbol{z}}^i = 0$

Finding equilibrium by projection. The subspace of income transfers

$$\langle \boldsymbol{W} \rangle = \left\{\boldsymbol{\tau} \in \ell_2^{\Delta}(\boldsymbol{\Sigma}) \mid \boldsymbol{\tau} = \boldsymbol{W}\boldsymbol{z}, \boldsymbol{z} \text{ satisfies } (TR)\right\}$$

which agents can achieve by trading on the financial markets, will be called the *market subspace*. Since $\langle \boldsymbol{W} \rangle$ is a closed subspace, the space $\ell_2^{\Delta}(\boldsymbol{\Sigma})$ can be decomposed into a direct sum of the market subspace $\langle \boldsymbol{W} \rangle$ and its orthogonal complement $\langle \boldsymbol{W} \rangle^{\perp\!\!\!\perp}$ i.e. $\ell_2^{\Delta}(\boldsymbol{\Sigma}) = \langle \boldsymbol{W} \rangle \oplus \langle \boldsymbol{W} \rangle^{\perp\!\!\!\perp}$, (see Dunford-Schwartz [10, p. 249]). Thus any income stream $\boldsymbol{x} \in \ell_2^{\Delta}(\boldsymbol{\Sigma})$ can be decomposed uniquely as

$$\boldsymbol{x} = \boldsymbol{x}^* + \hat{\boldsymbol{x}} \text{ with } \boldsymbol{x}^* \in \langle \boldsymbol{W} \rangle,\ \hat{\boldsymbol{x}} \in \langle \boldsymbol{W} \rangle^{\perp\!\!\!\perp} \tag{4}$$

Let $\text{Proj}_{\langle \boldsymbol{W} \rangle} : \ell_2^{\Delta}(\boldsymbol{\Sigma}) \longrightarrow \langle \boldsymbol{W} \rangle$ denote the projection operator which assigns to any $\boldsymbol{x}$ in $\ell_2^{\Delta}(\boldsymbol{\Sigma})$ its orthogonal projection $\boldsymbol{x}^*$ onto $\langle \boldsymbol{W} \rangle$. As is well-known, $\boldsymbol{x}^*$ is the vector in the subspace $\langle \boldsymbol{W} \rangle$ which best approximates $\boldsymbol{x}$ in the sense that it lies closest to $\boldsymbol{x}$ in the metric induced by $\| \quad \|$

$$\boldsymbol{x}^* = \arg\min \{\| \boldsymbol{x} - \boldsymbol{y} \| \mid \boldsymbol{y} \in \langle \boldsymbol{W} \rangle\}$$

Note that agent i's utility function can be written as

$$u^i(\boldsymbol{x}^i) = -\frac{1}{2}(\alpha^i \mathbf{1} - \boldsymbol{x}^i)^{\mathsf{T}} \boldsymbol{\Delta}(\alpha^i \mathbf{1} - \boldsymbol{x}^i) = -\frac{1}{2} \| \alpha^i \mathbf{1} - \boldsymbol{x}^i \|^2 \tag{1$'$}$$

where $\mathbf{1} = (1, 1, \ldots) \in \ell_2^{\Delta}(\boldsymbol{\Sigma})$ denotes the perpetual annuity of 1 unit of the good (income) for ever. Thus the utility of the income stream $\boldsymbol{x}^{\boldsymbol{i}}$ is the loss induced by the (square) of the distance of $\boldsymbol{x}^i$ from the agent's *ideal consumption stream* $\alpha^i \mathbf{1}$.

Consider the maximum problem (i) of agent i in an equilibrium: the agent's budget set $\mathbb{B}(\bar{\boldsymbol{q}}, \boldsymbol{\omega}^i)$ is the translation of the market subspace which passes through the agent's initial endowment stream $\boldsymbol{\omega}^i$,

$$\mathbb{B}(\bar{\boldsymbol{q}}, \boldsymbol{\omega}^i) = \boldsymbol{\omega}^i + \langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle$$

In view of (1$'$), agent i's most preferred consumption stream is the solution of the problem

$$\bar{\boldsymbol{x}}^i = \arg\min \left\{ \| \alpha^i \mathbf{1} - \boldsymbol{x}^i \| \mid \boldsymbol{x}^i \in \boldsymbol{\omega}^i + \langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle \right\} \tag{5}$$

and is thus the income stream in the budget set which lies closest to the agent's ideal income stream $\alpha^i \mathbf{1}$. If $\boldsymbol{\tau}^i = \boldsymbol{x}^i - \boldsymbol{\omega}^i$ denotes agent i's *net trade*, then the problem (5) can be written as

$$\bar{\boldsymbol{\tau}}^i = \arg\min \left\{ \| \alpha^i \mathbf{1} - \boldsymbol{\omega}^i - \boldsymbol{\tau}^i \| \mid \boldsymbol{\tau}^i \in \langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle \right\}$$

so that $\bar{\boldsymbol{\tau}}^i$ is the vector in the closed subspace $\langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle$ which lies closest to $(\alpha^i \mathbf{1} - \boldsymbol{\omega}^i)$

$$\bar{\boldsymbol{\tau}}^i = \operatorname{Proj}_{\langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle} (\alpha^i \mathbf{1} - \boldsymbol{\omega}^i) \tag{6}$$

and the utility of agent i at equilibrium is

$$u^i(\bar{\boldsymbol{x}}^i) = -\frac{1}{2} \| \alpha^i \mathbf{1} - \boldsymbol{\omega}^i - \operatorname{Proj}_{\langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle} (\alpha^i \mathbf{1} - \boldsymbol{\omega}^i) \|^2$$

By (4), this is the projection of $\alpha^i \mathbf{1} - \boldsymbol{\omega}^i$ onto the orthogonal complement (dual space) $\langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle^{\perp\!\!\perp}$

$$u^i(\bar{\boldsymbol{x}}^i) = -\frac{1}{2} \| \operatorname{Proj}_{\langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle^{\perp\!\!\perp}} (\alpha^i \mathbf{1} - \boldsymbol{\omega}^i) \|^2 \tag{7}$$

Agent i begins with a "deficiency" $\alpha^i \mathbf{1} - \boldsymbol{\omega}^i$ whose norm measures the loss of the agent at the initial endowment $\boldsymbol{\omega}^i$, and ends up after trade with the shorter deficiency $\operatorname{Proj}_{\langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle^{\perp\!\!\perp}} (\alpha^i \mathbf{1} - \boldsymbol{\omega}^i)$.

The equilibrium price vector $\bar{\boldsymbol{q}}$ is determined by the market clearing equations for the securities, $\sum_{i=1}^{I} \bar{\boldsymbol{z}}^i = 0$, which are equivalent to the market clearing equations for the good at every date-event $\sum_{i=1}^{I} (\bar{\boldsymbol{x}}^i - \boldsymbol{\omega}^i) = 0$, namely $\sum_{i=1}^{I} \bar{\boldsymbol{\tau}}^i = 0$. Thus setting the sum of the vectors in (6) equal to zero gives the equilibrium condition

$$\operatorname{Proj}_{\langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle} (\alpha \mathbf{1} - \boldsymbol{w}) = 0 \quad \text{with} \quad \alpha = \frac{1}{I} \sum_{i=1}^{I} \alpha^i, \ \boldsymbol{w} = \frac{1}{I} \sum_{i=1}^{I} \boldsymbol{\omega}^i \tag{8}$$

Define $\boldsymbol{\pi} = \alpha \mathbf{1} - \boldsymbol{w}$: this is just the vector of (instantaneous) marginal utilities of the average agent with utility function $u(\boldsymbol{x}) = -\frac{1}{2} \| \alpha \mathbf{1} - \boldsymbol{x} \|^2$ consuming the per-capita endowment $\boldsymbol{w}$. By (8), the vector of equilibrium prices $\bar{\boldsymbol{q}}$ must be such that the market subspace $\langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle$ is orthogonal[5] to the average agent's vector of node prices $\boldsymbol{\pi}$. This can be written as $\boldsymbol{\pi}^\top \boldsymbol{\Delta} \boldsymbol{W}(\bar{\boldsymbol{q}}) = 0$, which implies that each column of the matrix $\boldsymbol{W}(\bar{\boldsymbol{q}})$ is orthogonal to $\boldsymbol{\pi}$: thus for $j = 1, \ldots, J$ and for all $\sigma_t \in \boldsymbol{\Sigma}$

$$-\pi(\sigma_t) \delta^t \rho(\sigma_t) \bar{q}^j(\sigma_t) + \sum_{\sigma' \in \sigma_t^+} \pi(\sigma') \delta^{t+1} \rho(\sigma') R^j(\sigma') = 0 \tag{9}$$

which shows that the equilibrium security prices are obtained from the implicit *present value prices*

[5] (8) is equivalent to $\alpha \mathbf{1} - \boldsymbol{w} \in \langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle^{\perp\!\!\perp}$.

$$\Pi(\sigma_t) = \delta^t \rho(\sigma_t)\pi(\sigma_t), \quad \forall\ \sigma_t \in \boldsymbol{\Sigma}$$

Thus whatever the payoff structure or the number of securities traded – and in particular whether markets are complete or incomplete – the prices of the securities are derived from the vector of present-value prices $\boldsymbol{\Pi} = \boldsymbol{\Delta\pi}$, namely the vector of present values of the (instantaneous) marginal utilities of consumption of the average agent at $\boldsymbol{w}$. The recursive system of equations (9) has a unique solution if we require that the price of each security at any node is equal to the present value of its future dividends (its so-called *fundamental value* at this node). All other solutions involve *bubbles*, and although equilibria with bubbles can exist for infinite-lived securities in zero net supply (see Magill-Quinzii [27], Santos-Woodford [31]) we will not consider them here. We derive the unique equilibrium of the economy $\mathscr{E}$ in which each security is priced at its fundamental value i.e. for each $j = 1,\ldots,J$ and each $\sigma_t \in \boldsymbol{\Sigma}$ the equilibrium price is given by

$$\bar{q}_j(\sigma_t) = \sum_{\tau \geq 1} \delta^\tau \rho(\sigma_{t+\tau} \mid \sigma_t) \frac{\pi(\sigma_{t+\tau})}{\pi(\sigma_t)} F_j(\sigma_{t+\tau}) d^{h^{\tau-1}(j)}(\sigma_{t+\tau}) \tag{10}$$

where $h^0(j) = j$, $F_j(\sigma_{t+1}) = 1$, $\forall\ \sigma_{t+1} \in \sigma_t^+$ and for $\tau \geq 2$, $F_j(\sigma_{t+\tau}) = f_j(\sigma_{t+1}) f_{h(j)}(\sigma_{t+2}) \ldots f_{h^{\tau-1}(j)}(\sigma_{t+\tau})$, this latter expression being evaluated along the path from σ_t to $\sigma_{t+\tau}$. If, for some k, $h^k(j) = \emptyset$ then by convention $f_{h^k(j)}(\sigma_{t+k+1}) = 0$ (i.e. the security matures at date $t+k$ and delivers nothing thereafter). For short-lived securities ($h(j) = \emptyset$) and infinitely-lived securities which do not "shrink" over time ($h(j) = j,\ f_j(\sigma_t) = 1,\ \forall\ \sigma_t$), (10) is the standard present-value formula for the price of a security.

Complete markets. As a reference case we first calculate the consumption and utility of each agent in an Arrow-Debreu equilibrium. This is the same as the consumption and utility of agents in a financial market equilibrium in which at each node agents can trade at least S securities with linearly independent payoffs at the immediate successors – a case that we refer to as "complete" financial markets. In this case it can be shown that there is a unique vector (up to multiplication by a scalar) $\boldsymbol{\pi} \in \ell_2^{\Delta}(\boldsymbol{\Sigma})$ which is orthogonal to all the columns of $\boldsymbol{W}(\bar{\boldsymbol{q}})$, so that $\dim\langle \boldsymbol{W}(\bar{\boldsymbol{q}})\rangle^{\perp\!\!\perp} = 1$ and agents choose their net trades $\boldsymbol{\tau}^i$ from the hyperplane $[\![\, \boldsymbol{\pi}, \boldsymbol{\tau}^i \,]\!] = 0$ (see Magill-Quinzii [26, ch. 4] for the T-period case, which is easily extended to the infinite-horizon case). When markets are complete it is straightforward to derive an explicit solution for the equilibrium consumption and utility levels of agents, since the projections of $\alpha^i \mathbf{1} - \boldsymbol{\omega}^i$ onto $\langle \boldsymbol{W}(\bar{\boldsymbol{q}})\rangle$ and $\langle \boldsymbol{W}(\bar{\boldsymbol{q}})\rangle^{\perp\!\!\perp}$ are readily found. Let

$$\boldsymbol{\pi}^i = \alpha^i \mathbf{1} - \boldsymbol{\omega}^i$$

denote the vector of (instantaneous) marginal utilities of agent i at the initial endowment $\boldsymbol{\omega}^i$.

Proposition 1 (Complete markets equilibrium). *The Arrow–Debreu equilibrium consumption streams and utility levels of the agents are given by* ($i = 1,\ldots,I$).

(i) $\bar{\boldsymbol{x}}^i_{AD} \;=\; a^i_{AD}\mathbf{1} \;+\; b^i_{AD}\boldsymbol{w} \quad with \quad a^i_{AD} \;=\; \dfrac{\alpha^i \parallel \boldsymbol{\pi} \parallel^2 - \alpha \,[\![\, \boldsymbol{\pi}^i, \boldsymbol{\pi} \,]\!]}{\parallel \boldsymbol{\pi} \parallel^2},$

$b^i_{AD} \;=\; \dfrac{[\![\, \boldsymbol{\pi}^i, \boldsymbol{\pi} \,]\!]}{\parallel \boldsymbol{\pi} \parallel^2}$

(ii) $u^i_{AD} = -\frac{1}{2}\dfrac{[\![\, \boldsymbol{\pi}^i, \boldsymbol{\pi} \,]\!]^2}{\parallel \boldsymbol{\pi} \parallel^2}$

Proof. Since $\dim\langle \boldsymbol{W}(\bar{\boldsymbol{q}})\rangle^{\perp\!\!\!\perp} = 1$, by (8), the vector $\boldsymbol{\pi}$ is a basis for $\langle \boldsymbol{W}(\bar{\boldsymbol{q}})\rangle^{\perp\!\!\!\perp}$. The orthogonal decomposition of $\boldsymbol{\pi}^i$ can thus be written as

$$\boldsymbol{\pi}^i = \boldsymbol{\pi}^{i*} \;+\; \hat{\boldsymbol{\pi}}^i \;=\; \boldsymbol{\pi}^{i*} \;+\; \lambda^i \boldsymbol{\pi}$$

which implies

$$\lambda^i = \frac{[\![\, \boldsymbol{\pi}^i, \boldsymbol{\pi} \,]\!]}{\parallel \boldsymbol{\pi} \parallel^2}, \quad \hat{\boldsymbol{\pi}}^i \;=\; \lambda^i \boldsymbol{\pi}, \quad \boldsymbol{\pi}^{i*} = (\alpha^i - \lambda^i \alpha)\mathbf{1} - \boldsymbol{\omega}^i + \lambda^i \boldsymbol{w} \qquad (11)$$

By (6), $\bar{\boldsymbol{x}}^i_{AD} \;=\; \boldsymbol{\omega}^i + \boldsymbol{\pi}^{i*}$, and by (7), $u^i_{AD} \;=\; -\frac{1}{2} \parallel \hat{\boldsymbol{\pi}}^i \parallel^2$. Substituting the expressions (11), gives (i) and (ii). □

Incomplete markets. Although the Arrow-Debreu equilibrium is derived by assuming that the financial markets are complete, such a rich array of securities is not necessary to achieve this Pareto optimal allocation: all that is needed is that agents can trade from their initial endowment to a consumption stream which is a combination of a sure income stream and the average endowment. Whether or not such trades are achievable depends on the "span" of the markets from date 1 on. To express such conditions we introduce the following notation. Let $\boldsymbol{\Sigma}_+ = \boldsymbol{\Sigma}\setminus\{\sigma_0\}$ denote the event tree after date 0: for a stochastic income stream $\boldsymbol{x} \in \ell_2^{\Delta}(\boldsymbol{\Sigma})$, let

$$\boldsymbol{x}_+ = (x(\sigma_t), \sigma_t \in \boldsymbol{\Sigma}_+)$$

denote the associated income stream after date 0, obtained by deleting the date 0 component. The space of *after date 0 income streams* $\mathbb{R}^{\boldsymbol{\Sigma}_+}$ is endowed with the natural scalar product derived from the scalar product on $\mathbb{R}^{\boldsymbol{\Sigma}}$

$$[\![\, \boldsymbol{x}_+, \boldsymbol{y}_+ \,]\!] \;=\; \sum_{t \geq 1} \delta^t \rho(\sigma_t) x(\sigma_t) y(\sigma_t) = \boldsymbol{x}_+ \boldsymbol{\Delta}_+ \boldsymbol{y}_+$$

where $\boldsymbol{\Delta}_+$ is matrix obtained from $\boldsymbol{\Delta}$ by deleting the first row. For simplicity we use the same notation to denote the scalar product and norm on $\mathbb{R}^{\boldsymbol{\Sigma}}$ and $\mathbb{R}^{\boldsymbol{\Sigma}_+}$. Finally let $\boldsymbol{W}(\boldsymbol{q})_+$ denote the matrix obtained from $\boldsymbol{W}(\boldsymbol{q})$ by deleting the first row and let $\langle \boldsymbol{W}(\boldsymbol{q})_+\rangle$ denote the smallest closed subspace of $\ell_2^{\Delta}(\boldsymbol{\Sigma}_+)$ spanned by the columns of $\boldsymbol{W}(\boldsymbol{q})_+$. In economic terms the difference between $\langle \boldsymbol{W}(\boldsymbol{q})\rangle$ and $\langle \boldsymbol{W}(\boldsymbol{q})_+\rangle$ is that the date 0 cost is not taken into account in $\langle \boldsymbol{W}(\boldsymbol{q})_+\rangle$ while it is in $\langle \boldsymbol{W}(\boldsymbol{q})\rangle$: we refer to $\langle \boldsymbol{W}(\boldsymbol{q})_+\rangle$ as the *marketed subspace*. For some cost at date 0 agents can have access to all income streams in $\langle \boldsymbol{W}(\boldsymbol{q})_+\rangle$.

The Arrow-Debreu equilibrium allocation can be achieved under the following assumptions:

A1 (*Diversifiable individual risks*): $\boldsymbol{\omega}^i_+ \in \langle \boldsymbol{W}(\bar{\boldsymbol{q}})_+ \rangle, \quad i = 1, \ldots, I$

A2 (*Perfect consumption smoothing*): $\mathbf{1}_+ \in \langle \boldsymbol{W}(\bar{\boldsymbol{q}})_+ \rangle$.

A1 is an assumption about the diversifiability (marketability) of the agents' real individual risks; it is satisfied if output is generated by a finite number of firms (whose production plans are taken as fixed), if agents' endowments consist of ownership shares of the firms, and if the equity contracts of the firms are traded. **A2** is normally viewed as an assumption on the bond market. In a two-period model it is satisfied if there is a bond whose real payoff is constant across the states: this requires either that the bond is indexed and that perfect indexation is possible, or that the bond is nominal and that inflation is not variable (see Magill-Quinzii [28]). In a multiperiod model **A2** is a stronger assumption. For suppose that at each date-event there is a short-lived bond which gives one unit of real purchasing power at each of the immediate successors. This still does not permit a constant income stream over time to be achieved if the real interest rate is variable. For a portfolio strategy on a short-lived bond which delivers income at every date requires that a part of the payoff of the bond purchased at date t be used to finance the purchase of the bond at date $t + 1$: when the price of the bond varies across date-events, there is no portfolio strategy that delivers a constant income stream across date-events. Thus not only the *variability of inflation*, but also the *variability of the real interest rate* can lead to **A2** being violated. Since there are even more reasons for dispensing with assumption **A2** in a multiperiod model, we will study the agents' consumption and welfare in an economy in which real individual risks are diversifiable (**A1**), but in which **A2** is not necessarily satisfied. If agents do not have access to the sure income stream, it seems natural that they will seek to replace it by the least risky income stream in the marketed subspace $\langle \boldsymbol{W}(\bar{\boldsymbol{q}})_+ \rangle$. This leads to the following definition.

Definition. The orthogonal projection of $\mathbf{1}_+$ onto $\langle \boldsymbol{W}(\bar{\boldsymbol{q}})_+ \rangle$ is called the *least variable income stream* (LVI for short) in the marketed subspace, and is denoted by $\boldsymbol{\eta}$. Thus $\boldsymbol{\eta}$ is defined by the orthogonal decomposition

$$\mathbf{1}_+ = \boldsymbol{\eta} + \hat{\boldsymbol{\eta}}, \quad \boldsymbol{\eta} \in \langle \boldsymbol{W}(\bar{\boldsymbol{q}})_+ \rangle, \quad \hat{\boldsymbol{\eta}} \in \langle \boldsymbol{W}(\bar{\boldsymbol{q}})_+ \rangle^{\perp\!\!\perp} \tag{12}$$

Proposition 2 (Incomplete markets equilibrium). *If individual risks are diversifiable* **(A1)***, then the equilibrium consumption streams and utility levels of the agents* ($i = 1, \ldots, I$) *are given by*

(i) $\bar{\boldsymbol{x}}^i = a^i(1, \boldsymbol{\eta}) + b^i \boldsymbol{w}$ *with*

$$a^i = \frac{\alpha^i \parallel \boldsymbol{\pi} \parallel^2 - \alpha [\![\boldsymbol{\pi}^i, \boldsymbol{\pi}]\!]}{\parallel \boldsymbol{\pi} \parallel^2 - \alpha^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2}, \quad b^i = \frac{[\![\boldsymbol{\pi}^i, \boldsymbol{\pi}]\!] - \alpha\alpha^i \parallel \hat{\boldsymbol{\eta}} \parallel^2}{\parallel \boldsymbol{\pi} \parallel^2 - \alpha^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2}$$

(ii) $u^i(\bar{\boldsymbol{x}}^i) = \bar{u}^i_{AD} \left[1 + \dfrac{(\gamma^i)^2 \parallel \boldsymbol{\pi} \parallel^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2}{1 - B \parallel \hat{\boldsymbol{\eta}} \parallel^2} \right]$ *with*

$$B = \frac{\alpha^2}{\| \boldsymbol{\pi} \|^2}, \quad \gamma^i = \frac{\alpha^i}{[\![\boldsymbol{\pi}^i, \boldsymbol{\pi}]\!]} - \frac{\alpha}{\| \boldsymbol{\pi} \|^2}$$

Proof. As in Proposition 1, the proof consists in calculating the projections $\boldsymbol{\pi}^{i*}$ and $\hat{\boldsymbol{\pi}}^i$ of $\boldsymbol{\pi}^i$ onto $\langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle$ and $\langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle^{\perp\!\!\!\perp}$. However since $\langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle^{\perp\!\!\!\perp}$ is now larger than a one dimensional space, the calculation of the projections is more involved. We will compute the projections on $\langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle$ and $\langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle^{\perp\!\!\!\perp}$ by taking limits of projections on finite dimensional subspaces obtained by truncating trade at date T, for $T = 1, 2, \ldots$. For simplicity we omit $\bar{\boldsymbol{q}}$ when writing $\langle \boldsymbol{W} \rangle$. Let $\langle \boldsymbol{W} \rangle_{T}$ denote the subspace of $\langle \boldsymbol{W} \rangle$ obtained by using trading strategies which stop at date T i.e. $z(\sigma_t) = 0$ if $t \geq T$. $\langle \boldsymbol{W} \rangle_{T}$ lies in the subspace A^T of $\ell_2^{\Delta}(\boldsymbol{\Sigma})$ consisting of vectors with zero co-ordinates after date T: let B^T denote the complement in $\ell_2^{\Delta}(\boldsymbol{\Sigma})$ consisting of vectors with zero co-ordinates up to date T

$$A^T = \{\boldsymbol{x} \in \ell_2^{\Delta}(\boldsymbol{\Sigma}) \mid x(\sigma_t) = 0 \text{ if } t > T\}, \; B^T = \{\boldsymbol{x} \in \ell_2^{\Delta}(\boldsymbol{\Sigma}) \mid x(\sigma_t) = 0 \text{ if } t \leq T\}$$

Let A_+^T and B_+^T denote the associated subspaces of $\ell_2^{\Delta}(\boldsymbol{\Sigma}_+)$. A vector $\boldsymbol{x} \in \ell_2^{\Delta}(\boldsymbol{\Sigma})$ can be written as $\boldsymbol{x} = \boldsymbol{x}_{T} + \boldsymbol{x}_{>T}$ with $\boldsymbol{x}_{T} \in A^T$ and $\boldsymbol{x}_{>T} \in B^T$. A^T is a finite dimensional subspace of dimension N_{T} where N_{T} is the number of nodes up to date T (i.e. $N_{T} = 1 + S + \ldots + S^T$) and $\langle \boldsymbol{W} \rangle_{T}$ is a subspace of A^T of maximum dimension $(1 + S + \ldots + S^{T-1})J$. The vector $\boldsymbol{\pi}^i = \alpha^i \mathbf{1} - \boldsymbol{\omega}^i$ can be decomposed as $\boldsymbol{\pi}^i = \boldsymbol{\pi}_{T}^{i*} + \hat{\boldsymbol{\pi}}_{T}^i + \boldsymbol{\pi}_{>T}^i$ where $\boldsymbol{\pi}_{T}^{i*}$ is the projection of $\boldsymbol{\pi}^i$ (and also of $\boldsymbol{\pi}_{T}^i$) onto $\langle \boldsymbol{W} \rangle_{T}, \hat{\boldsymbol{\pi}}_{T}^i$ is the orthogonal complement of $\boldsymbol{\pi}_{T}^{i*}$ in A^T, and $\boldsymbol{\pi}_{>T}^i \in B^T$. Thus

$$\boldsymbol{\pi}_{T}^{i*} = \text{Proj}_{\langle \boldsymbol{W} \rangle_{T}}(\boldsymbol{\pi}^i), \quad \hat{\boldsymbol{\pi}}_{T}^i + \boldsymbol{\pi}_{>T}^i = \text{Proj}_{\langle \boldsymbol{W} \rangle_{T}^{\perp\!\!\!\perp}}(\boldsymbol{\pi}^i)$$

Since $\langle \boldsymbol{W} \rangle_{T} \subset \langle \boldsymbol{W} \rangle_{T+1} \subset \ldots \subset \langle \boldsymbol{W} \rangle, T = 1, 2, \ldots$ is an increasing sequence of subspaces of $\langle \boldsymbol{W} \rangle$, and since $\langle \boldsymbol{W} \rangle = \overline{\bigcup_{T=1}^{\infty} \langle \boldsymbol{W} \rangle_{T}}$,

$$\text{Proj}_{\langle \boldsymbol{W} \rangle}(\boldsymbol{\pi}^i) = \lim_{T \longrightarrow \infty} \text{Proj}_{\langle \boldsymbol{W} \rangle_{T}}(\boldsymbol{\pi}^i) \tag{13}$$

(see Lorch [21], theorem 5.1). To calculate $\boldsymbol{\pi}_{T}^{i*}$ it suffices to compute $\hat{\boldsymbol{\pi}}_{T}^i$. To this end, consider a basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ of the orthogonal complement of $\langle \boldsymbol{W} \rangle_{T}$ in A^T, which we denote by $\langle \boldsymbol{W} \rangle_{T}^{\perp}$, to differentiate it from $\langle \boldsymbol{W} \rangle_{T}^{\perp\!\!\!\perp}$, the orthogonal complement of $\langle \boldsymbol{W} \rangle_{T}$ in $\ell_2^{\Delta}(\boldsymbol{\Sigma})$, where $\langle \boldsymbol{W} \rangle_{T}^{\perp\!\!\!\perp} = \langle \boldsymbol{W} \rangle_{T}^{\perp} \oplus B^T$. We need to find the coefficients $(c_1, \ldots, c_m)$ such that

$$\hat{\boldsymbol{\pi}}_{T}^i = c_1 \boldsymbol{v}_1 + \ldots + c_m \boldsymbol{v}_m$$

To find these coefficients we use the property that $\boldsymbol{\pi}_{T}^i - \hat{\boldsymbol{\pi}}_{T}^i = \boldsymbol{\pi}_i^*$ is orthogonal to each $\boldsymbol{v}_j$ i.e. $[\![\hat{\boldsymbol{\pi}}_{T}^i, \boldsymbol{v}_j]\!] = [\![\boldsymbol{\pi}_{T}^i, \boldsymbol{v}_j]\!]$ for $j = 1, \ldots, m$, namely

$$\begin{array}{ccccc} c_1[\![\boldsymbol{v}_1, \boldsymbol{v}_1]\!] & + \ldots + & c_m [\![\boldsymbol{v}_m, \boldsymbol{v}_1]\!] & = & [\![\boldsymbol{\pi}_{T}^i, \boldsymbol{v}_1]\!] \\ \vdots & & \vdots & & \\ c_1[\![\boldsymbol{v}_1, \boldsymbol{v}_m]\!] & + \ldots + & c_m [\![\boldsymbol{v}_m, \boldsymbol{v}_m]\!] & = & [\![\boldsymbol{\pi}_{T}^i, \boldsymbol{v}_m]\!] \end{array} \tag{14}$$

Solving this system of m equations determines $(c_1,\dots,c_m)$: a careful choice of basis for $\langle \boldsymbol{W}\rangle_T^{\perp}$ will simplify the calculation.

Let $\boldsymbol{W}_T$ denote the finite matrix obtained by eliminating all but the first N_T rows and $(N_T - S^T)J$ columns of $\boldsymbol{W}$. Note that $\langle \boldsymbol{W}\rangle_T$ is isomorphic to the subspace of $\mathbb{R}^{N_T}$ spanned by the columns of $\boldsymbol{W}_T$. Since the vector of present value prices $\boldsymbol{\Pi} = \boldsymbol{\Delta}(\alpha\mathbf{1} - \boldsymbol{\omega})$ is orthogonal to every column of $\boldsymbol{W}$ and since $\boldsymbol{W}$ has only zeros below the matrix $\boldsymbol{W}_T$, it follows that $\boldsymbol{\Pi}_T^{\mathsf{T}}\boldsymbol{W}_T = \mathbf{0}, T = 1,2,\dots$. Since $\Pi_0 = \alpha - w_0 > 0$, the first row of the matrix $\boldsymbol{W}_T$ is a linear combination of its remaining rows: thus rank $\boldsymbol{W}_T$ = rank $\boldsymbol{W}_{+_T} \iff$ dim $\langle \boldsymbol{W}_T\rangle =$ dim $\langle \boldsymbol{W}_+\rangle_T$ so that

$$\dim\langle \boldsymbol{W}\rangle_T^{\perp} = 1 + \dim \langle \boldsymbol{W}_+\rangle_T^{\perp} \tag{15}$$

Case 1: $\mathbf{1}_+ \notin \langle \boldsymbol{W}_+\rangle$. Then there exists $\overline{T}$ such that $\mathbf{1}_{+T} \notin \langle \boldsymbol{W}_+\rangle_T$ for $T \geq \overline{T}$. Let $\mathbf{1}_{+_T} = \boldsymbol{\eta}_T + \hat{\boldsymbol{\eta}}_T$ where $\boldsymbol{\eta}_T = \mathrm{Proj}_{\langle \boldsymbol{W}_+\rangle_T}(\mathbf{1}_{+_T}), \hat{\boldsymbol{\eta}}_T = \mathrm{Proj}_{\langle \boldsymbol{W}_+\rangle_T^{\perp}}(\mathbf{1}_{+_T})$. For $T \geq \overline{T}, \hat{\boldsymbol{\eta}}_T \neq 0$. Let $\{\hat{\boldsymbol{\eta}}_T, \boldsymbol{e}_3,\dots,\boldsymbol{e}_m\}$ be an orthogonal basis for $\langle \boldsymbol{W}_+\rangle_T^{\perp}$. Since $\boldsymbol{\Pi}_T^{\mathsf{T}}\boldsymbol{W}_T = \boldsymbol{\pi}_T^{\mathsf{T}}\boldsymbol{\Delta}\boldsymbol{W}_T = 0$, $\boldsymbol{\pi}_T$ belongs to $\langle \boldsymbol{W}\rangle_T^{\perp}$ and since $\pi_0 = \alpha - w_0 > 0, \boldsymbol{\pi}_T$ and $(0,\hat{\boldsymbol{\eta}}_T)$ (respectively $(0,\boldsymbol{e}_j), j = 3,\dots,m$) are linearly independent. Thus in view of (15), the vectors

$$\boldsymbol{v}_1 = \boldsymbol{\pi}_T, \quad \boldsymbol{v}_2 = (0,\hat{\boldsymbol{\eta}}_T), \quad \boldsymbol{v}_3 = (0,\boldsymbol{e}_3),\dots,\boldsymbol{v}_m = (0,\boldsymbol{e}_m)$$

form a basis for $\langle \boldsymbol{W}\rangle_T^{\perp}$. With this basis, let us evaluate the inner products in (14): only a few are non-zero, so the system of equations (14) can be simplified. $[\![\boldsymbol{v}_1,\boldsymbol{v}_2]\!] = \alpha[\![\mathbf{1}_T,(0,\hat{\boldsymbol{\eta}}_T)]\!] - [\![\boldsymbol{w}_T,(0,\hat{\boldsymbol{\eta}}_T)]\!] = \alpha \| \hat{\boldsymbol{\eta}}_T \|^2$ since $[\![\mathbf{1}_T,(0,\hat{\boldsymbol{\eta}}_T)]\!] = [\![\mathbf{1}_{+_T},\hat{\boldsymbol{\eta}}_T]\!] = [\![\boldsymbol{\eta}_T + \hat{\boldsymbol{\eta}}_T,\hat{\boldsymbol{\eta}}_T]\!] = \| \hat{\boldsymbol{\eta}}_T \|^2$ and since $\boldsymbol{\omega}^i_{+_T} \in \langle \boldsymbol{W}_+\rangle_T \implies \boldsymbol{w}_{+_T} \in \langle \boldsymbol{W}_+\rangle_T \implies [\![\boldsymbol{w}_T,(0,\hat{\boldsymbol{\eta}}_T)]\!] = 0$. For $j \geq 3$, $[\![\boldsymbol{v}_1,\boldsymbol{v}_j]\!] = \alpha[\![\mathbf{1}_{+_T},(0,\boldsymbol{e}_j)]\!] + [\![\boldsymbol{\omega}^i_T,(0,\boldsymbol{e}_j)]\!] = 0, [\![\boldsymbol{v}_2,\boldsymbol{v}_j]\!] = 0$ by the choice of $\{\boldsymbol{e}_3,\dots,\boldsymbol{e}_m\}$ and $[\![\boldsymbol{v}_{j'},\boldsymbol{v}_j]\!] = 0$ if $j' \geq 3$, $j' \neq j$. For $j \geq 3$, the right-hand side $[\![\boldsymbol{\pi}^i_T,\boldsymbol{v}_j]\!]$ of (14) satisfies $[\![\boldsymbol{\pi}^i_T,\boldsymbol{v}_j]\!] = \alpha^i[\![\mathbf{1}_{+_T},(0,\boldsymbol{e}_j)]\!] - [\![\boldsymbol{\omega}^i_T,(0,\boldsymbol{e}_j)]\!] = 0$ and since $[\![v_j,v_j]\!] \neq 0$ the coefficients $c_3,\dots,c_m$ are zero. Thus (14) reduces to the pair of equations in the coefficients (c_1^T,c_2^T) (recalling that they depend on T)

$$\begin{aligned} c_1^T \| \boldsymbol{\pi}_T \|^2 + c_2^T\alpha \| \hat{\boldsymbol{\eta}}_T \|^2 &= [\![\boldsymbol{\pi}^i_T,\boldsymbol{\pi}_T]\!] \\ c_1^T\alpha \| \hat{\boldsymbol{\eta}}_T \|^2 + c_2^T \| \hat{\boldsymbol{\eta}}_T \|^2 &= \alpha^i \| \hat{\boldsymbol{\eta}}_T \|^2 \end{aligned}$$

with solution

$$c_1^T = \frac{[\![\boldsymbol{\pi}^i_T,\boldsymbol{\pi}_T]\!] - \alpha\alpha^i \| \hat{\boldsymbol{\eta}}_T \|^2}{\| \boldsymbol{\pi}_T \|^2 - \alpha^2 \| \hat{\boldsymbol{\eta}}_T \|^2}, \quad c_2^T = \frac{\alpha^i \| \boldsymbol{\pi}_T \|^2 - \alpha [\![\boldsymbol{\pi}^i_T,\boldsymbol{\pi}_T]\!],}{\| \boldsymbol{\pi}_T \|^2 - \alpha^2 \| \hat{\boldsymbol{\eta}}_T \|^2} \tag{16}$$

Since

$$\lim_{T\longrightarrow\infty} \boldsymbol{\eta}_T = \lim_{T\longrightarrow\infty} \mathrm{Proj}_{\langle \boldsymbol{W}_+\rangle_T}(\mathbf{1}_+) = \mathrm{Proj}_{\langle \boldsymbol{W}_+\rangle}(\mathbf{1}_+) = \boldsymbol{\eta}$$

$\mathbf{1}_+ = \boldsymbol{\eta}_T + \hat{\boldsymbol{\eta}}_T + \mathbf{1}_{>_T}$ implies $\lim_{T\longrightarrow\infty}(\hat{\boldsymbol{\eta}}_T + \mathbf{1}_{>_T}) = \mathbf{1}_+ - \lim_{T\longrightarrow\infty} \boldsymbol{\eta}_T = \mathbf{1} - \boldsymbol{\eta} = \hat{\boldsymbol{\eta}}$ and since $\mathbf{1}_{>_T} \longrightarrow 0$ in $\ell_2^{\Delta}(\boldsymbol{\Sigma}), \lim_{T\longrightarrow\infty} \hat{\boldsymbol{\eta}}_T = \hat{\boldsymbol{\eta}}$. By a similar argument, using

$$\lim_{T\longrightarrow\infty} \boldsymbol{\pi}_T^{i*} = \lim_{T\longrightarrow\infty} \mathrm{Proj}_{\langle \boldsymbol{W}\rangle_T}(\boldsymbol{\pi}^i) = \mathrm{Proj}_{\langle \boldsymbol{W}\rangle}(\boldsymbol{\pi}^i) = \boldsymbol{\pi}^{i*}$$

$\boldsymbol{\pi}^i = \boldsymbol{\pi}_{_T}^{i*} + \hat{\boldsymbol{\pi}}_{_T}^i + \boldsymbol{\pi}_{>T}^i$ and $\boldsymbol{\pi}^i = \boldsymbol{\pi}^{i*} + \hat{\boldsymbol{\pi}}^i$, we deduce $\lim_{T\longrightarrow\infty} \hat{\boldsymbol{\pi}}_{_T}^i = \hat{\boldsymbol{\pi}}^i$. Since $\boldsymbol{\pi}_{_T}^i \longrightarrow \boldsymbol{\pi}^i$ and $\boldsymbol{\pi}_{_T} \longrightarrow \boldsymbol{\pi}$, it follows from (16) that

$$\hat{\boldsymbol{\pi}}^i = \frac{[\![\boldsymbol{\pi}^i, \boldsymbol{\pi}]\!] - \alpha\alpha^i \parallel \hat{\boldsymbol{\eta}} \parallel^2}{\parallel \boldsymbol{\pi} \parallel^2 - \alpha^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2}\boldsymbol{\pi} + \frac{\alpha^i \parallel \boldsymbol{\pi} \parallel^2 - \alpha\, [\![\boldsymbol{\pi}^i, \boldsymbol{\pi}]\!]}{\parallel \boldsymbol{\pi} \parallel^2 - \alpha^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2}(0, \hat{\boldsymbol{\eta}}) \qquad (17)$$

and $\boldsymbol{\pi}^{i*} = \boldsymbol{\pi}^i - \hat{\boldsymbol{\pi}}^i$. Since the equilibrium consumption and utility level of agent i satisfy $\overline{\boldsymbol{x}}^i = \boldsymbol{\omega}^i + \boldsymbol{\pi}^{i*}$ and $u^i(\overline{\boldsymbol{x}}^i) = -\frac{1}{2} \parallel \hat{\boldsymbol{\pi}}^i \parallel^2$, substituting from (17) gives (i) and (ii) of the proposition.
Case 2: $\mathbf{1}_+ \in \langle \boldsymbol{W}_+ \rangle$. Let $\{\boldsymbol{e}_2, \ldots, \boldsymbol{e}_m\}$ be an orthogonal basis for $\langle \boldsymbol{W}_+ \rangle_{_T}^{\perp}$. Then $\boldsymbol{v}_1 = \boldsymbol{\pi}_{_T}, \boldsymbol{v}_2 = (0, \boldsymbol{e}_2), \ldots, \boldsymbol{v}_m = (0, \boldsymbol{e}_m)$ is a basis for $\langle \boldsymbol{W}_{_T}^{\perp} \rangle$. Solving the system of equations (14) in this case, leads to $c_1^{^T} = \dfrac{[\![\boldsymbol{\pi}_{_T}^i, \boldsymbol{\pi}_{_T}]\!]}{\parallel \boldsymbol{\pi}_{_T} \parallel^2}, c_j^{^T} = 0,\ j = 2, \ldots, m$, so that after taking limits, $\hat{\boldsymbol{\pi}}^i$ and $\boldsymbol{\pi}^*$ are given by (11). □

Explaining equilibrium consumption. The expressions for the coefficients (a^i, b^i) in Proposition 2, which determine the agents' equilibrium consumption streams, are those that fall most naturally out of the proof of the proposition and are useful for computation. There is however an alternative way of writing these coefficients which brings out better the economic logic underlying the agents' equilibrium consumption streams. For $i = 1, \ldots, I$, define the variables

$$Y^i = [\![\boldsymbol{\pi}, \boldsymbol{\omega}^i]\!], \quad Y = [\![\boldsymbol{\pi}, \boldsymbol{w}]\!], \quad \beta^i = \frac{\alpha^i}{[\![\boldsymbol{\pi}, \boldsymbol{\omega}^i]\!]}, \quad \beta = \frac{\alpha}{[\![\boldsymbol{\pi}, \boldsymbol{w}]\!]} \qquad (18)$$

By substituting the values of $\boldsymbol{\pi}^i$ and $\boldsymbol{\pi}$ it is easy to see that (a^i, b^i) can be expressed as the following functions of the variables in (18)

$$a^i = \frac{Y^i}{Y}\,\frac{\beta - \beta^i}{\parallel \boldsymbol{\pi} \parallel^2 - \alpha^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2}, \quad b^i = \frac{Y^i}{Y}\,\frac{\beta^i([\![\boldsymbol{\pi}, \mathbf{1}]\!] - \alpha \parallel \hat{\boldsymbol{\eta}} \parallel^2) - 1}{\beta([\![\boldsymbol{\pi}, \mathbf{1}]\!] - \alpha \parallel \hat{\boldsymbol{\eta}} \parallel^2) - 1} \qquad (19)$$

The coefficients (a_{AD}^i, b_{AD}^i) for the Arrow-Debreu equilibrium are obtained by setting $\hat{\boldsymbol{\eta}}$ equal to $\mathbf{0}$.

Since $\boldsymbol{\Pi} = \boldsymbol{\Delta\pi}$ is the equilibrium vector of present value prices, $\boldsymbol{\Pi\omega^i} = \boldsymbol{\pi}^\top \boldsymbol{\Delta\omega^i} = [\![\boldsymbol{\pi}, \boldsymbol{\omega}^i]\!] = Y^i$ is the present value of $\boldsymbol{\omega}^i$, namely the equilibrium wealth of agent i. The ratio $\frac{[\![\boldsymbol{\pi}, \alpha^i \mathbf{1}]\!]}{[\![\boldsymbol{\pi}, \boldsymbol{\omega}^i]\!]} = \beta^i [\![\boldsymbol{\pi}, \mathbf{1}]\!]$, which measures the wealth required to purchase the ideal stream $\alpha^i \mathbf{1}$ relative to agent i's wealth, can be interpreted as a measure of the agent's "ambition". The more ambitious agent i is, the less curved is the indifference surface passing through $\boldsymbol{\omega}^i$, and thus the less curved are the indifference surfaces in the relevant consumption region[6]

$$\{\boldsymbol{x}^i \in \mathbb{R}^{\boldsymbol{\Sigma}} \mid \boldsymbol{\Pi}^\top \boldsymbol{x}^i \leq \boldsymbol{\Pi}^\top \boldsymbol{\omega}^i, \quad u^i(\boldsymbol{x}^i) \geq u^i(\boldsymbol{\omega}^i)\}$$

[6] We know, from the theory of incomplete markets, that the budget set $\mathbb{B}(\overline{\boldsymbol{q}}, \boldsymbol{\omega}^i)$ is a subset of the Arrow-Debreu budget set $\{x^i \in \mathbb{R}^{\boldsymbol{\Sigma}} \mid \boldsymbol{\Pi x}^i \leq \boldsymbol{\Pi\omega}^i\}$. Thus agents' choices need not be considered outside this set.

Thus a high ambition is equivalent to a high level of risk tolerance and $\beta^i = \alpha^i/[\![\boldsymbol{\pi}, \boldsymbol{\omega}^i]\!]$ is the measure of the risk tolerance of agent i which is natural in this model. Given the homogeneity of the expressions (19) in Y^i, the equilibrium consumption stream of agent i can be written as

$$\bar{\boldsymbol{x}}^i = \frac{Y^i}{Y}(a(\beta^i)(1, \boldsymbol{\eta}) + b(\beta^i)\boldsymbol{w}) \tag{20}$$

where $(a(\beta^i), b(\beta^i))$ are the expressions in (19) for the normalized wealth level $Y^i = Y$. It follows from (20) that an agent's risk tolerance β^i completely determines the relative weights of the agent's consumption stream on the LVI and the per-capita endowment. An agent who is less risk tolerant than average ($\beta^i < \beta$) has $a(\beta^i) > 0$ and $b(\beta^i) < 1$, with a^i increasing and b^i decreasing in the difference $\beta - \beta^i$: thus, since $(1, \boldsymbol{\eta})$ is less variable than $\boldsymbol{w}$, an agent who is less risk tolerant than average has an equilibrium consumption stream which is less variable than the per-capita endowment process. The inequalities are reversed for an agent who is more risk tolerant than average, the increased variability being compensated by a higher average consumption (or more accurately by a higher expected discounted consumption).[7]

To interpret the equilibrium utility levels (ii) in Proposition 2, note that $\| \hat{\boldsymbol{\eta}} \| = \| \mathbf{1} - \boldsymbol{\eta} \|$ measures the distance of the LVI from the constant annuity $\mathbf{1}_+$: it gives an average measure (across all possible realizations) of the inability of the market to provide intertemporal consumption smoothing. Since $u^i_{AD} < 0$, and since the function $\dfrac{\| \hat{\boldsymbol{\eta}} \|^2}{\| \boldsymbol{\pi} \|^2 - \alpha^2 \| \hat{\boldsymbol{\eta}} \|^2}$ is an increasing function of $\| \hat{\boldsymbol{\eta}} \|^2$, the equilibrium utility of any agent who differs from the average agent ($\gamma^i \neq 0$) is a decreasing function of $\| \hat{\boldsymbol{\eta}} \|^2$. Thus the greater the inability of the financial markets to provide consumption smoothing, the smaller the welfare of the agents in equilibrium.

Ensuring non-negative consumption. The equilibria in Propositions 1 and 2 can be obtained by projecting each agent's vector $\alpha^i \mathbf{1} - \boldsymbol{\omega}^i$ onto the market subspace $\langle \boldsymbol{W}(\boldsymbol{q}) \rangle$, because two simplications are made in the definition of equilibrium. First, the non-negativity constraints on consumption are omitted, and second, the free disposal of income is not assumed, since there is a utility penalty for consumption in excess of α^i. These simplifications are warranted if the resulting equilibrium is such that the consumption streams are non-negative and do not exceed the ideal points of the agents. We can deduce from Proposition 1 the set of parameter values $(\alpha^i, \boldsymbol{\omega}^i)_{i=1}^I$ for which the equilibrium with complete markets satisfies these two requirements (namely $0 \leq \bar{x}^i_{AD}(\sigma_t) < \alpha^i$ for all $\sigma_t \in \boldsymbol{\Sigma}$ and all

[7] This can be shown by evaluating the ratio of an agent's expected discounted consumption to his wealth $E(\sum_{t=0}^{\infty} \delta^t x^i_t)/Y^i = [\![\mathbf{1}, \bar{\boldsymbol{x}}^i]\!]/[\![\boldsymbol{\pi}, \boldsymbol{\omega}^i]\!]$. This ratio is equal (after some calculations) to a fraction with a positive denominator and a numerator equal to: $\alpha(1 + \| \boldsymbol{\eta} \|^2) + \beta^i(\| \boldsymbol{w} \|^2 (1 + \| \boldsymbol{\eta} \|^2) - [\![\mathbf{1}, \boldsymbol{w}]\!]^2)$. Since $\boldsymbol{w}_+$ is in $\langle \boldsymbol{W}_+ \rangle$, $[\![\mathbf{1}, \boldsymbol{w}]\!] = [\![(1, \boldsymbol{\eta}), \boldsymbol{w}]\!]$ and, by the Cauchy-Schwartz inequality, the coefficient of β^i is positive. Thus, for a given wealth level, an agent's average discounted consumption increases with his risk tolerance.

$i = 1, \ldots, I$): thus when markets are complete ensuring non-negative and non-satiated consumption does not pose a problem (Proposition 3(i) below). When markets are incomplete we can obtain an approximate result: for the equilibrium consumption stream $\bar{\boldsymbol{x}}^i$ of agent i in Proposition 2 involves a component on $(1, \boldsymbol{\eta})$, and the LVI $\boldsymbol{\eta}$ is close to $\mathbf{1}_+$ in the $\ell_2^\Delta(\boldsymbol{\Sigma}_+)$ norm, but this does not imply that $\eta(\sigma_t)$ is uniformly close to 1 for all $\sigma_t \in \boldsymbol{\Sigma}_+$; that is, the fact that $\sum_{t=1}^{\infty} \delta^t \rho(\sigma_t)(1 - \eta(\sigma_t))^2$ is small does not prevent $\eta(\sigma_t)$ from being far from 1 for date-events σ_t for which $\rho(\sigma_t)$ is small and/or t is large. Thus bounds for $\eta(\sigma_t)$ can only be obtained for events of sufficiently high probability which do not lie too far in the future. In Proposition 3 (ii) below we obtain restrictions on the agents' characteristics which ensure that with probability greater than $\bar{\rho}$ the equilibrium consumption stream of each agent satisfies $0 \leq \bar{x}^i(\sigma_t) < \alpha^i$ for all dates $t \leq \bar{T}$: the greater $\bar{\rho}$ and $\bar{T}$, the smaller the admissible dispersion of agents' characteristics. To state this condition, note that all economies with the same average-agent characteristics $(\alpha, \boldsymbol{w})$ and the same security structure $\boldsymbol{D}$ have the same equilibrium security prices $\bar{\boldsymbol{q}}$, the same market subspace $\langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle$, and the same LVI $\boldsymbol{\eta}$. Let $\omega^i_{\text{inf}} = \inf \{\omega^i(\sigma_t), \sigma_t \in \boldsymbol{\Sigma}\}$ and $\omega^i_{\text{sup}} = \sup \{\omega^i(\sigma_t), \sigma_t \in \boldsymbol{\Sigma}\}$.

Proposition 3 (Restrictions on parameters). *Let $(\alpha, \boldsymbol{w}, \boldsymbol{D})$ be the average-agent characteristics and security structure of an economy with $\alpha > w_{\text{sup}}$. Let $\boldsymbol{\pi} = \alpha \mathbf{1} - \boldsymbol{w}$ and let $\boldsymbol{\eta}$ be the LVI. For $\bar{T} \geq 1$ and $\bar{\rho} \in (0, 1)$ define*

$$\kappa = \frac{\| \hat{\boldsymbol{\eta}} \|}{\sqrt{1 - \bar{\rho}}\, \delta^{\bar{T}/2}} \tag{21}$$

$$\underline{\beta}_\kappa = \frac{\alpha(1 + \kappa) - w_{\text{sup}}}{(\alpha - w_{\text{sup}})([\![\boldsymbol{\pi}, \mathbf{1}]\!] - \alpha \| \hat{\boldsymbol{\eta}} \|^2) - \kappa [\![\boldsymbol{\pi}, \boldsymbol{w}]\!]},$$

$$\bar{\beta}_\kappa = \frac{\alpha(1 + \kappa) - w_{\text{inf}}}{[\![\boldsymbol{\pi}, \boldsymbol{w}]\!](1 + \kappa) - ([\![\boldsymbol{\pi}, \mathbf{1}]\!] - \alpha \| \hat{\boldsymbol{\eta}} \|^2) w_{\text{inf}}}$$

(i) *Let (**A1**, **A2**) be satisfied (so that $\hat{\boldsymbol{\eta}} = 0$). If the agents' characteristics $(\alpha^i, \boldsymbol{\omega}^i)_{i=1}^I$ satisfy $\frac{1}{I}\sum_{i=1}^I \alpha^i = \alpha$, $\frac{1}{I}\sum_{i=1}^I \boldsymbol{\omega}^i = \boldsymbol{w}$ and if the implied risk tolerance β^i of each agent satisfies $\underline{\beta}_0 < \beta^i < \bar{\beta}_0$, then the equilibrium consumption in Proposition 1 satisfies $0 < \bar{x}^i_{AD}(\sigma_t) < \alpha^i$ for all $\sigma_t \in \boldsymbol{\Sigma}$, for $i = 1, \ldots, I$.*

(ii) *Let **A1** be satisfied and let $\hat{\boldsymbol{\eta}} \neq 0$. If $(\bar{T}, \bar{\rho})$ are chosen so that $\alpha(1-\kappa) > \omega_{\text{sup}}$ and if $(\alpha^i, \boldsymbol{\omega}^i)_{i=1}^I$ satisfy $\frac{1}{I}\sum_{i=1}^I \alpha^i = \alpha$, $\frac{1}{I}\sum_{i=1}^I \boldsymbol{\omega}^i = \boldsymbol{w}$, $\underline{\beta}_\kappa < \beta^i < \bar{\beta}_\kappa$ for $i = 1, \ldots, I$, then with probability greater than $\bar{\rho}$ the equilibrium consumption in Proposition 2 satisfies $0 < \bar{x}^i(\sigma_t) < \alpha^i$ for all σ_t with $t \leq \bar{T}$, for $i = 1, \ldots, I$.*

Proof. (See appendix).

Note that $\underline{\beta}_0 = 1/[\![\boldsymbol{\pi}, \mathbf{1}]\!]$, so that with complete markets the condition $\alpha^i > \omega^i_{\text{sup}}$, which implies $\alpha^i [\![\boldsymbol{\pi}, \mathbf{1}]\!] > [\![\boldsymbol{\pi}, \boldsymbol{\omega}^i]\!]$ is sufficient to ensure that $\bar{x}^i(\sigma_t) < \alpha^i$ for all σ_t. $\bar{\beta}_0$ depends on ω_{inf} and is equal to the risk tolerance of the average agent $\alpha/[\![\boldsymbol{\pi}, \boldsymbol{w}]\!]$ if $w_{\text{inf}} = 0$: this is natural, since an agent who is more risk tolerant than average takes more risk and has lower than average consumption in unfavorable states. If the average (and aggregate) output is zero in the worst date-event, then any agent who is more risk-tolerant than average will have negative consumption:

in this case, negative consumption can be avoided only if agents have the same risk tolerance. In typical economies $\omega_{\inf} > 0$, so that there is room for differences in the risk tolerance of agents without violating the non-negativity constraints. The allowable interval $[\underline{\beta}_\kappa, \overline{\beta}_\kappa]$ for agents' risk tolerance with incomplete markets is smaller than the interval with complete markets $[\underline{\beta}_0, \overline{\beta}_0]$: furthermore, the greater the incompleteness of markets as measured by $\| \hat{\boldsymbol{\eta}} \|^2$ (i.e. the lack of consumption smoothing) and the larger κ (i.e. the probability and/or the length of time over which consumption is to be positive), the smaller the allowable interval.

(ii) essentially asserts that with high probability the consumption of all agents is positive on the first $\overline{T}$ periods: this implies that the equilibrium is a good approximation of a "true" equilibrium in which default occurs with small probability and in which social arrangements exist (bankruptcy) for sharing the losses incurred by default among the lenders. On the side of borrowers there are utility penalties for bankruptcy, since the utility loss $-\frac{1}{2}(\alpha^i - x^i(\sigma_t))^2$ increases if planned consumption is negative. On the side of lenders, the payoffs of the securities should be corrected by the effect of default for those date-events at which some borrowers have negative consumption. If these events have small discounted probability $\delta^t \rho(\sigma_t)$, then the prices of the securities when appropriately corrected for default will be close the prices $\bar{\boldsymbol{q}}$ given by (10), and the linear map $\widetilde{\boldsymbol{W}(\tilde{\boldsymbol{q}})}$ in which the prices and payoffs of the securities are appropriately corrected will be close to $\boldsymbol{W}(\bar{\boldsymbol{q}})$ (in the $\ell_2^\Delta(\boldsymbol{\Sigma})$ norm). Then by continuity, the projection $\tilde{\boldsymbol{\tau}}^i$ of $\alpha^i \mathbf{1} - \boldsymbol{\omega}^i$ will be close to the net trade $\bar{\boldsymbol{\tau}}^i$ given in Proposition 2: in short, since bankruptcy only occurs for distant and/or low probability events, agents plans would not be much affected if they in fact took into account the true consequences of default by borrowers. The trading strategies of agents are shaped by the most likely scenarios on the significant part $[0, \overline{T}]$ of the future: much less weight is placed on events in the distant future $(\overline{T}, \infty)$, and on those of low probability within the immediate horizon $[0, \overline{T}]$, even if they lead to bankruptcy. The projection technique thus leads to a concept of equilibrium which approximates an equilibrium with bankruptcy, but in which the likelihood and consequences of bankruptcy for agents is small.

Measuring loss from imperfect consumption smoothing. An Arrow–Debreu equilibrium provides a natural measure of the maximum gain that each agent can obtain from trade: if $u_0^i = u^i(\boldsymbol{\omega}^i)$ denotes agent i's utility with no trade, then $\bar{u}_{AD}^i - u_0^i$ measures the agent's maximum gain from trade. These gains will be achieved if the market structure is complete or well-adapted to the characteristics of the economy (assumptions **A1** and **A2**). If assumption **A2** is not satisfied, the market structure is imperfect and

$$\phi^i = \frac{u_{AD}^i - u^i(\bar{\boldsymbol{x}}^i)}{u_{AD}^i - u_0^i}$$

measures the proportion of the potential gains from trade for agent i which are left unexploited at the equilibrium, or more briefly, the *unexploited gains from* *trade* for agent i. Equivalently $1 - \phi^i$ measures the proportion of the potential gains from trade captured by the given market structure. Under assumption **A1** the gains from trade for the agents in the economy come from two sources: (i) the gains from diversifying (sharing) their individual risks; (ii) the gains from using the least risky income stream $\boldsymbol{\eta}_+$ to adjust their share of the aggregate risk to their risk characteristic measured by $\beta^i = \alpha^i / \boldsymbol{\Pi}\boldsymbol{\omega}^{\boldsymbol{i}}$. It is useful to think of the equilibrium as being reached in two steps: $\boldsymbol{\omega}^{\boldsymbol{i}} \longrightarrow \theta^i \boldsymbol{w} \longrightarrow \overline{\boldsymbol{x}}^i$. The first step can be achieved by trading on the financial (equity) markets, since $\boldsymbol{\omega}^i_+ \in \langle \boldsymbol{W}_+ \rangle$ for all i implies $\boldsymbol{w}_+ = \frac{1}{I}\sum_{i=1}^{I} \boldsymbol{\omega}^{\boldsymbol{i}}{}_+ \in \langle \boldsymbol{W}_+ \rangle$; furthermore $\theta^i \boldsymbol{w}$ satisfies the agent's budget constraints if θ^i is chosen so that the present value of the agent's income is unchanged: $\boldsymbol{\Pi}\boldsymbol{\omega}^{\boldsymbol{i}} = \boldsymbol{\Pi}(\theta^i \boldsymbol{w})$. This step generates the agent's *individual risk-sharing* (or diversification) *gain*. In the second step, agent i trades from $\theta^i \boldsymbol{w}$ to $\overline{\boldsymbol{x}}^i = a^i(1, \boldsymbol{\eta}) + b^i \boldsymbol{w}$, either reducing or increasing the variability of his consumption stream relative to the variability of per-capita output $\boldsymbol{w}$, depending on his risk tolerance β^i. This second step generates what we may call the agent's *aggregate risk-sharing gain*: note that there are gains to trade at this second stage only if there is aggregate risk ($\boldsymbol{w}_+$ is not collinear to $\mathbf{1}_+$), for otherwise all agents would already have achieved a riskless income stream by sharing their individual risks and there would be no further gains from trade.

Under assumption **A1** the gains from sharing individual risks are fully captured; the market imperfection ($\mathbf{1}_+ \notin \langle \boldsymbol{W}_+ \rangle$) only affects the agents' ability to share aggregate risk. To measure the impact of this imperfection it is natural to measure the unexploited gains in the second step, assuming that there is aggregate risk and that agents' endowments are collinear. The next proposition shows that with these assumptions ϕ^i is the same for all agents and leads to a natural measure of the loss due to this imperfection.

Proposition 4 (Unexploited gains from trade). *In an economy in which there is aggregate risk, if individual risks are diversifiable (***A1***) and if agents' endowments are collinear ($\boldsymbol{\omega}^i = \theta^i \boldsymbol{w},\ i = 1, \ldots, I$), then the unexploited gains from trade arising from the absence of perfect consumption smoothing are the same for all agents, and are given by the increasing function*

$$\phi(\| \hat{\boldsymbol{\eta}} \|) = k(\boldsymbol{w}) \frac{\| \hat{\boldsymbol{\eta}} \|^2}{\| \boldsymbol{\pi} \|^2 - \alpha^2 \| \hat{\boldsymbol{\eta}} \|^2} \quad \textit{where} \quad k(\boldsymbol{w}) = \frac{[\![\boldsymbol{\pi}, \boldsymbol{w}]\!]^2}{\| \mathbf{1} \|^2 \| \boldsymbol{w} \|^2 - [\![\mathbf{1}, \boldsymbol{w}]\!]^2}$$

Proof. (See appendix)

In Section 4 the measure ϕ will be used to evaluate the relative benefits of trading with different market structures.

3 Calculating the LVI in a Markov economy

Proposition 2 gives an expression for the equilibrium of the economy $\mathscr{E}(\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{D})$ in terms of the LVI $\boldsymbol{\eta}$. In this section we show how $\boldsymbol{\eta}$ can be calculated when the exogenous shock process has a recursive Markov structure. In so doing we show that in an economy with quadratic preferences equilibria can be calculated with other assumptions on the stochastic processes than the linear autoregressive assumption used in the macroeconomic literature and in particular by Hansen–Sargent [14]: a discrete state space Markovian assumption works equally well.

To model the exogenous shocks as a Markov process, let

$$\boldsymbol{K} = \left[K(s, s')\right]_{s,s'=1,\ldots,S}$$

denote a transition matrix, where $K(s, s') > 0$ is the probability that shock s' occurs in the next period, given that the current shock is s. The probability ρ defined in Section 2 is thus assumed to satisfy

$$\rho\Big(\sigma_{t+1} = (s_0, \ldots, s_{t+1}) \,|\, \sigma_t = (s_0, \ldots, s_t)\Big) = K(s_t, s_{t+1})$$

so that the transition probability to state s_{t+1} at date $t+1$ depends only on the current state s_t. The initial shock $\bar{s}_0$ is taken as exogenously given. Agents' endowments, the dividends on securities at date t, and the coefficients $f(\sigma_t)$ in (2) are assumed to depend only on the current state

$$\omega^i(\bar{s}_0, \ldots, s_t) = \omega^i(s_t), \quad d^j(\bar{s}_0, \ldots, s_t) = d^j(s_t) \quad f^j(\bar{s}_0, \ldots, s_t) = f^j(s_t)$$

Under these assumptions, it follows from (10) that the prices of the securities at date t in the infinite horizon economy depend only on the current state. Since in a Markov economy the basic variables depend only on the current state, we can adopt a simpler notation. Let ω^i_s denote agent i's endowment if the current state is s and let

$$\omega^i = (\omega^i_1, \ldots, \omega^i_S)$$

denote the associated S-vector: the vector $\omega^i \in \mathbb{R}^S$ is to be distinguished from the vector $\boldsymbol{\omega}^i \in \mathbb{R}^{\boldsymbol{\Sigma}}$ (written in boldface) which denotes the agent's stochastic endowment over the whole event tree. Similarly let

$$w = (w_1, \ldots, w_S), \;\; q^j = (q^j_1, \ldots, q^j_S), \;\; d^j = (d^j_1, \ldots, d^j_S), \quad f^j = (f^j_1, \ldots, f^j_S)$$

be the vectors denoting the S-possible values of per-capita output, and the price and dividend of each security $j = 1, \ldots, J$. The matrix of possible payoffs of the securities in the different states is thus an $S \times J$ matrix

$$R = \begin{bmatrix} d^1_1 + f^1_1 q^1_1 & \ldots & d^J_1 + f^J_1 q^J_1 \\ \vdots & & \vdots \\ d^1_S + f^1_S q^1_S & \ldots & d^J_S + f^J_S q^J_S \end{bmatrix} \tag{22}$$

R^j denotes the j^{th} *column* of R (the vector of payoffs of security j across the states), while R_s denotes the s^{th} *row* of R (payoffs of the J securities in state s). In the same way, q^j denotes the vector of prices of security j across the states (and is considered as a column vector), while $q_s = (q^1_s, \ldots, q^J_s)$ denotes the prices of the J securities in state s (and is considered as a row vector).

By (9) the security prices are solutions of the stationary pricing equations

$$(\alpha - w_s)\bar{q}_s = \delta \sum_{s'=1}^{S} K(s,s')(\alpha - w_{s'})R_{s'}(\bar{q}), \quad s = 1., \ldots, S \tag{23}$$

Let $\bar{R}$ denote the matrix in (22) evaluated with the equilibrium prices $\bar{q}$ satisfying (23). We assume that at these equilibrium prices there are no redundant securities, an assumption which is satisfied generically if $J < S$ and the dividend vectors of the securities are linearly independent.

A3 (*No Redundant Securities*): rank($\bar{R}$)=J

To calculate $\boldsymbol{\eta}$ we follow the strategy used in the proof of Proposition 2: first calculate $\boldsymbol{\eta}_T = \text{Proj}_{\langle \boldsymbol{W}_+ \rangle_T}(\mathbf{1}_+)$, then take the limit when $T \to \infty$. The projection $\boldsymbol{\eta}_T$ of $\mathbf{1}_+$ onto $\langle \boldsymbol{W}(\bar{\boldsymbol{q}})_+ \rangle_T$ (defined in the proof of Proposition 2) is obtained by finding the portfolio strategy $\boldsymbol{z}$ up to date $T-1$ which solves the problem

$$\min_{\boldsymbol{z} \in \mathbb{R}^{JN_{T-1}}} \sum_{\sigma_t \in \boldsymbol{\Sigma}_T \setminus \sigma_0} \delta^t \rho(\sigma_t) \left(1 - \bar{R}_{s_t} z(\sigma_t^-) + \bar{q}_{s_t} z(\sigma_t)\right)^2, \quad z(\sigma_T) = 0 \tag{24}$$

where $N_{T-1} = 1 + S + \ldots + S^{T-1}$ is the number of date-events at which the components of $\boldsymbol{z}$ are chosen and where $\rho(\sigma_t) = K(\bar{s}_0, s_1) \ldots K(s_{t-1}, s_t)$ for $\sigma_t = (\bar{s}_0, \ldots, s_t)$. If $\boldsymbol{z}^*$ is the solution of (24), extended by setting $z^*(\sigma_t) = 0$ if $t > T$, then the income stream $\boldsymbol{\eta}_T = \boldsymbol{W}(\bar{\boldsymbol{q}})_+ \boldsymbol{z}^*$ is the projection of $\mathbf{1}_+$ onto $\langle \boldsymbol{W}(\bar{\boldsymbol{q}})_+ \rangle_T$, and the minimum value in (24) is equal to $\| \; \mathbf{1}_{+T} - \boldsymbol{\eta}_T \|^2 = \| \; \hat{\boldsymbol{\eta}}_{\perp} \|^2$ (using the notation introduced in the proof of Proposition 2). Even though $\bar{R}_{s_t}$ and $\bar{q}_{s_t}$ only depend on the exogenous state s_t at date t, the optimal choice $z(\sigma_t)$ depends on the portfolio $z(\sigma_t^-)$ chosen at the previous date $t-1$. Thus the optimal portfolio at date t depends on three variables: the portfolio inherited from the previous period, the current state and the number of periods to go $(T-t)$. Let $V^T(z,s)$ denote the *value function*

$$V^T(z,s) \;=\; \min_{z \in \mathbb{R}^{JN_{T-1}}} \left\{ \sum_{\sigma_t \in \boldsymbol{\Sigma}_T} \delta^t \rho(\sigma_t) \left(1 - \bar{R}_{s_t} z(\sigma_t^-) + \bar{q}_{s_t} z(\sigma_t)\right)^2 \;\middle|\; z(\sigma_0^-) = z, \sigma_0 = s, z(\sigma_T) = 0 \right\} \tag{25}$$

which gives the minimum distance from $\mathbf{1}$ that can be achieved between date 0 and date T starting in state s with an initial portfolio z. $V^T(z,s)$ is a solution of the Bellman (functional) equation

$$V^T(z,s) = \min_{z' \in \mathbb{R}^J} \left\{ (1 - \bar{R}_s z + \bar{q}_s z')^2 + \delta \sum_{s'=1}^{S} K(s,s') V^{T-1}(z',s') \right\} \tag{26}$$

with boundary condition $V^0(z,s) = (1 - \bar{R}_s z)^2$. The solution to the date 0 unconstrained problem (24) can be derived from $V^{T-1}(z_0, s)$ by optimizing over the choice of the initial portfolio z_0

$$\| \hat{\boldsymbol{\eta}}_T \|^2 = \min_{z_0 \in \mathbb{R}^J} \delta \sum_{s=1}^{S} K(\bar{s}_0, s) V^{T-1}(z_0, s) \tag{27}$$

In view of (27), to find $\| \hat{\boldsymbol{\eta}}_T \|^2$ it suffices to calculate the value function which solves (26).

Solution of Bellman equation. A solution $V^T(\cdot, s)$ of (26) must be a quadratic function of z for each $s = 1, \ldots, S$: since $V^0(\cdot, s')$ is a quadratic function for all $s' = 1, \ldots, S$ and since the the FOC for the minimizing z' in the computation of $V^1(\cdot, s)$ gives a linear relation between z' and z, substituting the resulting expression for z' leads to a quadratic expression for $V^1(\cdot, s)$. By backward induction this property is true for $V^T(\cdot, s)$.

The next proposition gives the recurrence equations which are satisfied by the coefficients of the quadratic expression for $V^T(\cdot, s)$ and proves that they converge when $T \to \infty$. We use the following notation: if $x : \boldsymbol{S} \to \mathbb{R}^m$ or $x : \boldsymbol{S} \to \mathbb{R}^{nm}$ is a real-valued, vector-valued or matrix-valued function on $\boldsymbol{S}$ i.e. a random scalar, vector or matrix, let $E_s x$ denote the conditional expectation of x given the current state s

$$E_s x = \sum_{s'=1}^{S} K(s, s') x_{s'}$$

Proposition 5 (Value function for the LVI). (i) *For $T = 0, 1, 2, \ldots$ the finite horizon value function $V^T : \mathbb{R}^J \times \boldsymbol{S} \longrightarrow \mathbb{R}$ which is a solution of the Bellman equation (26) is given by the S linear-quadratic functions*

$$V^T(z, s) = a_s^T + 2 b_s^T \bar{R}_s z \; + \; c_s^T z^\intercal \bar{R}_s^\intercal \bar{R}_s z, \quad s = 1, \ldots, S \tag{28}$$

where the coefficients

$$(a^T, b^T, c^T) \; = \; (a_s^T, b_s^T, c_s^T, \; s = 1, \ldots, S) \in \mathbb{R}^{3,S}$$

are solutions of the system of difference equations, for $s = 1, \ldots, S$

$$\begin{aligned} a_s^T &= 1 + \delta E_s a^{T-1} - (\bar{q}_s + \delta E_s b^{T-1} \bar{R})[G_s^{T-1}]^{-1} (\bar{q}_s + \delta E_s b^{T-1} \bar{R})^\intercal \\ b_s^T &= -1 + \bar{q}_s [G_s^{T-1}]^{-1} (\bar{q}_s + \delta E_s b^{T-1} \bar{R})^\intercal \\ c_s^T &= 1 - \bar{q}_s [G_s^{T-1}]^{-1} \bar{q}_s^\intercal \end{aligned} \tag{29}$$

with initial condition $(a_s^0, b_s^0, c_s^0) = (1, -1, 1)$ *and where*

$$G_s^{T-1} = \bar{q}_s^\intercal \bar{q}_s + \delta E_s c^{T-1} \bar{R}^\intercal \bar{R} \tag{30}$$

$$E_s b \bar{R} = \sum_{s=1}^{S} K(s, s') b_{s'} \bar{R}_{s'}, \quad E_s c^{T-1} \bar{R}^\intercal \bar{R} = \sum_{s'=1}^{S} K(s, s') c_{s'}^{T-1} \bar{R}_{s'}^\intercal \bar{R}_{s'}$$

(ii) *The coefficients converge*

$$(a^T, b^T, c^T) \; \longrightarrow \; (a, b, c,) \text{ as } T \longrightarrow \infty$$

and the resulting value function $V(z,s)$, defined by (28) with the coefficients (a,b,c), satisfies

$$V(z,s) \;=\; \min_{z'\in\mathbb{R}^J}\left\{(1-\bar{R}_s z+\bar{q}_s z')^2+\delta\sum_{s'=1}^{S}K(s,s')V(z',s')\right\} \tag{31}$$

Proof. (i) We prove by induction that the value function has the form given by (28) - (30). If $T=0$, there is no choice of portfolio since there is no period which follows and $V^0(z,s)=(1-\bar{R}_s z)^2$ so that $(a_s^0,b_s^0,c_s^0)=(1,-1,1)$ for all $s=1,\ldots,S$. Suppose V^{T-1} has the form (28); let us show that V^T has the same form, with the coefficients (a^T,b^T,c^T) given by (29)-(30) as functions of $(a^{T-1},b^{T-1},c^{T-1})$. The first order conditions for the minimizing problem

$$V^T(z,s)=\min_{z'\in\mathbb{R}^J}\left\{(1-\bar{R}_s z+\bar{q}_s z')^2+\delta E_s(a^{T-1}+2b^{T-1}\bar{R}z+c^{T-1}z'^{\top}\bar{R}^{\top}\bar{R}z')\right\} \tag{32}$$

are given by the system of linear equations

$$(1-\bar{R}_s z+\bar{q}_s z')\bar{q}_s^{\top}+\delta E_s b^{T-1}\bar{R}^{\top}+\delta E_s c^{T-1}\bar{R}^{\top}\bar{R}z'=0$$

If the matrix G_s^{T-1} defined by (30) is invertible, then the optimal portfolio is given by

$$z'=-[G_s^{T-1}]^{-1}[(1-\bar{R}_s z)\bar{q}_s^{\top}+\delta E_s b^{T-1}\bar{R}^{\top}] \tag{33}$$

Substituting (33) into (32) shows that the new coefficients (a^T,b^T,c^T) are given by the system of equations (29).

To prove that G_s^{T-1} is invertible for all $T\geq 1$, it suffices to prove by induction that $c_s^T>0$ for all $T\geq 0$ and all s. For if $c_s^T>0$, then

$$z^{\top}G_s^T z=(\bar{q}_s z)^2+\delta\sum_{s'=1}^{S}K(s,s')c_{s'}^T(\bar{R}_{s'}z)^2$$

is non-negative and equal to zero if and only if $q_s z=0$ and $\bar{R}_{s'}z=0\ \forall\ s'=1,\ldots,S$. (Recall that $K(s,s')>0$ for all s, s'). Since, by **A3**, rank $\bar{R}=J$, $\bar{R}z=0$ is possible only if $z=0$. Thus G_s^T is positive definite.

To prove by induction that $c^T\gg 0$ for all $T\geq 0$, first note that $c^0=(1,\ldots,1)\gg 0$. Then consider for all $s=1,\ldots,S$ and all $T\geq 0$ the functions $g_s^T:\ (0,+\infty)\to\mathbb{R}^{J\times J}$ and $\phi_s^T:\ (0,+\infty)\to\mathbb{R}$ defined by

$$g_s^T(\theta)=\bar{q}_s^{\top}\bar{q}_s+\delta\theta E_s c^T\bar{R}^{\top}\bar{R},\quad \phi_s^T(\theta)=1-\bar{q}_s[g_s^T(\theta)]^{-1}\bar{q}_s$$

By the same reasoning as above, if $c^T\gg 0$ and $\theta>0$, $g_s^T(\theta)$ is positive definite and $\phi_s^T(\theta)$ is well defined. By (29), $c^{T+1}=\phi_s^T(1)$. Thus the property will be proved by induction if we show that $c^T\gg 0$ implies $\phi_s^T(1)>0$ for all $s=1,\ldots,S$. If $c^T\gg 0$, then the derivative

$$\frac{d\phi_s^T}{d\theta}(\theta)=\bar{q}_s[g_s^T(\theta)]^{-1}[\delta E_s c^T\bar{R}^{\top}\bar{R}][g_s^T(\theta)]^{-1}\bar{q}_s^{\top}$$

is positive for $\theta > 0$. Thus $\phi_s^T(1) > 0$ can be deduced from the property

$$\lim_{\theta \to 0^+} \phi_s(\theta) = 0$$

which is a consequence of the following Lemma, proved in the Appendix.

Lemma 6. *If $x \in \mathbb{R}^J, x \neq 0$ and A is a $J \times J$ positive definite symmetric matrix, then*

$$\lim_{\theta \to 0^+} x^\top [xx^\top + \theta A]^{-1} x = 1$$

.

To prove (ii), consider the function

$$V(z,s) = \inf \left\{ \sum_{\sigma_t \in \boldsymbol{\Sigma}} \delta^t \rho(\sigma_t) \left(1 - \bar{R}_{s_t} z(\sigma_t^-) + \bar{q}_{s_t} z(\sigma_t)\right)^2 \;\middle|\; \begin{array}{l} z(\sigma_0^-) = z, \sigma_0 = s, z(\sigma_t) \neq 0 \text{ for} \\ \text{at most a finite number of nodes} \end{array} \right\} \tag{34}$$

(34) is the greatest lower bound of the (square of the) distances to **1** which can be attained by trading at most at a finite number of nodes, when starting in state s with the inherited portfolio z. Since all trading strategies which stop at date T are admissible for V

$$V(z,s) \leqq V^T(z,s) + \frac{\delta^{T+1}}{1-\delta}$$

Since the constraint that the portfolio be zero from date T on is less severe than the equivalent constraint from date $T-1$ on, the sequence $T \mapsto V^T(z,s) + \frac{\delta^{T+1}}{1-\delta}$ is decreasing (for any fixed (z,s)). Since the sequence is bounded below, it converges and the limit is $V(z,s)$. For suppose not, $V(z,s) < \lim_{T\to\infty} V^T(z,s)$, then there would be a trading strategy satisfying the constraints of (34) – which can be considered as a trading strategy admissible for V^T for some T – which gives a distance to **1** smaller than $V^T(z,s) + \frac{\delta^{T+1}}{1-\delta}$, contradicting the definition of V^T. Finally, to show that the coefficients (a^T, b^T, c^T) converge, note that by (28), $V^T(0,s) = a_s^T$ and since V^T converges, $a_s^T \to a$ as $T \to \infty$. Similarly for $\bar{R}_s z = 1$, $2b_s^T + c_s^T \to \alpha_s$ and for $\bar{R} z = -1, -2b_s^T + c_s^T \to \beta_s \Longrightarrow c_s^T \to \alpha_s + \beta_s = c_s$, $b_s^T \to \alpha_s - \beta_s = b_s$ as $T \to \infty$. It is easy to check that the limiting function $V(z,s)$ defined by (28) with $(a^T, b^T, c^T) = (a,b,c)$ satisfies (31). □

The function V defines the minimum distance from **1** to $\langle \boldsymbol{W}(\bar{\boldsymbol{q}}) \rangle$ which can be achieved if there is an inherited portfolio z at date 0. When $\mathbf{1}_+$ is projected onto $\langle \boldsymbol{W}(\bar{\boldsymbol{q}})_+ \rangle$, there is no inherited portfolio – the date 0 portfolio can be chosen freely. Thus $\| \hat{\boldsymbol{\eta}} \|$ is given by the analog of (27) for the infinite horizon problem

$$\| \hat{\boldsymbol{\eta}} \|^2 = \min_{z_0 \in \mathbb{R}^J} \delta \sum_{s=1}^{S} K(\bar{s}_0, s) V(z_0, s) \tag{35}$$

which is consistent with the result $\hat{\boldsymbol{\eta}} = \lim_{T\to\infty} \hat{\boldsymbol{\eta}}_T$, established in the proof of Proposition 2. Taking the first order conditions for the minimization problem (35) leads to the the optimal date 0 portfolio

$$z_0 = -\left[E_{\bar{s}_0}(c\overline{R}^\intercal \overline{R})\right]^{-1} E_{\bar{s}_0}(b\overline{R}^\intercal) \tag{36}$$

Substituting (36) into (35) gives the following corollary of Proposition 5.

Corollary 7. $\| \hat{\boldsymbol{\eta}} \|^2 = \delta \left(E_{\bar{s}_0} a - E_{\bar{s}_0}(b\overline{R}) \left[E_{\bar{s}_0}(c\overline{R}^\intercal \overline{R})\right]^{-1} E_{\bar{s}_0}(b\overline{R}^\intercal) \right)$ *where* (a, b, c) *are given by Proposition 5(ii).*

4 Calibrating infinite-horizon CAPM

In this section we report briefly on the equilibrium of the equity and bond markets generated by a calibrated version of the infinite-horizon CAPM model fitted to postwar data for the US economy. Agents' individual risks are assumed to arise from ownership shares of firms whose equity contracts are traded on the stock market: to simplify the calibration we assume that the gains from diversifying individual risks have already been realized by trading equity contracts so that each agent begins with an ownership share of aggregate output. The focus of the analysis is then on how individual agents subsequently realign their portfolios to hold their desired optimal proportions of equity and bonds.

To make growth of output compatible with quadratic preferences, we assume that aggregate output satisfies a trend growth process with growth rate g and stochastic deviation from trend γ, where γ is a three-state Markov process $(\gamma_H, \gamma_M, \gamma_L)$ with associated transition matrix K_y. The inflation rate i also satisfies a three-state Markov process (i_H, i_M, i_L) with transition matrix K_i. The economy is thus driven by a Markov process with nine states

$$\boldsymbol{S} = \{s_1, \ldots, s_9\} = \{(\gamma_H, i_H), (\gamma_H, i_M), (\gamma_H, i_L), (\gamma_M, i_H), \ldots, (\gamma_L, i_L)\}$$

If $\sigma_t = (s_0, \ldots, s_t)$ denotes the history up to date t and if $Y(\sigma_t)$ and $P(\sigma_t)$ denote the aggregate output and the money (dollar) price of one unit of the good respectively, then

$$Y(\sigma_t) = Y_0\, g^t \gamma_{s_t}, \quad P(\sigma_t) = P(\sigma_t^-)(1 + i_{s_t}) \tag{37}$$

Again to simplify the calibration, we assume that the output and inflation processes are independent: thus the transition matrix K for the combined output-inflation process defining the state of the economy is given by $K\,((\gamma_k, i_\ell), (\gamma_{k'}, i_{\ell'})) = K_y(\gamma_k, \gamma_{k'}) K_i(i_\ell, i_{\ell'})$, where $k, k', \ell, \ell' \in \{H, M, L\}$. The values of g, γ, i, K_y and K_i are obtained by calibrating to US data for the period 1959-1995 (Economic Report of the President, 1996):

$$g = 1.023 \quad \gamma = \begin{bmatrix} 1.04 \\ 1 \\ 0.96 \end{bmatrix} \quad K_y = \begin{bmatrix} 0.786 & 0.214 & 0 \\ 0.256 & 0.54 & 0.204 \\ 0 & 0.182 & 0.818 \end{bmatrix}$$

$$i = \begin{bmatrix} 0.1 \\ 0.05 \\ 0.02 \end{bmatrix} \quad K_i = \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.175 & 0.696 & 0.129 \\ 0 & 0.375 & 0.625 \end{bmatrix} \tag{38}$$

Since the scale of the economy grows by the factor g^t we adjust agents' utility functions to reflect this change of scale[8]

$$u^i(X^i) = E\left[\sum_{t=0}^{\infty} \delta^t (\alpha^i g^t - X_t^i)^2\right]$$

with $\delta g^2 < 1$ (to ensure convergence of the sum), so that each agent's ideal consumption level $\alpha^i g^t$ grows at the growth rate of the economy.

The financial markets consist of equity and bonds. There is one equity contract whose dividend stream is aggregate output, and each agent' s initial endowment consists of a share of this contract: $\boldsymbol{\omega}^i = \theta^i \boldsymbol{Y}$, $i = 1, \ldots, I$. There are two *nominal* bonds representing the two extreme maturities, a *short-lived* bond delivering one unit of money in the period after it is issued, and a *consol* promising delivery of one unit of money for ever. When the budget equations (3) are written out for this security structure, it is clear that the model (as it stands) is not a special case of the model introduced in Sections 2 and 3: for the dividends of the bonds are expressed in nominal rather than real terms, the dividends of the equity grow over time, and the rate of change of the bonds' dividends rather that the dividends themselves satisfies the Markov assumption. Under a change of variable explained in Appendix B the model reverts to being a special case of the model studied in Sections 2 and 3 in which the dividends of the securities and the agents' endowments are real, bounded and Markov. This change of variable amounts to expressing the dividends of all securities in real terms (purchasing power of money), factoring out growth, and shrinking the capital values of the long-term bond by the factor[9] $f_s = 1/g(1 + i_s)$.

To determine the stationary prices of the securities using equations (23), the parameters δ and $\alpha = \sum_{i=1}^{I} \alpha^i / I$ need to be specified: these parameters are chosen so as to generate a "reasonable" real interest rate and equity premium, with the additional proviso that the nominal interest rate is positive even in the high-output low-inflation state. This leads to the choice $\delta = 0.92$, $\widetilde{\delta} = \delta g^2 = 0.963$, $\alpha = 1.13 w_0$ (with $w_0 = Y_0/I$). When growth is factored out, the consumption (equal to output) of the average agent varies between $1.04 w_0$ and $.96 w_0$: the above value of α implies a coefficient of relative risk aversion between 5.6 and 11.5. The

[8] We use capital letters to denote variables in the model with growth: a change of variable is needed to map the model into the framework of Sections 2 and 3, and lower-case notation is used to denote the transformed variables.

[9] The factor f introduced in equation (2) of Section 2.

(implicit) real interest rate is then -3% when γ is high, 8% when γ is medium and 11% when γ is low, giving an average of 5.1%. The equity premium, equal to the expected real rate of return on equity minus the real interest rate, is 2% when γ is high, 4.5% when γ is medium and 1% when γ is low, giving an average of 2.4%. As in the model of Mehra and Prescott [29] (with power utility) and in many later models (see Kocherlakota [19]) the real interest rate is somewhat high and the equity premium low relative to US historical levels. One of the realistic features of the calibrated model is the behavior of the annual rates of return of the securities: because of the variability of the real interest rate, the prices of both equity and bonds vary substantially despite the mild variability of their dividends. The (annual) real rate of return on the short-term bond varies between -7% and 14.7%, the real rate of return on equity varies between -24% and 47.3%, and the real rate of return on the consol varies between -27% and 49.7%, duplicating at least approximately the historical data reported in Ibbotson and Sinquefield [18].

Agents' portfolios of equity and bonds. Since one advantage of the infinite-horizon CAPM model is that it yields explicit solutions to agents' equilibrium consumption streams, trading strategies and welfare, we focus attention on this aspect of the model and study the predicted behavior of agents on the equity and bond markets, as well as the welfare consequences of inflation. Since equilibria are readily calculated for different security structures, in addition to the calibrated economy described above, we report the outcomes for several alternative security structures with different scenarios for inflation. This permits us to explore the robustness of the rather surprising result that, in this model, the bond market is of marginal importance. For a reasonable interval of parameter values the magnitudes of agents' trades on the bond market relative to the equity market are very small: indeed closing down the bond market altogether would cause only small welfare losses.

The four market structures that we consider are: (A) equity only, (B) equity and consol, (C) equity and short bond, (D) equity, consol and short bond, and the inflation scenarios consist of increasing levels of constant inflation (- 2.2, 0, 6, 12 and 24 percent) and increasing levels of variable inflation [10] (US inflation i in (38), $2i$ and $4i$). The average value[11] of ϕ (the measure of welfare loss introduced in Proposition 4) is given in Table 1. The average proportion[12] of the total value of the LVI portfolio invested in each security is shown in Table 2.

The market structure A where equity is the only financial instrument is used as the reference case. The payoff of equity is independent of inflation and depends only on the real shocks. The LVI strategy z_t defined by (33) and (36) depends

[10] In each case the persistence of inflation summarized by the matrix K_i in (38) is left unchanged.

[11] Since the norm $\|\quad\|$ and inner product $[\![\ ,\ ]\!]$ appearing in the expression for ϕ in Proposition 4 depend on the state s in which the economy starts out at date 0, we use the steady-state probabilities of the matrix $K^* = \lim_{T\to\infty} K^T$ to evaluate the average value of ϕ.

[12] The average proportions of the securities are obtained as a time average over a one thousand period realization of the Markov process defined by (38); the actual portfolio proportions lie within the bounds $\pm$ 1% to $\pm$ 2% of these mean values.

Table 1. Percentage unexploited gains from trade

	Constant inflation					Variable inflation			
	-2.2%	0%	6%	12%	24%	US	2×US	4×US	
A	21.3	21.3	21.3	21.3	21.3	21.3	21.3	21.3	equity only
B	0	0.06	0.07	0.08	0.1	12.2	14.4	15.9	equity-consol
C	-	-	0.3	0.3	0.3	0.8	2	4.7	equity-bond
D	-	-	0	0	0	0.35	0.4	0.5	equity-consol-bond

Table 2. Portfolio proportions for least variable income stream

	Constant inflation					Variable inflation			
	-2.2%	0%	6%	12%	24%	US	2×US	4×US	
A	100	100	100	100	100	100	100	100	equity
B	0	79.6	92	94.3	95.8	97.6	99	99	equity
	100	20.4	8	5.7	4.2	2.4	1	1	consol
C	-	-	98	98	98	98	98	99	equity
	-	-	2	2	2	2	2	1	bond
	-	-	85	89.4	92.3	98.4	98.4	98.4	equity
D	-	-	17.7	13.9	11.9	-1.1	-1.3	-1.6	consol
	-	-	-2.7	-3.3	-4.2	2.7	2.9	3.2	bond

on the previous period portfolio z_{t-1} by a simple, state-dependent, linear rule[13]: after a certain amount is withdrawn for "consumption" (this gives η_t), when the state is favorable (unfavorable) the holding of equity z_t is increased (decreased), remaining approximately unchanged in the medium state. Constructing the LVI reduces in essence to a carryover problem:[14] what is striking in this setting is how successful such a carryover strategy is at creating a smooth consumption stream – repeated trading of equity permits 80% of the potential gains from trade to be realized.

The extent to which adding a particular nominal security permits the remaining gains from trade to be captured depends on how much closer it brings the LVI to the constant annuity $\mathbf{1}_+$: the annuity $\mathbf{1}_+$ in the transformed economy corresponds to a stream which grows at the constant rate g in the original economy. This stream can be achieved by the consol in the idealized case where the monetary policy ensures that the purchasing power of money grows at the rate g, namely when there is constant deflation at the rate $i = -2.2\%$. In this case the

[13] In state s the portfolio rule is $z_t = \lambda_s z_{t-1} - \mu_s$ where (λ_s, μ_s) are (1.01, - 0.01), (1.002, -0.001) and (0.99, 0.01) for $s = H, M, L$ respectively. The three lines intersect the diagonal at the same point $z^* = \frac{\mu_s}{1-\lambda_s} = 0.885$.

[14] See for example Gustafson [13], Schechtman [32], Yaari [35], Bewley [1].

ideal LVI, is achieved by purchasing the consol at date 0 and keeping it for ever, and this permits the Arrow-Debreu equilibrium to be achieved.[15]

When there is inflation, the constant nominal dividend of the consol does not provide the growth in purchasing power required by the LVI and repeated retrading is necessary. Generating the LVI with the consol alone becomes impossible: for the real shocks to output imply that the real interest rate varies, and this in turn leads to variations in the price of the consol. The same is true of the (short) bond which would need to be constantly repurchased. Table 2 shows that the solution to consumption smoothing does not lie in investing mainly in the nominal security despite the constancy of its dividend: rather, it is better to invest in equity for growth and to smooth the 4% variations up and down in its dividend by exploiting the differences in price changes across the states between the equity and the bond. This strategy is very successful at creating a smooth income stream as shown by the less than 1/10% (resp. 3/10%) unexploited gains for the consol and bond respectively with constant inflation. Thus, as is familiar from other settings (see Woodford [34]), a strict adherence to the Friedman rule[16] of constant deflation is not essential, what is important is that inflation should not be variable.

The consol is particularly vulnerable to variable inflation: with security structure B, more than half the gains from trade vanish with the rather mild variability of US inflation. The effect is much less marked with the short-term bond (structure C). Variable inflation has more impact on the long-term bond because it creates variability both in its dividend and in its capital value (this latter effect coming from the permanence in the transition matrix K_i), while for the short bond it only affects its dividend. This conforms with the observation that when inflation increases – and typically this means that the variability of inflation increases – agents retreat from long-term to short-term instruments.

The last row of Table 1 with security structure D illustrates the effectiveness of hedging: since the payoffs of the consol and the bond are positively correlated, going short on the riskier security (the bond with constant inflation or the consol with variable inflation) and long on the less risky security creates a less risky income stream out of the two nominal securities: forming appropriate proportions of this strategy and the purchase and sale of equity leads to the least risky security. With constant inflation there are 3 states and 3 securities: the markets are complete, and all gains from trade are realized.

[15] This setting of constant deflation in which the consol acts as an ideal security is however strictly speaking not within the framework of the model considered here, which assumes that the velocity of circulation is one. For in the high output state the real rate of interest is negative (-3%): this would induce agents to hold onto some of their money balances (earned from the sale of their endowment) as a store of value for use in the subsequent period and this would reduce the velocity of circulation. Handling this case with variable velocity of circulation is somewhat complex and we shall not enter into it here: for an analysis of this case see Magill-Quinzii [24]. Note that the same problem arises in Table 2 for the cases of constant inflation of -2.2% and 0% for the security structures C and D which involve the bond: the nominal interest rate cannot be negative if agents can carry money balances from one period to the next.

[16] See Friedman [11].

The bond market puzzle. In this model the most striking consequence of inflation lies not so much in its effect on welfare, which is limited, as in its impact on the role of bonds, and especially long-term bonds, as instruments for dynamically smoothing income. The proportion of both long and short-term bonds in the portfolio of a typical investor falls rapidly as inflation increases and becomes virtually negligible when inflation is variable. For example, the coefficient a^i on the LVI of the most risk-averse agent[17] is approximately 0.6 in all market structures, and multiplying the proportion of the nominal security in Table 2 by 0.6 gives a good approximation of the average proportion of bonds in such an agent' s optimal portfolio. Thus even in the most favorable setting of constant (but non-negative) inflation the most risk-averse agent never has more than 13% invested in the consol. As soon as inflation is variable – and in the real world positive inflation is typically accompanied by variability of inflation – the proportion of bonds, whether long-term or short-term, in the optimal portfolio falls dramatically to less than 2%.

Given that in the US economy the value of the private sector bond holdings exceeds the total value of firms' equity[18] the small amount of trade on the bond market predicted by the model is another "puzzle" of the equilibrium model with infinitely-lived agents, to be added to the equity premium puzzle. For we conjecture that this "irrelevance" of the bond market will hold in most equilibrium models with infinitely-lived agents and stationary shocks[19]. For in these models risk-averse agents will seek a smooth consumption stream, while more risk-tolerant agents will seek higher expected consumption at the cost of greater variability. In the infinite-horizon model the "serious" risks on the security markets arise from the variability of the security prices, not the variability of their dividends. As a result, growth and the shrinking value of the nominal payoffs of bonds due to inflation give a substantial advantage to equity over bonds for consumption smoothing: with equity, a basic "buy and hold" strategy with limited subsequent retrading to smoothe the dividends – that is, a carryover strategy combined with some hedging in bonds – leads to excellent consumption smoothing. Except in the unrealistic case of constant deflation such a "buy and hold" strategy cannot work with bonds: to create a growing consumption stream with bonds requires frequent trading either because the bonds mature or because their dividends shrink in value.

The intuition derived from the two-period model that risk-averse agents invest a a significant part of their portfolios in bonds while risk-tolerant agents borrow on the bond market to invest in equity is not supported by the infinite-horizon

[17] As explained in Section 2, the parameters of risk aversion of the agents need to be limited to avoid negative or satiated consumption. In the calibrated model, we limit the risk-aversion parameters to $1.056 \leq \alpha^i/\theta^i w_0 \leq 2.26$. These restrictions imply that, with probability 0.96, agents' equilibrium consumption streams satisfy $0 \leq x^i(\sigma_t) \leq \alpha^i$ for the first 100 periods ($t \leq 100$), for the most risky security structure A.

[18] See the Flow of Funds Accounts, Table 796 of the Statistical Abstract of the United States, 1998.

[19] We do not believe that the presence of idiosyncratic risks will importantly change the result, as long as the risks are stationary: for such risks can be successfully smoothed by the same type of carryover strategies as those underlying the construction of the LVI with only equity.

model. In this latter model the difference between risk-averse agents and risk-tolerant agents is that risk-averse agents buy equity "high" and sell "low" to smoothe their consumption, while risk-tolerant agents buy "low" and sell "high" to achieve higher expected consumption.

Appendix A

Proof of Proposition 3. Since (i) is readily deduced from (ii), it suffices to prove (ii). $\| \hat{\boldsymbol{\eta}} \|^2 = \sum_{t=1}^{\infty} \delta^t \rho(\sigma_t)(1-\eta(\sigma_t))^2 \implies$

$$\begin{aligned} \| \hat{\boldsymbol{\eta}} \|^2 &\geq \textstyle\sum_{\{\sigma_t \in \boldsymbol{\Sigma} | t \leq T,\ |1-\eta(\sigma_t)|>\kappa\}} \delta^t \rho(\sigma_t)(1-\eta(\sigma_t))^2 \\ &\geq \kappa^2 \delta^T \textstyle\sum_{\{\sigma_T \in \boldsymbol{\Sigma}_T \mid |1-\eta(\sigma_t)|>\kappa \text{ for some node on the path } [\sigma_0,\sigma_T]\}} \rho(\sigma_T) \end{aligned}$$

which implies

$$\| \hat{\boldsymbol{\eta}} \|^2 \geq \kappa^2 \delta^T \mathbb{P}\ (|\ 1-\eta_t\ | > \kappa \text{ for some } t \leq T)$$

where $\mathbb{P}$ denotes the probability that a set of paths of $\boldsymbol{\Sigma}$ satisfy a given condition. Thus if $(\overline{T}, \overline{\rho})$ are fixed and κ is given by (21), then

$$\begin{aligned} \mathbb{P}(|1-\eta(\sigma_t)| > \kappa \text{ for some } t \leq \overline{T}) &\leq 1-\overline{\rho} \iff \\ \mathbb{P}\left(|1-\eta(\sigma_t)| \leq \kappa,\ \forall\, t \leq \overline{T}\right) &> \overline{\rho} \end{aligned}$$

Thus with probability greater than $\bar{\rho}$, $1-\kappa < \eta(\sigma_t) < 1+\kappa,\ \forall\ t \leq \overline{T}$.

Upper bound on β^i. $a^i \eta(\sigma_t) + b^i w(\sigma_t) > 0$

$$\begin{aligned} &\iff \left(\alpha^i \ \| \boldsymbol{\pi} \|^2 - \alpha [\![\boldsymbol{\pi}^i, \boldsymbol{\pi}]\!]\right) \eta(\sigma_t) + \left([\![\boldsymbol{\pi}^i, \boldsymbol{\pi}]\!] - \alpha\alpha^i \ \| \hat{\boldsymbol{\eta}} \|^2\right) w(\sigma_t) > 0 \\ &\iff [\![-\alpha^i \boldsymbol{w} + \alpha \boldsymbol{\omega}^i, \boldsymbol{\pi}]\!] \eta(\sigma_t) + \left([\![\alpha^i \mathbf{1} - \boldsymbol{\omega}^i, \boldsymbol{\pi}]\!] - \alpha\alpha^i \ \| \hat{\boldsymbol{\eta}} \|^2\right) w(\sigma_t) > 0 \\ &\iff \alpha^i \left([\![\boldsymbol{\pi}, \boldsymbol{w}]\!] \eta(\sigma_t) - ([\![\boldsymbol{\pi}, \mathbf{1}]\!] - \alpha \ \| \hat{\boldsymbol{\eta}} \|^2) w(\sigma_t)\right) \\ &\qquad\qquad < [\![\boldsymbol{\pi}, \boldsymbol{\omega}^i]\!] (\alpha\eta(\sigma_t) - w(\sigma_t)) \end{aligned}$$

Since we have assumed that $(\bar{\rho}, \overline{T})$ are chosen so that $\alpha(1-\kappa) > w_{\text{sup}}$, when $\eta(\sigma_t) \geq 1-\kappa$, the right side is positive. If the coefficient of α^i is negative then the inequality is satisfied. Thus when $\eta(\sigma_t) \geq 1-\kappa$, $\overline{x}^i(\sigma_t) > 0$ is equivalent to

$$\beta^i < \frac{\alpha\eta(\sigma_t) - w(\sigma_t)}{[\![\boldsymbol{\pi}, \boldsymbol{w}]\!] \eta(\sigma_t) - \left([\![\boldsymbol{\pi}, \mathbf{1}]\!] - \alpha \ \| \hat{\boldsymbol{\eta}} \|^2\right) w(\sigma_t)}$$

for all σ_t such that the denominator is positive. Consider the function

$$f(x,y) = \frac{\alpha y - x}{[\![\boldsymbol{\pi}, \boldsymbol{w}]\!] y - \left([\![\boldsymbol{\pi}, \mathbf{1}]\!] - \alpha \ \| \hat{\boldsymbol{\eta}} \|^2\right) x}$$

Computing the partial derivatives gives

$$\text{sign}\left(\frac{\partial f}{\partial x}\right) = \text{sign}\left(\alpha[\![\boldsymbol{\pi},\mathbf{1}]\!] - [\![\boldsymbol{\pi},\boldsymbol{w}]\!] - \alpha^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2\right) y$$
$$= \text{sign}\left(\parallel \boldsymbol{\pi} \parallel^2 - \alpha^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2\right) y = \text{sign}(y)$$

and sign $\left(\frac{\partial f}{\partial y}\right)$ = -sign (x), since[20] $\parallel \boldsymbol{\pi} \parallel^2 - \alpha^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2 > 0$. Thus if $x \in [w_{\inf}, w_{\sup}]$ and $y \in [1-\kappa, 1+\kappa]$ then the minimum of $f(x,y)$ is attained for $(x,y) = (w_{\inf}, 1+\kappa)$.

Lower bound on β^i. $a^i\eta(\sigma_t) + b^i w(\sigma_t) < \alpha^i$

$$\iff \left(\alpha^i \parallel \boldsymbol{\pi} \parallel^2 - \alpha[\![\boldsymbol{\pi}^i,\boldsymbol{\pi}]\!]\right)\eta(\sigma_t) + \left([\![\boldsymbol{\pi}^i,\boldsymbol{\pi}]\!] - \alpha\alpha^i \parallel \hat{\boldsymbol{\eta}} \parallel^2\right) w(\sigma_t) < \alpha^i \left(\parallel \boldsymbol{\pi} \parallel^2 - \alpha^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2\right)$$
$$\iff \alpha^i \left((\alpha - w(\sigma_t))([\![\boldsymbol{\pi},\mathbf{1}]\!] - \alpha \parallel \hat{\boldsymbol{\eta}} \parallel^2) + [\![\boldsymbol{\pi},\boldsymbol{w}]\!](\eta(\sigma_t) - 1)\right) > [\![\boldsymbol{\pi},\boldsymbol{w}]\!]\,(\alpha\eta(\sigma_t) - w(\sigma_t))$$

after some calculation. The coefficient of α^i can be written as

$$\alpha[\![\boldsymbol{\pi},\mathbf{1}]\!] - \alpha^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2 - [\![\boldsymbol{\pi},\boldsymbol{w}]\!] - w(\sigma_t)\left([\![\boldsymbol{\pi},\mathbf{1}]\!] - \alpha \parallel \hat{\boldsymbol{\eta}} \parallel^2\right) + \eta(\sigma_t)[\![\boldsymbol{\pi},\boldsymbol{w}]\!]$$

and is larger than $(\parallel \boldsymbol{\pi} \parallel^2 - \alpha^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2)(1 - w(\sigma_t))/\alpha)$ since $\eta(\sigma_t) \geq w(\sigma_t)/\alpha$, and is thus positive. Thus $a^i\eta(\sigma_t) + b^i w(\sigma_t) < \alpha^i$ for all σ_t is equivalent to

$$\beta^i > \frac{\alpha\eta(\sigma_t) - w(\sigma_t)}{(\alpha - w(\sigma_t))\left([\![\boldsymbol{\pi},\mathbf{1}]\!] - \alpha \parallel \hat{\boldsymbol{\eta}} \parallel^2\right) + [\![\boldsymbol{\pi},\boldsymbol{w}]\!]\,(\eta(\sigma_t) - 1)}$$

for all σ_t. Studying the partial derivatives of the function

$$g(x,y) = \frac{\alpha y - x}{(\alpha - x)\left([\![\boldsymbol{\pi},\mathbf{1}]\!] - \alpha \parallel \hat{\boldsymbol{\eta}} \parallel^2\right) + [\![\boldsymbol{\pi},\boldsymbol{w}]\!](y-1)}$$

shows that if $x \in [w_{\inf}, w_{\sup}]$ and $y \in [1-\kappa, 1+\kappa]$ then the maximum of $g(x,y)$ is attained for $(x,y) = (w_{\sup}, 1+\kappa)$. □

Proof of Proposition 4. Substituting $u_0^i = -\frac{1}{2} \parallel \boldsymbol{\pi}^i \parallel^2$ and the expressions for u_{AD}^i and $u^i(\bar{\boldsymbol{x}}^i)$ in Propositions 1 and 2 gives

$$\phi^i = \frac{(\alpha^i \parallel \boldsymbol{\pi} \parallel^2 - \alpha[\![\boldsymbol{\pi}^i,\boldsymbol{\pi}]\!])^2}{\parallel \boldsymbol{\pi}^i \parallel^2 \parallel \boldsymbol{\pi} \parallel^2 - [\![\boldsymbol{\pi}^i,\boldsymbol{\pi}]\!]^2} \frac{\parallel \hat{\boldsymbol{\eta}} \parallel^2}{\parallel \boldsymbol{\pi} \parallel^2 - \alpha^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2} = k^i \frac{\parallel \hat{\boldsymbol{\eta}} \parallel^2}{\parallel \boldsymbol{\pi} \parallel^2 - \alpha^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2}$$

We want to show that k^i is independent of i. Define $\widetilde{\alpha}^i$ by $\widetilde{\alpha}^i\theta^i = \alpha^i$ then

$$k^i = \frac{\left(\widetilde{\alpha}^i \parallel \alpha\mathbf{1} - \boldsymbol{w} \parallel^2 - \alpha[\![\widetilde{\alpha}^i\mathbf{1} - \boldsymbol{w},\ \alpha\mathbf{1} - \boldsymbol{w}]\!]\right)^2}{\parallel \widetilde{\alpha}^i\mathbf{1} - \boldsymbol{w} \parallel^2 \parallel \alpha\mathbf{1} - \boldsymbol{w} \parallel^2 - [\![\widetilde{\alpha}^i\mathbf{1} - \boldsymbol{w},\ \alpha\mathbf{1} - \boldsymbol{w}]\!]^2}$$

Developing the terms in the numerator and denominator and assuming $\alpha - \widetilde{\alpha}^i \neq 0$, the term $(\alpha - \widetilde{\alpha}^i)^2$ can be factored out and the expression for k^i reduces to

[20] By Pythagoras theorem, $\parallel \boldsymbol{\pi} \parallel^2 = \parallel \alpha\mathbf{1} - \boldsymbol{w} \parallel^2 = \parallel \alpha(1,\boldsymbol{\eta}) + \alpha(0,\hat{\boldsymbol{\eta}}) - \boldsymbol{w} \parallel^2 = \parallel \alpha(1,\boldsymbol{\eta}) - \boldsymbol{w} \parallel^2 + \alpha^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2 > \alpha^2 \parallel \hat{\boldsymbol{\eta}} \parallel^2$.

$$k^i = \frac{[\![\boldsymbol{\pi}, \boldsymbol{w}]\!]^2}{\| \mathbf{1} \|^2 \| \boldsymbol{w} \|^2 - [\![\mathbf{1}, \boldsymbol{w}]\!]^2}, \quad i = 1, \ldots, I$$

For an agent with the average characteristics ($\boldsymbol{\omega}^i = \boldsymbol{w}/I$, $\alpha^i = \alpha/I$) $\theta^i = 1/I$ so that $\widetilde{\alpha}^i = \alpha$: thus ϕ^i is not defined and must be understood as the limit of ϕ^i when $\widetilde{\alpha}^i \longrightarrow \alpha/I$. Note also that k^i and hence ϕ^i is not well-defined when there is no aggregate risk, since in this case the denominator of k^i is zero. □

Proof of Lemma 6. The proof consists in computing the expression $x^\top[xx^\top + \theta A]^{-1}x$ and letting $\theta \longrightarrow 0^+$. To find the matrix inverse we begin by computing $\det[xx^\top + \theta A]$. Let A^j denote the j^{th} column of A ($j = 1, \ldots, J$) and note that the j^{th} column of $xx^\top$ is xx_j. Develop $\det[xx^\top + \theta A]$ as a polynominal of order J in θ using the multi-linearity of the determinant

$$\det[xx^\top + \theta a] = \sum_{j=0}^{J} a_j \theta^j = \det[xx_1 + \theta A^1,\ xx_2 + \theta A^2, \ldots, xx_J + \theta A^J]$$

$a_J = \det A$, $a_{J-1} = \sum_{j=1}^{J} \det[A^1, \ldots, A^{j-1}, xx_j, A^{j+1}, \ldots, A^J]$, $a_j = 0$ if $j \leq J-2$, since at least two vectors of the form xx_j, xx_i are linearly dependent. Thus

$$\det[xx^\top + \theta A] = (\det A)\theta^J + \sum_{j=1}^{J} \det[A^1, \ldots, A^{j-1}, xx_j, A^{j+1}, \ldots, A^J]\theta^{J-1}$$

Note that the coefficient of θ^{J-1} is non-zero: for the coefficient to be zero, x would have to belong to the intersection of all the $J-1$ dimensional subspaces generated by $J-1$ vectors from the basis $\{A^1, \ldots, A^J\}$ and this intersection is the zero vector, contradicting $x \neq 0$. For a matrix B, let $\operatorname{cof}(B)$ denote the matrix of cofactors of B. We need to compute $\operatorname{cof}(xx^\top + \theta A)$. (Note that we do not need to take the transpose of $xx^\top + \theta A$ to compute the inverse, since A is symmetric.). Let c_ℓ^j denote the cofactor corresponding to column j and row ℓ of $[xx^\top + \theta A]$, then

$$\begin{aligned} x^\top \operatorname{cof}(xx^\top + \theta A)x &= \sum_{\ell=1}^{J} \sum_{j=1}^{J} x_j x_\ell c_\ell^j \\ &= \sum_{\ell=1}^{J} \sum_{j=1}^{J} x_j x_\ell \det[xx_1 + \theta A^1, \ldots, xx_{j-1} + \theta A^{j-1}, e_\ell, \\ &\quad xx_{j+1} + \theta A^{j+1}, \ldots, xx_J + \theta A^J] \end{aligned}$$

where e_ℓ is the column vector with 1 in row ℓ and zero elsewhere. Summing over ℓ with j fixed gives

$$\sum_{j=1}^{J} x_j \det[xx_1 + \theta A^1, \ldots, xx_{j-1} + \theta A^{j-1},\ x,\ xx_{j+1} + \theta A^{j+1}, \ldots, xx_J + \theta A^J]$$

$$= \sum_{j=1}^{J} \det[xx_1 + \theta A^1, \ldots, xx_{j-1} + \theta A^{j-1}, xx_j, xx_{j+1} + \theta A^{j+1}, \ldots, xx_J + \theta A^J]$$
$$= a_{j-1}\theta^{J-1}$$

Thus since $a_{J-1} \neq 0$,

$$x^{\top}[xx^{\top} + \theta A]^{-1}x = \frac{a_{j-1}\theta^{J-1}}{a_{J-1}\theta^{J-1} + a_J\theta^J} \rightarrow 1 \text{ as } \theta \rightarrow 0$$

Appendix B

Transforming the calibrated economy to an economy satisfying the assumptions of Sections 2 and 3. Let $\big(Q_y(\sigma_t), Q_c(\sigma_t), Q_b(\sigma_t)\big)$ denote the money price of the equity, consol and short bond respectively at node σ_t. Agent i's budget equation at node σ_t can be written as

$$\begin{aligned} & P(\sigma_t)(X^i(\sigma_t) - \omega^i(\sigma_t)) \\ & = \big(P(\sigma_t)Y(\sigma_t) + Q_y(\sigma_t)\big) Z_y^i(\sigma_t^-) + (1 + Q_c(\sigma_t)) Z_c^i(\sigma_t^-) + Z_b^i(\sigma_t^-) \\ & \quad - Q_y(\sigma_t)Z_y^i(\sigma_t) - Q_c(\sigma_t)Z_c^i(\sigma_t) - Q_b(\sigma_t)Z_b^i(\sigma_t) \end{aligned} \tag{B.1}$$

where (Z_y^i, Z_c^i, Z_b^i) denote the portfolio holdings of the equity, consol, and short bond respectively. When the market structure is restricted to a subset of the securities, then the corresponding component of the portfolio is set equal to zero: thus if there is only trade in equity, we set $Z_c^i = Z_b^i = 0$, or if there is only trade in the equity and consol, then $Z_b^i = 0$. This budget equation differs from the budget equation of a standard cash-in-advance model (Lucas [23]) in that at node σ_t the agent can spend the value of the current endowment $P(\sigma_t)\omega^i(\sigma_t)$ rather than the value of the endowment at the predecessor $P(\sigma_t^-)\omega^i(\sigma_t^-)$. It is compatible with the transaction technology introduced in Magill-Quinzii [24, 26]: money circulates in exchange for goods within periods, but is held in coffers of a "Central Exchange" across periods. This approach factors out the seignorage tax on money balances, focusing instead on the effects of inflation on the real payoffs of nominal securities. To revert to the model of Sections 2 and 3, which is a real model without growth, we need to divide by the price level $P(\sigma_t)$ and factor out growth by dividing by g^t. Let $\nu(s_t) = 1/P(\sigma_t)$ denote the *purchasing power of money* (*ppm*) and let $n_s = 1/(1 + i_s)$ denote the loss of value of the *ppm* in state s. The inflation process in (37) can be written as an equivalent process for the *ppm*: if $\sigma_t = (\bar{s}_0, \ldots, s_t)$ then $\nu(\sigma_t) = \nu(\sigma_t^-)n_{s_t}$. Given that it is the rate of decrease of the *ppm* which is Markov and not the *ppm* itself (which is the real value of the bonds' dividends), an additional transformation is needed to map the economy into a Markov economy: it is based on the idea that it is equivalent to invest in a security which costs q and gives a payoff of R or to invest in a security which costs λq and has the payoff λR, since it suffices to divide the amount invested in the security by λ. Using this observation, we may replace a nominal security which costs $Q(\sigma_t)$ units of money (e.g. the consol with price $Q_c(\sigma_t)$) by a security which costs $Q(\sigma_t)$ growth-factored units of the good. This amounts to multiplying the nominal price of the security by $\lambda = g^t/\nu(\sigma_t)$. The nominal payoff of the new security at date $t+1$ must be $\lambda(1+Q(\sigma_t))$ and its real value in the rescaled units is

$$\frac{\nu(\sigma_{t+1})}{g^{t+1}}\left[\frac{g^t}{\nu(\sigma_t)}(1+Q(\sigma_t))\right] = \frac{n_{s_{t+1}}}{g}(1+Q(\sigma_t))$$

This leads to the change of variable

$$\begin{aligned} x^i(\sigma_t) &= \frac{X^i(\sigma_t)}{g^t}, \quad z_y^i(\sigma_t) = Z_y^i(\sigma_t), \quad z_c^i(\sigma_t) = \frac{\nu(\sigma_t)}{g^t}Z_c^i(\sigma_t), \\ z_b^i(\sigma_t) &= \frac{\nu(\sigma_t)}{g^t}Z_b^i(\sigma_t), \quad q_y(\sigma_t) = \frac{\nu(\sigma_t)}{g^t}Q_y(\sigma_t), \quad q_c(\sigma_t) = Q_c(\sigma_t), \\ q_b(\sigma_t) &= Q_b(\sigma_t) \end{aligned}$$

in terms of which the budget equation (B.1) becomes

$$\begin{aligned} & x^i(\sigma_t) - \theta^i Y_0\gamma_{s_t} \\ &= (Y_0\gamma_{s_t} + q_y(\sigma_t))z_y^i(\sigma_t^-) + \left(\frac{n_{s_t}}{g} + \frac{n_{s_t}}{g}q_c(\sigma_t)\right)z_c^i(\sigma_t^-) + \frac{n_{s_t}}{g}z_b^i(\sigma_t^-) \\ & \quad -q_y(\sigma_t)z_y^i(\sigma_t) - q_c(\sigma_t)z_c^i(\sigma_t) - q_b(\sigma_t)z_b^i(\sigma_t) \end{aligned} \tag{B.2}$$

In these new variables, agent i's maximum problem consists in finding a portfolio strategy $z^i = (z_y^i, z_c^i, z_b^i)$ which maximizes

$$E\left[\sum_{t=0}^{\infty}\widetilde{\delta}^t(\alpha^i - x_t^i)^2\right], \quad \widetilde{\delta} = \delta g^2$$

subject to the budget equations (B.2) and the transversality condition (TR): thus the analysis of Section 2 applies. Moreover in these variables, the payoffs of the securities satisfy the Markov assumption of Section 3: the dividends of the equity contract and the bonds ($Y_0\gamma_{s_t}$ and n_{s_t}/g respectively) depend only on the current state, and the consol "shrinks" by the factor $f^c(\sigma_t) = n_{s_t}/g$ each period.

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