Econometrica, Vol. 49, No. 3 (May, 1981)

INFINITE HORIZON PROGRAMS

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This paper presents a general framework for the analysis of programs over an infinite horizon in continuous time. Sufficient conditions for the *existence* of an optimal program are derived and are shown to reduce to the condition that the underlying preference ordering exhibit *impatience* in a topology determined by the underlying technology.

1. INTRODUCTION

THIS PAPER PROVIDES a solution to a basic problem of capital theory posed by the open-endedness of the future. *Under what conditions do optimal infinite horizon programs exist*? It might be argued that such a question is little more than an intellectual curiosum, leading to just another abstract mathematical exercise. I shall argue that this is not the case and indeed that a proper understanding of the practical economic consequences of the open-endedness of the future can only be obtained by giving a precise answer to this question. It is true that we are led to adopt a fairly abstract approach to the problem. But the abstraction serves the best possible purpose—it focuses attention on the few basic ideas that lead directly to the solution of the problem. Once the abstract framework is understood (the degree of abstraction is always a relative concept) the solution of the problem is both simple and natural.

It might be thought that in the years since Ramsey [**44**] first formalized capital theory such a basic problem would have been thoroughly resolved. We shall find however, that this is not the case, subject to one qualification. For in considering the existence problem over an infinite horizon we need to distinguish two cases, depending on whether the underlying preference ordering leads to a *partial ordering* or a *complete ordering*. It is true that the first case has been rather thoroughly examined, including as it does the original analysis of Ramsey [**44**] and the subsequent work of Von Weisäcker [**54**], Gale [**24**], Brock [**8**], and finally in continuous time, the rather complete analysis of Brock and Haurie [**10**]. I refer therefore to the case of a complete ordering. As recently as 1960 Tinbergen [**51**] examined a problem of this kind in which he found that for certain natural parameter values of the underlying utility function the limit of the finite horizon optimal paths is in a sense the worst possible path, for consumption is zero for every finite time and infinite at infinity. This paper prompted Chakravarty [**15**] to a perceptive, albeit largely verbal discussion of the problem of existence. It began to be recognized that Ramsey's approach while ingenious in many respects involved a device that was artificial and inapplicable in many cases and indeed served to hide some important economic problems posed by the open-endedness

[1]This research was supported by the National Science Foundation Grant SOC 78-12589 and in part by the Grant SOC 77-0600-A01 to Stanford University. I am grateful to Truman Bewley, William Brock, Don Brown, David Gale, and Terry Rockafellar for helpful discussions. I am grateful to Mordecai Kurz for the opportunity of spending a stimulating year at the Institute for Mathematical Studies in the Social Sciences at Stanford University.

of the future. This was brought out clearly in the penetrating analysis of Koopmans [**29**], the first major attempt to resolve the existence problem in the case of a complete ordering.

Koopmans' analysis is however incomplete (no rigorous proof of the existence of the crucial infinite horizon path [**29**, p. 278], a fault common to virtually all the literature of this period), uses special assumptions (single good, differentiable utility and production functions, and bounded technology set), and does not seem to be readily amenable to generalization. This last point is important. For one natural generalization of Koopmans' approach to the case of many commodities would involve finding conditions that ensure the existence of a positive stationary solution which is globally stable. It is clear however from recent analysis [**10**, **14**, **32**, **33**, **34**, **47**] that such conditions may have little or nothing to do with the problem of existence. It would seem that existence results which depend upon qualitative properties of an optimal solution are unlikely to lead to results of any generality (the undiscounted stationary problem is rather special in this respect [**8**, **10**, **24**]).

The first rigorous and complete analysis of existence over an infinite horizon in the case of a complete ordering is the discrete time one-dimensional case analyzed by Brock and Gale [**9**]. Their analysis has the further merit that it contains the germ of a general method that ultimately forms the basis for the approach adopted in this paper. An n-dimensional version of the method of Brock-Gale was considered by McFadden [**38**]. A more general case was subsequently analyzed by Evstigneev [**21**].

In the economic literature there has been some misunderstanding as to the basic causes of existence or nonexistence. Both Chakravarty [**15**, p. 179] and Koopmans [**27**, p. 8] attribute it to a lack of compactness. McFadden [**37**, p. 403; **38**, p. 260] argues that in the infinite horizon case existence cannot be based on considerations of continuity and compactness except in a trivial case [**37**, p. 411]. If this conclusion were true it would be disturbing indeed since virtually all existence results in economic theory depend at root on continuity and compactness. I shall argue, in brief, that compactness is on the whole readily secured by the appropriate choice of a program space and topology and that the true force of existence conditions lies in the requirement of upper semicontinuity of the preference ordering, this latter condition being tantamount to a precise form of impatience.

Section 2 outlines the basic framework for the analysis of the problem of choice over an infinite horizon. I define a *program space* $\mathscr{W}$ as a certain infinite dimensional space (a locally convex space) endowed with a suitable *topology* Υ (Section 3). A subset $\mathscr{F} \subset \mathscr{W}$ denotes the set of *feasible programs*. The basic axiom is that choice among programs in $\mathscr{F}$ can be represented by a preference ordering. In this paper I am concerned with the subset of preference orderings that are representable by a certain class of *integral functionals*: this class includes all the standard preference orderings that arise in capital theory. *Under certain conditions (Assumptions* 4–6 *in Section* 6) *such preference orderings can be shown to* *be complete, transitive, and upper semicontinuous on* $\mathcal{F}$ *relative to a certain topology* Υ (*Proposition* 7.1).

The choice of the program space $\mathcal{W}$ *and its topology* Υ *is crucial.* I allow the choice of the program space to be dictated by the properties of the instantaneous technology set $\Gamma(t)$, the set which represents the production opportunities at any moment of time (see Sections 4, 5). I choose the topology in such a way that the set of feasible programs $\mathcal{F}$ induced by $\Gamma(t)$ is compact. This leads to two distinct cases. In the first case $\Gamma(t)$ is *uniformly bounded* for all $t \in I = [0, \infty)$: this case arises when there are *decreasing returns to scale* (Section 8). In this case it is natural to let $\mathcal{W}$ be the space of *bounded, vector valued measurable functions* defined on I and to let Υ be the *weak* topology*. In the second case $\Gamma(t)$, $t \in I$, is *unbounded*: this case arises when there are *constant returns to scale* (Section 9) or when there is *technical change* (Section 10). In this case a preliminary analysis is needed to find out how fast feasible programs can grow when viewed as a function of time (see Section 5, Definition 5.1). Given this information I show that the program space may be chosen as the space of vector valued functions defined on I whose components are *integrable*, the Lebesgue measure on I being replaced by a measure which is in essence the reciprocal of the *maximal rate of growth* of any feasible program. In this case I let the topology Υ be the *weak topology*. Given this choice of program space and topology, *Assumptions* 1–3 *in Sections* 4 *and* 5 *ensure that the set of feasible programs* $\mathcal{F}$ *is compact* (Proposition 7.5).

I recall in Section 2 (Theorem 2.5) that if a preference ordering is complete, transitive, and upper semicontinuous on $\mathcal{F}$ relative to a topology Υ and if $\mathcal{F}$ is compact in the topology Υ and nonempty, then there exists an optimal program. This leads us to the basic Existence Theorem 7.6. If additional assumptions are made concerning the asymptotic properties of the instantaneous utility function, then more precise existence conditions (Corollaries 7.8, 7.9) can be deduced. Sections 8–10 establish existence in the context of the three main types of technology set that arise in capital theory. The problem reduces to showing that Assumptions 1–6 are satisfied. Since Assumptions 1, 2, 4, and 5 are readily established, it is Assumptions 3 and 6 that require the additional analysis.

Koopmans [**30**] first uncovered the basic relation between the concept of *impatience* and the property of *continuity* of the preference ordering. Brown and Lewis [**12**] have subsequently substantially refined the analysis. Section 11 considers these ideas in a preliminary way in the continuous time case. The upper semicontinuity condition leads in the simplest case to a certain inequality on underlying parameters which characterize the asymptotic form of the instantaneous utility function, the asymptotic rate of discounting and the maximal asymptotic rate of growth of consumption. This condition is nothing but the requirement that the asymptotic instantaneous rate of impatience in the sense of Irving Fisher [**22**, p. 62], along a path of maximal growth of consumption, exceed the maximal rate of growth of output. In the class of problems considered in this paper the requirement of upper semicontinuity of the preference ordering and its relation to the concept of impatience is the central idea in terms of which the essential economic aspects of the open-endedness of the future may be summarized.

In the framework of this paper impatience is a sufficient condition for existence. It is clear however that optimal programs can exist without the preference ordering exhibiting impatience. We have only to consider a preference ordering that is upper semicontinuous in the norm topology on the space of bounded measurable functions and a technology set compact in the norm topology. More generally the partial orderings considered by Von Weisäcker [**54**], Gale [**24**], and Brock-Haurie [**10**] do not exhibit impatience. These are however, in a sense, hairline cases. For as soon as the preference ordering exhibits *patience* optimal programs cease to exist. This result is not established in the general case in this paper but is a result that is readily established in the simplest cases.[2]

Yaari [**55**] and Aumann-Perles [**3**] have considered a related variational problem with a simpler integral structure. By exploiting this simpler integral structure, in conjunction with the Liapunov convexity theorem, Aumann-Perles are able to establish existence without the assumption of *convexity*. There are essentially two parts to the convexity assumption in this paper: convexity in the *stocks* and convexity in the *flows*. The former assumption can be dropped in Sections 4–7 at the cost of some complication in the analysis. I have chosen not to enter into this. An attempt to drop the latter assumption, however, leads to the phenomenon of *chattering*. To deal satisfactorily with this case requires an extension of the concept of a solution curve to the concept of *generalized curves* along the lines originally introduced by Young [**56**].

In a framework closely related to the analysis of this paper Chichilnisky [**16**] has established existence without the assumption of convexity in either stocks or flows. To avoid the problem of chattering she imposes bounds on the degree to which flows can fluctuate over time [**16**, vi p. 512]. It is clear however that in general the basic statement of the economic problem may not permit the introduction of such an assumption. Just as mixed strategies are forced to make an appearance in the theory of games, so generalized curves need to be introduced to cope with the problem of nonconvexities in the flows. When convexity is introduced into Chichilnisky's analysis, her existence result appears as a special case of the results of Section 7.

A well developed existence theory is applicable in the case of a *finite horizon*[3] [**20, 45**]. This paper will explore ways of extending these earlier results to the case of an infinite horizon, with particular emphasis on the class of problems that arise naturally in intertemporal economics. This paper is part of a broader project that seeks to integrate capital theory and equilibrium theory. Indeed there

[2] If the criterion of optimality is replaced by the criterion of *weak maximality* introduced by Brock [**8**], then preference orderings which exhibit a small degree of *patience* are consistent with existence. A precise bound on the degree of patience in the one good case has been given by Mitra [**42**, Theorem 1] exploiting the efficiency criterion of Cass [**13**, Theorem 3].

[3] Much of the early development of this theory is due to Tonelli [**52**], who first recognized the crucial importance of semicontinuity in variational problems.

is a close connection between the results of this paper and the infinite dimensional version of equilibrium theory developed by Bewley [7].

2. UPPER SEMICONTINUITY AND COMPACTNESS

The first basic notion is that of a *program space*.

DEFINITION 2.1: Let $(\mathcal{W}, \Upsilon)$ be a *locally convex space* endowed with a *topology* Υ. $\mathcal{W}$ will be called the *program space*. $\mathcal{F} \subset \mathcal{W}$ will denote the subset of *feasible programs*.

The second basic notion is that of a *preference ordering* $\succsim$. I take this notion to be understood. A solution of the problem of choice among programs in $\mathcal{W}$ will depend on the following basic assumption.

ASSUMPTION P.1: *Choice among programs in $\mathcal{W}$ can be represented by a preference ordering $\succsim$.*

DEFINITION 2.2: The preference ordering $\succsim$ is said to be *complete on* $\mathcal{F}$ if for every w, $w' \in \mathcal{F}$ either $w \succsim w'$ or $w' \succsim w$ and *transitive* if $w \succsim w'$, $w' \succsim w''$ implies $w \succsim w''$.

ASSUMPTION P.2: *The preference ordering $\succsim$ is complete and transitive on $\mathcal{F}$.*

DEFINITION 2.3: The preference ordering $\succsim$ is *upper semicontinuous* on $\mathcal{F}$ in the Υ topology if the preferred sets

$$(2.1) \qquad \mathcal{U}_w = \{w' \in \mathcal{F} \mid w' \succsim w\}, \quad \forall w \in \mathcal{F}$$

are *closed* in the Υ topology.

ASSUMPTION P.3: *The preference ordering $\succsim$ is upper semicontinuous on $\mathcal{F}$ in the Υ topology.*

DEFINITION 2.4: If $w^* \in \mathcal{F}$ satisfies $w^* \succsim w$, $\forall w \in \mathcal{F}$, then w^* will be called an *optimal program*. If $\mathcal{W}$ is a space of programs over an infinite horizon, then w^* will be called an *optimal infinite horizon program*.

In the sections that follow the proof of the existence of an optimal infinite horizon program will be reduced by a sequence of steps to the following standard result.

THEOREM 2.5: *If Assumptions P.1–P.3 are satisfied, if $\mathcal{F}$ is compact in the Υ topology and $\mathcal{F} \neq \emptyset$, then there exists an optimal program.*

PROOF: Since the proof is simple, and basic to what follows, I shall remind the reader of it. Since $\mathcal{F} \neq \emptyset$, consider a finite collection of points $\{w_1, \ldots, w_k\} \subset \mathcal{F}$. Transitivity implies $\bigcap_{i=1}^{k} \mathcal{U}_{w_i} \neq \emptyset$. Since the finite intersection property is satisfied for all finite subsets $\{w_1, \ldots, w_k\} \subset \mathcal{F}$, since $\mathcal{U}_w$ is Υ-closed $\forall w \in \mathcal{F}$ by P.3, and since $\mathcal{F}$ is Υ-compact, $\bigcap_{w \in \mathcal{F}} \mathcal{U}_w \neq \emptyset$. $v^* \in \bigcap_{w \in \mathcal{F}} \mathcal{U}_w$ is optimal. *Q.E.D.*

3. LOCALLY CONVEX SPACES

Let $I = [0, \infty)$ denote the *infinite time interval* and $(I, \mathcal{G}, \lambda)$ the associated measure space, $\mathcal{G}$ the Lebesgue measurable sets, λ the Lebesgue measure. I shall be concerned with linear subspaces of the space $\mathfrak{M}^m$ of all *R^m-valued* $(m \geqq 1)$ *$\mathcal{G}$-measurable functions* defined on the measurable space $(I, \mathcal{G})$.

Let $\mathcal{L}_\infty^m(\lambda) = \mathcal{L}_\infty^m(I, \mathcal{G}, \lambda)$ and $\mathcal{L}_1^m(\lambda) = \mathcal{L}_1^m(I, \mathcal{G}, \lambda)$ denote the subspaces of $\mathfrak{M}^m$ for which

$$(3.1) \qquad \|v\|_\infty = \operatorname{ess\,sup}\|v(\tau)\| < \infty, \quad \|v\|_1 = \int_I \|v(\tau)\| \, d\tau < \infty, \quad v \in \mathfrak{M}^m,$$

where $\operatorname{ess\,sup}\|v(t)\| = \inf\{\alpha \in R \mid \mu(t \| \|v(t)\| > \alpha) = 0\}$ and where $\| \ \|$ is the standard Euclidean norm. $\mathcal{L}_1^m(\lambda)$ induces a locally convex topology on $\mathcal{L}_\infty^m(\lambda)$ by means of the family of seminorms

$$(3.2) \qquad \Phi_f(v) = \left| \int_I f(\tau) v(\tau) \, d\tau \right|, \quad \forall f \in \mathcal{L}_1^m(\lambda), \quad v \in \mathcal{L}_\infty^m(\lambda).$$

This topology, which is written as $\Upsilon = \sigma(\mathcal{L}_\infty^m, \mathcal{L}_1^m)$, is called the *weak* topology* of $\mathcal{L}_\infty^m(\lambda)$. A sequence $\{v_n\}_{n=1}^\infty \subset \mathcal{L}_\infty^m(\lambda)$ is said to *converge* to v in the weak* topology if

$$(3.3) \qquad \Phi_f(v_n - v) \to 0 \quad \text{as} \quad n \to \infty, \quad \forall f \in \mathcal{L}_1^m(\lambda).$$

DEFINITION 3.1: A function $\theta \in \mathfrak{M}$ satisfying

$$(3.4) \qquad 0 < \theta(\tau) < \infty \quad \text{a.e.}, \quad \int_I \theta(\tau) \, d\tau = 1,$$

will be called a *density function* for I. θ induces a *measure* on the measurable space $(I, \mathcal{G})$ defined by

$$(3.5) \qquad \mu(S) = \int_S \theta(\tau) \, d\tau, \quad S \in \mathcal{G}.$$

Let θ denote a density function for I and let μ be its associated measure; then I shall let $\mathcal{L}_1^m(\mu) = \mathcal{L}_1^m(I, \mathcal{G}, \mu)$ and $\mathcal{L}_\infty^m(\mu) = \mathcal{L}_\infty^m(I, \mathcal{G}, \mu)$ denote the subspaces of $\mathfrak{M}^m$ for which

$$(3.6) \qquad \|v\|_1 = \int_I \|v(\tau)\| \, d\mu(\tau) < \infty, \quad \|v\|_\infty = \operatorname{ess\,sup}\|v(\tau)\| < \infty, \quad v \in \mathfrak{M}^m,$$

respectively. $\mathfrak{L}_\infty^m(\mu)$ induces a locally convex topology on $\mathfrak{L}_1^m(\mu)$ by means of the family of seminorms

$$(3.7) \qquad \Phi_f(v) = \left| \int_I f(\tau) v(\tau)\, d\mu(\tau) \right|, \quad \forall f \in \mathfrak{L}_\infty^m(\mu), \quad v \in \mathfrak{L}_1^m(\mu).$$

This topology, which is written as $\Upsilon = \sigma(\mathfrak{L}_1^m, \mathfrak{L}_\infty^m)$, is called the *weak topology* of $\mathfrak{L}_1^m(\mu)$. A sequence $\{v_n\}_{n=1}^\infty \subset \mathfrak{L}_1^m(\mu)$ is said to *converge* to v in the weak topology if

$$(3.8) \qquad \Phi_f(v_n - v) \to 0 \quad \text{as} \quad n \to \infty, \quad \forall f \in \mathfrak{L}_\infty^m(\mu).$$

REMARK 3.2: In view of Definition 3.1, $\mu(S) = 0$ if and only if $\lambda(S) = 0$, $\forall S \in \mathcal{G}$. Thus $\mathfrak{L}_\infty^m(\mu) = \mathfrak{L}_\infty^m(\lambda)$.

REMARK 3.3: It will be clear from the context whether the norms are being defined by (3.1) or (3.6).

The locally convex spaces defined by the seminorms (3.2) and (3.7) are the two basic program spaces that I shall consider in the sections that follow.

4. TECHNOLOGY

The description of productive activity in the economy will be based on the following stylized facts concerning the process of intertemporal production. Output of commodities increases through the accumulation of *stocks of capital goods* and through the *process of invention*, a process by which existing techniques of production are made more productive. If there is no underlying process of invention, the process of capital accumulation leads to *decreasing returns* due to the presence of *nonproduced primary resources* such as labor and land. In certain instances it is useful to set aside the effect of nonproduced primary resources and to allow *constant returns* to increases in capital. This allows one to focus attention on the way in which goods are produced from one another and to obtain insights into the structural interdependence of the economy. In the general case where a *process of invention* or *technical change* is present the decreasing returns are offset by new inventions which increase the marginal product of capital. These three broadly defined categories will be referred to as the cases of *decreasing returns*, *constant returns*, and *technical change*, respectively.

The stocks of *n capital goods* available at time t are denoted by the vector $z(t) = (z_1(t), \ldots, z_n(t))$, $n \geqq 1$. The stocks of *r nonproduced resources* and a vector of *s parameters* indicating the current *state of technology* are summarized in a single vector $\xi(t) = (\xi_1(t), \ldots, \xi_m(t))$ where $m = r + s \geqq 0$. It will be useful to let

$$(4.1) \qquad \bar{z}(t) = (z(t), \xi(t)), \quad t \in I.$$

The current flow output of the n goods available for *consumption* is denoted by $c(t)=(c_1(t),\ldots,c_n(t))$ and the flow output available for *investment* is denoted by $v(t)=(v_1(t),\ldots,v_n(t))$. It will be useful to let

$$(4.2) \qquad w(t)=(c(t),v(t)), \quad t\in I.$$

The vector of consumption-investment goods $w(t)$ is produced by means of processes employing stocks of capital goods and nonproduced resources, each process depending also on the current state of technical change.

DEFINITION 4.1: A *process* is represented by a pair of nonnegative vectors $(\bar{z}(t),w(t))$. The set of all processes available at time t is called the *technology set* $\Gamma(t)$.

I need to be more precise about the properties assumed for $\Gamma(t)$ and about the relation between $\bar{z}(t)$ and $w(t)$ in (4.1) and (4.2).

ASSUMPTION 1: $\Gamma(t): I\rightarrow R_+^{n+m}\times R_+^{2n}$ *is a closed, convex-valued measurable correspondence.*

ASSUMPTION 2: *There exists an $(n+m)\times(n+m)$ matrix-valued measurable function $a_0(t)$, with $a_0(0)=I_{n+m}$ (the identity matrix), and an $(n+m)\times n$ matrix-valued measurable function $a(t,\tau)$, such that on every finite interval $[0,s]\subset I$, $a_0(\cdot)$, $a(\cdot,\tau)$ $\forall\tau\in I$, $a(t,\cdot)$ $\forall t\in I$ are essentially bounded. For every $v\in\mathfrak{M}^n$ such that*

$$(4.3) \qquad \int_0^t \|v(\tau)\|\,d\tau<\infty, \quad \forall t\in I,$$

and for every $\bar{z}_0\in R_+^{n+m}$, $\bar{z}(t)$ is determined by the relation

$$(4.4) \qquad \bar{z}(t)=a_0(t)\bar{z}_0+\int_0^t a(t,\tau)v(\tau)\,d\tau, \qquad t\in I.$$

REMARK 4.2:

$$\|\bar{z}(t)\|\leqq\|a_0(t)\|\cdot\|\bar{z}_0\|+M_a(t)\int_0^t\|v(\tau)\|\,d\tau<\infty, \quad t\in I.$$

In most applications and all those considered in this paper, $a_0(\cdot)$ and $a(\cdot,\tau)$ are continuous. In this case $\bar{z}(t): I\rightarrow R_+^{n+m}$ is *continuous.*

REMARK 4.3: It will be convenient to write the affine transformation in (4.4) as

$$(4.5) \qquad \bar{z}(t)=(\mathcal{a}v)(t), \qquad t\in I,$$

where $\mathcal{a}$ depends on the *technology kernel* $a(t,\tau)$, $a_0(t)$ and the *initial condition* $\bar{z}_0$.

REMARK 4.4: The three basic categories of intertemporal production stated above correspond to certain properties of the family of technology sets $\{\Gamma(t)\}_{t\in I}$.

In the decreasing returns case, the technical change component of $\xi(t)$ is absent and the presence of the nonproduced resource component leads to decreasing returns. In this case there exists a bounded set $B \subset R_+^{n+m} \times R_+^{2n}$ such that

$$(4.6) \qquad \Gamma(t) \subset B, \quad \forall t \in I.$$

This is the classical case considered by Ramsey [**44**] and Koopmans [**29**]. It also corresponds in equilibrium theory to the case considered by Bewley [**7**].

In the constant returns case $\xi(t)$ is absent altogether. In this case there exists a *closed convex cone* $C \subset R_+^n \times R_+^{2n}$ such that

$$(4.7) \qquad \Gamma(t) = C, \quad \forall t \in I.$$

This is the very classical technology of Walras and Cassel, revived by Leontief and extended to the temporal context by von Neumann [**53**]. It was subsequently extensively used by Koopmans [**26**], Gale [**23**], Radner [**43**], McKenzie [**39**], and McFadden [**38**] among others.

In the technical change case although the nonproduced resource may be present, it is the presence of the technical change component of $\xi(t)$ which leads to an increasing family of technology sets

$$(4.8) \qquad \Gamma(\tau) \subset \Gamma(t), \quad \tau < t, \forall t \in I.$$

This is the case considered by Mirrlees [**41**], Brock-Gale [**9**], and Arrow-Kurz [**2**], among others.

5. PROGRAM SPACE

The idea is now to choose a program space that reflects the capacity of the economy for producing a stream of flow output over time which is either bounded or unbounded.

DEFINITION 5.1: I shall call the pair $(\Gamma, \mathcal{Q})$ where $\mathcal{Q}$ is defined by (4.4) and (4.5), the *technology* of the economy. Consider a function $\gamma \in \mathfrak{M}$ satisfying the following conditions:

(i) $0 < \gamma(\tau) < \infty$ a.e.

(ii) For all $w \in \mathfrak{M}^{2n}$ for which v satisfies (4.3) and for which

$$(5.1) \qquad ((\mathcal{Q}v)(t), w(t)) \in \Gamma(t) \quad \text{a.e.}$$

we have

$$(5.2) \qquad \|w(t)\| \leqq \gamma(t) \quad \text{a.e.}$$

Such a function will be called an *expansion function for the technology* $(\Gamma, \mathcal{Q})$.

I make the following basic compactness assumption.

ASSUMPTION 3: *$(\Gamma, \mathcal{Q})$ has an expansion function γ.*

The three principal types of technology considered in the previous section now fall into two categories depending on whether the associated expansion function is *bounded* or *unbounded*. If γ is bounded it is natural to choose the program space

$$\mathcal{W} = \mathcal{L}_{\infty}^{2n}(\lambda). \tag{5.3}$$

This case is examined in Section 8. If γ is unbounded it is natural to choose the program space

$$\mathcal{W} = \mathcal{L}_{1}^{2n}(\mu), \tag{5.4}$$

μ being the measure associated with a density function θ satisfying

$$\int_I \gamma(\tau)\theta(\tau)\,d\tau < \infty. \tag{5.5}$$

This case is examined in Sections 9 and 10. Note that in this latter case if there exists $r > 1$ such that

$$\int_I \gamma(\tau)^{1-r}\,d\tau < \infty, \tag{5.6}$$

then the density function

$$\theta(t) = \alpha\gamma(t)^{-r}, \quad \alpha > 0, \tag{5.7}$$

satisfies (5.5), where α is a normalization constant.

REMARK 5.2: *It is useful to introduce notation which simultaneously covers the program spaces* (5.3) *and* (5.4). To this end I shall write $\mathcal{W} = \mathcal{L}_p^{2n}(\nu)$ with $p = 1$ or ∞, $\nu = \lambda$ or μ, integrating with respect to the measure ν in the standard way.

DEFINITION 5.3: Let $w \in \mathfrak{M}^{2n}$. If w is a program satisfying (4.4) and (5.1) then w will be called *production feasible from* $\bar{z}_0$. Let $\mathcal{G}$ denote the set of all such programs.

I shall now introduce an important function that leads directly to an *integral functional* whose *effective domain* is $\mathcal{G}$.

DEFINITION 5.4: The extended real-valued function $\psi_\Gamma: R_+^{n+m} \times R_+^{2n} \times I \to \bar{R}$ defined by

$$\psi_\Gamma(\zeta, \omega, t) = \begin{cases} 0 & \text{if } (\zeta, \omega) \in \Gamma(t) \\ -\infty & \text{if } (\zeta, \omega) \notin \Gamma(t) \end{cases} \tag{5.8}$$

is called the *indicator function* of the technology correspondence $\Gamma(t)$.

DEFINITION 5.5: Given an extended real-valued function $g: R^k \times I \to \bar{R}$ we define

$$E_g(t) = \{(\chi, \alpha) \in R^k \times R \mid \alpha \leqq g(\chi, t)\}$$

as the *subgraph correspondence* of g. The function g is said to be a *normal integrand* if $E_g(t)$ is a closed-valued measurable correspondence.

REMARK 5.6: *Under Assumption* 1 *the indicator function* ψ_Γ *is a normal integrand* [**46**, p. 177].

REMARK 5.7: *If* $g: R^k \times I \to \bar{R}$ *is a normal integrand and* $v \in \mathfrak{M}^k$, *then* $g(v(t), t) \in \mathfrak{M}$ [**46**, p. 174].

REMARK 5.8: By Remarks 5.6 and 5.7 if Assumptions 1 and 2 are satisfied and if $w \in \mathfrak{M}^{2n}$ satisfies (4.3), then $\psi_\Gamma((\mathcal{Q}v)(t), w(t), t) \in \mathfrak{M}$. If in addition Assumption 3 is satisfied, then ψ_Γ induces an extended real-valued *integral functional* on the program space $\mathcal{L}_p^{2n}(\nu)$ defined by

$$(5.9) \qquad \Psi(w) = \int_I \psi_\Gamma((\mathcal{Q}v)(\tau), w(\tau), \tau) d\nu(\tau).$$

It follows from Definition 5.3 and from standard properties of the Lebesgue integral that

$$(5.10) \qquad \mathcal{G} = \operatorname{dom}\Psi = \{ w \in \mathcal{L}_p^{2n}(\nu) \mid \Psi(w) > -\infty \}.$$

6. PREFERENCES

I shall be concerned with preference orderings over infinite horizon programs which can be represented by *integral functionals*. An integral is the simplest functional operation. Being in the present context a temporal average, it has only limited power to discriminate among programs. The class of preference orderings representable by integral functionals is thus a small subset of the set of all preference orderings satisfying P.2 and P.3. This issue and the related representation problem has been examined by Leontief [**31**], Debreu [**17**, **18**], and Koopmans [**30**]. I shall not enter into this further. I shall however attempt to overcome some of the *independence* that characterizes standard integral representations, along lines that generalize the approach of Ryder-Heal [**49**].

Let $y(t) = (y_1(t), \ldots, y_n(t))$ denote the vector of stocks of *durable consumption goods* available at time t and let $\sigma(t) = (\sigma_1(t), \ldots, \sigma_k(t))$ summarize the current *state of tastes* at time t. It will be useful to let

$$(6.1) \qquad \bar{y}(t) = (y(t), \sigma(t)), \quad t \in I.$$

Current utility $u(t)$ depends on the stocks of the durable consumption goods, the current state of tastes, and the current flow of consumption

$$(6.2) \qquad u(t) = u(\bar{y}(t), c(t), t), \quad t \in I.$$

ASSUMPTION 4: (i) $u(\chi, \zeta, t): R^{n+k} \times R^n \times I \to \bar{R}$ *is upper semicontinuous and concave in* (χ, ζ), $\forall t \in I$. (ii) $C(t) = \operatorname{dom} u(\cdot, t)$ *has a nonempty interior* $\forall t \in I$. (iii) $u(\chi, \zeta, \cdot) \in \mathfrak{M}$, $\forall (\chi, \zeta) \in R^{n+k} \times R^n$.

REMARK 6.1: $C(t)$ denotes the *consumption set* at time t and has been included for simplicity into the effective domain of $u(t)$.

REMARK 6.2: *Under Assumption* 4, $u(\chi, \zeta, t)$ *is a normal integrand* [**46**, p. 176].

I need to make the relation between $\bar{y}(t)$ and $c(t)$ more precise.

ASSUMPTION 5: *There exists an* $(n+k) \times (n+k)$ *matrix-valued measurable function* $b_0(t)$, *with* $b_0(0) = I_{n+k}$, *and an* $(n+k) \times n$ *matrix-valued measurable function* $b(t,\tau)$, *such that on every finite interval* $[0,s] \subset I$, $b_0(\cdot)$, $b(\cdot,\tau)$ $\forall \tau \in I$, $b(t,\cdot)$ $\forall t \in I$, *are essentially bounded. The ith row* $b_i(t,\tau)$ *satisfies for each* $t \in I$

$$(6.3) \qquad \frac{b_i(t,\cdot)}{\theta(\cdot)} \in \mathcal{L}^n_{p'}(\nu), \quad i = 1, \ldots, n+k,$$

where $(1/p) + (1/p') = 1$. *For every* $c \in \mathcal{L}^n_p(\nu)$, $\bar{y}_0 \in R^{n+k}_+$, $\bar{y}(t)$ *is determined by the relation*

$$(6.4) \qquad \bar{y}(t) = b_0(t)\bar{y}_0 + \int_I b(t,\tau)c(\tau)\,d\tau, \quad t \in I.$$

REMARK 6.3: For fixed $t \in I$, let $g_i(\tau) = b_i(t,\tau)/\theta(\tau)$, $i = 1, \ldots, n+k$. If $c \in \mathcal{L}^n_p(\nu)$ and $g_i \in \mathcal{L}^n_{p'}(\nu)$ by (6.3), then by Hölder's inequality [**48**, p. 244]

$$\left| \int_I c(\tau) g_i(\tau)\,d\nu(\tau) \right| \leqq \|c\|_p \|g_i\|_{p'} < \infty, \quad i = 1, \ldots, n+k,$$

so that the integral in (6.4) is well-defined. The restriction (6.3) on the *preference kernel* $b(t,\tau)$ implies that the weight attached to future consumption must go to zero sufficiently rapidly.

REMARK 6.4: It will be convenient to write the affine transformation in (6.4) as

$$(6.5) \qquad \bar{y}(t) = (\mathcal{B}c)(t), \quad t \in I,$$

where $\mathcal{B}$ depends on the *preference kernel* $b(t,\tau)$, $b_0(t)$, and the *initial condition* $\bar{y}_0$.

DEFINITION 6.5: A program $(c,v) \in \mathcal{L}^{2n}_p(\nu)$ satisfying

$$(6.6) \qquad ((\mathcal{B}c)(t), c(t)) \in C(t) \quad \text{a.e.}$$

will be called a program *consumption feasible from* y_0. Let $\mathcal{C}$ denote the set of all such programs. A program (c,v) will be called *feasible* if it is both production and consumption feasible. The *set of all feasible programs* is denoted by $\mathcal{F}$. By definition

$$(6.7) \qquad \mathcal{F} = \mathcal{C} \cap \mathcal{G}.$$

DEFINITION 6.6: I define a *discount factor* as a function $\Delta \in \mathfrak{M}$ such that

$$(6.8)\quad 0 < \Delta(t) < \infty \quad \text{a.e.}$$

In the analysis that follows a crucial role is played by the existence of *bounds* on the current utility function (6.2). Such bounds can be obtained in a variety of ways. It is convenient to begin with the following formulation.

ASSUMPTION 6: *There exists functions* $\underline{\alpha}$, $\bar{\alpha} \in \mathfrak{M}$ *satisfying*

$$(6.9)\quad -\infty < \int_I \underline{\alpha}(\tau)\Delta(\tau)\,d\tau, \qquad \int_I \bar{\alpha}(\tau)\Delta(\tau)\,d\tau < \infty$$

such that, recalling $\mathcal{G}$ *from* (5.10),

$$(6.10)\quad u((\mathfrak{B}c)(t), c(t), t) < \bar{\alpha}(t) \quad \text{a.e.} \quad \textit{for all } (c,v) \in \mathcal{G},$$

$$(6.11)\quad \underline{\alpha}(t) < u((\mathfrak{B}\underline{c})(t), \underline{c}(t), t) \quad \text{a.e.} \quad \textit{for some } (\underline{c}, \underline{v}) \in \mathcal{G}.$$

REMARK 6.7: In view of Remarks 5.7, 6.2, Definition 6.6, and Assumptions 4–6, the current utility function u and discount factor Δ may be combined with the operator $\mathfrak{B}$ to form the extended real-valued *integral functional*

$$(6.12)\quad U(w) = U(c) = \int_I u((\mathfrak{B}c)(\tau), c(\tau), \tau)\Delta(\tau)\,d\tau$$

defined on the feasible subset (6.7) of the program space $\mathfrak{W} = \mathfrak{L}_p^{2n}(\nu)$. The *preference ordering* on $\mathfrak{F}$, that U represents, by the definition

$$(6.13)\quad w' \succsim w \quad \textit{if and only if} \quad U(w') \geqq U(w)$$

clearly satisfies Assumption P.2. In the following section it will be shown to satisfy Assumption P.3 for the locally convex topologies introduced in Section 3. As in (5.10) Definition 6.5 implies

$$(6.14)\quad \mathcal{C} = \operatorname{dom} U = \left\{ w \in \mathfrak{L}_p^{2n}(\nu) \mid U(w) > -\infty \right\}.$$

7. EXISTENCE THEOREM

In this section I shall show that the preference function $U(w)$ is upper semicontinuous and the feasible set $\mathfrak{F}$ is compact in the weak* and weak topologies (3.2) and (3.7).

PROPOSITION 7.1: *Let Assumptions* 1–6 *be satisfied and let*

$$U(w) = \int_I u((\mathfrak{B}c)(\tau), c(\tau), \tau)\Delta(\tau)\,d\tau, \quad w \in \mathfrak{F}.$$

(i) *If the expansion function* γ *is bounded, then* $\mathcal{F} \subset \mathcal{L}_{\infty}^{2n}(\lambda)$, $\mathcal{F} \neq \emptyset$, *and the preference function* $U(w)$ *is upper semicontinuous on* $\mathcal{F}$ *in the* $\sigma(\mathcal{L}_{\infty}^{2n}, \mathcal{L}_{1}^{2n})$ *topology.*
(ii) *If the expansion function* γ *is unbounded, then* $\mathcal{F} \subset \mathcal{L}_{1}^{2n}(\mu)$, $\mathcal{F} \neq \emptyset$, *and the preference function* $U(w)$ *is upper semicontinuous on* $\mathcal{F}$ *in the* $\sigma(\mathcal{L}_{1}^{2n}, \mathcal{L}_{\infty}^{2n})$ *topology.*

PROOF: I shall prove (i) and (ii) simultaneously using the notation of Remark 5.2 so that

$$\mathcal{W} = \mathcal{L}_p^{2n}(\nu) \quad \text{with} \quad (p,\nu) = \begin{cases} (\infty, \lambda), \\ (1, \mu), \end{cases}$$

with topology $\Upsilon = \sigma(\mathcal{L}_p^{2n}, \mathcal{L}_{p'}^{2n})$, $(1/p) + (1/p') = 1$. $\mathcal{F} \subset \mathcal{L}_p^{2n}(\nu)$ follows at once from Assumption 3, while $\mathcal{F} \neq \emptyset$ follows from (6.11) in Assumption 6. Let $x = (\bar{y}, \bar{z})$. Since the technology kernel $a(t,\tau)$ is a Volterra kernel, $a(t,\tau) \equiv 0$ for $\tau > t$, if we define

$$k_0(t) = \begin{bmatrix} b_0(t) & 0 \\ 0 & a_0(t) \end{bmatrix}, \quad k(t,\tau) = \begin{bmatrix} b(t,\tau) & 0 \\ 0 & a(t,\tau) \end{bmatrix},$$

then (4.4) and (6.4) may be combined into the single equation

$$(7.1) \qquad x(t) = k_0(t)x_0 + \int_I k(t,\tau)w(\tau)\,d\tau, \quad t \in I.$$

The affine transformation in (7.1) may be written as

$$x(t) = (\mathcal{K}w)(t), \quad t \in I.$$

Let $\chi = (\chi_1, \chi_2) \in R^{n+k} \times R^{n+m}$, $\zeta = (\zeta_1, \zeta_2) \in R^n \times R^n$. Consider the extended real-valued function f defined by

$$f(\chi, \zeta, t) = u(\chi_1, \zeta_1, t)\frac{\Delta(t)}{\theta(t)} + \psi_\Gamma(\chi_2, \zeta, t).$$

It is clear that f is a normal integrand. The integral functional

$$(7.2) \qquad F(w) = \int_I f((\mathcal{K}w)(\tau), w(\tau), \tau)\,d\nu(\tau)$$

is defined on all of $\mathcal{L}_p^{2n}(\nu)$. Proving (i) and (ii) reduces to showing that $F(w)$ is upper semicontinuous on $\mathcal{L}_p^{2n}(\nu)$. Consider the function

$$h(\chi, \zeta, t) = \beta(t) - f(\chi, \zeta, t), \quad \beta(t) = \bar{\alpha}(t)\frac{\Delta(t)}{\theta(t)}.$$

(5.8) and (6.10) in Assumption 6 imply

$$(7.3) \qquad h((\mathcal{K}w)(t), w(t), t) > 0 \quad \text{a.e.} \quad \forall w \in \mathcal{L}_p^{2n}(\nu).$$

Since

$$H(w) = \int_I h((\mathcal{K}w)(\tau), w(\tau), \tau)\, d\nu(\tau) = \beta^* - F(w)$$

where by (6.9)–(6.11)

$$-\infty < \beta^* = \int_I \beta(\tau)\, d\nu(\tau) < \infty,$$

it suffices to show that $H(w)$ is lower semicontinuous on $\mathcal{L}_p^{2n}(\nu)$. Let $\{w_n\}_{n=1}^{\infty} \subset \mathcal{L}_p^{2n}(\nu)$ be such that

(7.4) $\quad w_n \to w^* \quad \text{as } n \to \infty, \quad \sigma\left(\mathcal{L}_p^{2n}, \mathcal{L}_{p'}^{2n}\right)$

and let $H_n = H(w_n)$, $\underline{H} = \underline{\lim}_{n\to\infty} H_n$, $H^* = H(w^*)$. (7.3) implies $0 < \underline{H} \leqq \infty$. If $\underline{H} = \infty$ then $H^* \leqq \underline{H}$. Suppose therefore $0 < \underline{H} < \infty$. Let $\{w_m\}_{m=1}^{\infty} \subset \{w_n\}_{n=1}^{\infty}$ be a subsequence such that

(7.5) $\quad H(w_m) = H_m \to \underline{H} \quad \text{as } m \to \infty.$

By (3.3) and (3.8), (7.4) is equivalent to

(7.6) $\quad \lim_{m\to\infty} \int_I (w_m(\tau) - w^*(\tau)) g(\tau)\, d\nu(\tau) = 0, \quad \forall g \in \mathcal{L}_{p'}^{2n}(\nu).$

Let $k_i(t,\tau)$ denote the ith row of $k(t,\tau)$. Since by Assumption 5 for $i = 1, \ldots, n+k$, (6.3) is satisfied and since by Assumption 2 for $i = n+k+1, \ldots, (n+k)+(n+m)$, $k_i(t,\tau)$ is a Volterra kernel, if for each $t \in I$ and each $i = 1, \ldots, (n+k)+(n+m)$ we let

$$g(\tau) = \frac{k_i(t,\tau)}{\theta(\tau)}, \quad \tau \in I,$$

then (7.6) implies

$$\lim_{m\to\infty} \int_I k(t,\tau) w_m(\tau)\, d\tau = \int_I k(t,\tau) w^*(\tau)\, d\tau, \quad \forall t \in I,$$

so that by (7.1)

(7.7) $\quad \lim_{m\to\infty} x_m(t) = \lim_{m\to\infty} (\mathcal{K}w_m)(t) = (\mathcal{K}w^*)(t) = x^*(t), \quad \forall t \in I.$

To be able to apply Fatou's Lemma we need to improve on the weak* (weak) convergence (7.4). Let $W_k = \{w_m\}_{m=k}^{\infty}$, $k = 1, 2, \ldots$, and let $\overline{W}_k^w$, co W_k denote the weak closure and the convex hull of W_k. By (7.4), $w^* \in \overline{W}_k^w \subseteq \text{co}\, \overline{W}_k^w$, $k = 1, 2, \ldots$. By [**19**, Theorem V.3.13, p. 422] *the weak and strong closures of a convex subset of a locally convex space are identical*, $\overline{\text{co}\, W_k}^w = \overline{\text{co}\, W_k}^s$, so that $w^* \in \overline{\text{co}\, W_k}^s$, $k = 1, 2, \ldots$. Thus there exist $w_k' \in \text{co}\, W_k$, $k = 1, 2, \ldots$ such that

(7.8) $\quad \|w_k' - w^*\|_p \to 0 \quad \text{as } k \to \infty.$

Let $[w_m] = \{w_m\}_{m=1}^{\infty}$; then $w_k' = \Omega_k[w_m]$ where

$$\Omega_k[w_m] = \sum_{m=k}^{\bar{k}} \lambda_{km} w_m, \quad \lambda_{km} \geqq 0, \qquad \sum_{m=k}^{\bar{k}} \lambda_{km} = 1, \quad k \leqq \bar{k} < \infty, \qquad k = 1, 2, \ldots \tag{7.9}$$

Let $p = 1$. Since convergence in the *norm* topology in $\mathfrak{L}_1^{2n}(\nu)$ implies convergence in *measure* [**40**, p. 15] and since by the theorem of Riesz a sequence which converges in measure contains a subsequence which converges *almost everywhere* [**48**, p. 92], a subsequence $\{\Omega_r[w_m]\}_{r=1}^{\infty} \subset \{\Omega_k[w_m]\}_{k=1}^{\infty}$ can be selected in such a way that

$$\lim_{r\to\infty} \Omega_r[w_m(t)] = w^*(t), \quad \nu\text{—a.e.} \tag{7.10}$$

(7.10) follows at once from (7.8) when $p = \infty$. (7.10) is the desired improvement in the weak* (weak) convergence (7.4). (7.9) applied to (7.7) gives

$$\lim_{r\to\infty} \Omega_r[x_m(t)] = x^*(t), \quad \nu\text{—a.e.} \tag{7.11}$$

By Assumptions 1 and 4 and by the definitions of f and h, it follows that $h(\cdot, \cdot, t)$ is *lower semicontinuous* for all $t \in I$. Thus by virtue of (7.10) and (7.11)

$$h(x^*(t), w^*(t), t) \leqq \varliminf_{r\to\infty} h(\Omega_r[x_m(t)], \Omega_r[w_m(t)], t), \quad \nu\text{—a.e.}$$

Integration yields

$$\begin{aligned} H(w^*) &= \int_I h(x^*(t), w^*(t), t)\, d\nu(t) \\ &\leqq \int_I \varliminf_{r\to\infty} h(\Omega_r[x_m(t)], \Omega_r[w_m(t)], t)\, d\nu(t). \end{aligned} \tag{7.12}$$

(7.3), (7.10), and (7.11) allow us to apply Fatou's Lemma [**48**, p. 226]:

$$\begin{aligned} &\int_I \varliminf_{r\to\infty} h(\Omega_r[x_m(t)], \Omega_r[w_m(t)], t)\, d\nu(t) \\ &\leqq \varliminf_{r\to\infty} \int_I h(\Omega_r[x_m(t)], \Omega_r[w_m(t)], t)\, d\nu(t). \end{aligned} \tag{7.13}$$

From Assumptions 1 and 4 and the definitions of f and h, it follows that $h(\cdot, \cdot, t)$ is *convex* for all $t \in I$. Taken in conjunction with the *linearity* of the integral this yields

$$\begin{aligned} &\varliminf_{r\to\infty} \int_I h(\Omega_r[x_m(t)], \Omega_r[w_m(t)], t)\, d\nu(t) \\ &\leqq \varliminf_{r\to\infty} \Omega_r\left[\int_I h(x_m(t), w_m(t), t)\, d\nu(t)\right] \end{aligned} \tag{7.14}$$

so that by (7.5), (7.9), and (7.12)–(7.14),

$$H^* \leqq \varliminf_{r\to\infty} \Omega_r[H_m] = \underline{H}.$$

Q.E.D.

DEFINITION 7.2: $\mathfrak{F} \subset \mathfrak{L}_1^{2n}(\mu)$ is said to be *uniformly integrable* if for any $\epsilon > 0$ there exists $\alpha > 0$ such that

$$\int_{I(w,\alpha)} \|w(\tau)\|\, d\mu(\tau) < \epsilon, \quad \forall w \in \mathfrak{F},$$

where $I(w,\alpha) = \{t \in I \mid \|w(t)\| \geqq \alpha\}$.

I shall use the following classical characterization of weak compactness in $\mathfrak{L}_1^{2n}(\mu)$ [**40**, p. 20]:

LEMMA 7.3: *Let $\mathfrak{F}$ be a $\sigma(\mathfrak{L}_1^{2n}, \mathfrak{L}_\infty^{2n})$ closed subset of $\mathfrak{L}_1^{2n}(\mu)$. $\mathfrak{F}$ is $\sigma(\mathfrak{L}_1^{2n}, \mathfrak{L}_\infty^{2n})$ compact if and only if $\mathfrak{F}$ is uniformly integrable.*

To establish weak* compactness in $\mathfrak{L}_\infty^{2n}(\lambda)$ we use the theorem of Alaoglu [**19**, p. 424].

LEMMA 7.4: *Let V be a normed linear space with dual V^* and let $\mathfrak{F}$ be a $\sigma(V^*, V)$ closed subset of V^*. If $\mathfrak{F}$ is bounded in the norm topology of V^*, then $\mathfrak{F}$ is $\sigma(V^*, V)$ compact.*

PROPOSITION 7.5: *Let Assumptions* 1–3 *be satisfied.* (i) *If the expansion function γ is bounded, then $\mathfrak{F} \subset \mathfrak{L}_\infty^{2n}(\lambda)$ is compact in the $\sigma(\mathfrak{L}_\infty^{2n}, \mathfrak{L}_1^{2n})$ topology.* (ii) *If the expansion function γ is unbounded, then $\mathfrak{F} \subset \mathfrak{L}_1^{2n}(\mu)$ is compact in the $\sigma(\mathfrak{L}_1^{2n}, \mathfrak{L}_\infty^{2n})$ topology.*

PROOF: Proposition 7.1 applied to the integral functional $\Psi(w)$ in (5.9) implies that $\Psi(w)$ is upper semicontinuous on $\mathfrak{L}_p^{2n}(\nu)$ in the $\sigma(\mathfrak{L}_p^{2n}, \mathfrak{L}_{p'}^{2n})$ topology. In view of (5.10), $\mathfrak{G}$ is $\sigma(\mathfrak{L}_p^{2n}, \mathfrak{L}_{p'}^{2n})$ closed. Proposition 7.1 and (6.14) imply $\mathfrak{C}$ is $\sigma(\mathfrak{L}_p^{2n}, \mathfrak{L}_{p'}^{2n})$ closed. By (6.7) $\mathfrak{F}$ is $\sigma(\mathfrak{L}_p^{2n}, \mathfrak{L}_{p'}^{2n})$ closed. Lemma 7.4 with $V = \mathfrak{L}_1^{2n}(\lambda)$, $V^* = \mathfrak{L}_\infty^{2n}(\lambda)$ and Assumption 3 imply (i). To establish (ii) note that Assumption 3 implies $\gamma \in \mathfrak{L}_1(\mu)$. Thus by definition, for any $\epsilon > 0$ there exists $\alpha > 0$ such that

$$\int_{I(\gamma,\alpha)} \gamma(\tau)\, d\mu(\tau) < \epsilon.$$

Since by (5.2) $\|w(t)\| \leqq \gamma(t)$ a.e. $\forall w \in \mathfrak{F}$, $I(w,\alpha) \subset I(\gamma,\alpha)$, $\forall w \in \mathfrak{F}$. Thus

$$\int_{I(w,\alpha)} \|w(\tau)\|\, d\mu(\tau) \leqq \int_{I(\gamma,\alpha)} \|w(\tau)\|\, d\mu(\tau) \leqq \int_{I(\gamma,\alpha)} \gamma(\tau)\, d\mu(\tau) < \epsilon,$$

$$\forall w \in \mathfrak{F}.$$

By definition (7.2), $\mathfrak{F}$ is uniformly integrable. By Lemma 7.3 $\mathfrak{F}$ is $\sigma(\mathfrak{L}_1^{2n}, \mathfrak{L}_\infty^{2n})$ compact. *Q.E.D.*

THEOREM 7.6: *Let Assumptions* 1–6 *and P.*1 *be satisfied. If the preference ordering of Assumption P.*1 *is represented by* (6.12) *and* (6.13), *then there exists an optimal infinite horizon program.*

PROOF: P.2 is immediate. Proposition 7.1 implies $\mathfrak{F} \neq \emptyset$ and P.3 with $\Upsilon = \sigma(\mathfrak{L}_p^{2n}, \mathfrak{L}_{p'}^{2n})$, $(p, p') = (\infty, 1)$ or $(1, \infty)$. Proposition 7.5 implies $\mathfrak{F}$ is compact in the same topologies. Theorem 2.5 gives the result. *Q.E.D.*

DEFINITION 7.7: Let $C(t) = R_+^{n+k} \times R_+^n$ a.e. The utility function $u(\chi, \zeta, t)$ has a *stationary homogeneous decomposition* if (i) there exists a continuous increasing real-valued function $\phi(s)$ and (ii) there exists an upper semicontinuous concave function $h(\chi, \zeta)$ satisfying: (a) $h(\chi, \zeta) > 0$ whenever $0 \neq (\chi, \zeta) \geqq 0$; (b) $h(\lambda\chi, \lambda\zeta) = \lambda h(\chi, \zeta)$, $\lambda > 0$, $0 \neq (\chi, \zeta) \geqq 0$ such that

$$(7.15) \qquad u(\chi, \zeta, t) = \begin{cases} \phi(h(\chi, \zeta)), & (\chi, \zeta) \geqq 0, \\ -\infty, & \text{elsewhere.} \end{cases}$$

I shall let

$$(7.16) \qquad h^* = h(\chi^*, \zeta^*) = \sup_{\substack{(\chi,\zeta) \geqq 0 \\ \|(\chi,\zeta)\| \leqq 1}} h(\chi, \zeta).$$

COROLLARY 7.8: *If Assumption 6 is replaced by the following conditions:* (i) *the utility function has a stationary homogeneous decomposition*; (ii) $\|\bar{y}(t)\| \leqq \gamma(t)$ *a.e. for all* $(c, v) \in \mathfrak{F}$; (iii) *there exists* $(c, v) \in \mathfrak{F}$ *such that* $(\underline{y}(t), \underline{c}(t)) \geqq \underline{\gamma}(t)(\chi^*, \zeta^*)$ *a.e.*;

$$(\text{iv}) \qquad -\infty < \int_I \phi(\underline{\gamma}(\tau)h^*)\Delta(\tau)\, d\tau, \quad \int_I \phi(\gamma(\tau)h^*)\Delta(\tau)\, d\tau < \infty;$$

then there exists an optimal infinite horizon program.

PROOF: We need to establish (6.9)–(6.11). By (5.2) and (ii), $\|\bar{y}(t), c(t)\| \leqq \gamma(t)$ a.e. $\forall (c, v) \in \mathfrak{F}$. Then (i) and (7.16) imply

$$(7.17) \qquad u(\bar{y}(t), c(t)) \leqq \phi(\gamma(t)h(\chi^*, \zeta^*)), \quad \forall (c, v) \in \mathfrak{F};$$

(i) and (iii) imply

$$(7.18) \qquad u(\underline{y}(t), \underline{c}(t)) \geqq \phi(\underline{\gamma}(t)h(\chi^*, \zeta^*));$$

(iv), (7.17), and (7.18) give the result. *Q.E.D.*

COROLLARY 7.9: *Let* $\phi(s) = s^\beta/\beta$, $\beta \in (-\infty, 1)$, $\beta \neq 0$ *or* $\phi(s) = \log_k s$, $k > 1$ *and let* (*iv*) *in Corollary* 7.8 *be replaced by*

$$(7.19) \qquad -\infty < \beta\int_I \underline{\gamma}(\tau)^\beta \Delta(\tau)\, d\tau, \quad \beta\int_I \gamma(\tau)^\beta \Delta(\tau)\, d\tau < \infty,$$

$$(7.20) \qquad -\infty < \int_I \log_k(\underline{\gamma}(\tau))\Delta(\tau)\, d\tau, \quad \int_I \log_k(\gamma(\tau))\Delta(\tau)\, d\tau < \infty;$$

then there exists an optimal infinite horizon program.

REMARK 7.10: If $\beta > 0$ (< 0) only the second (first) inequality in (7.19) imposes a restriction. The reason is straightforward. When $\beta > 0$ (< 0) the utility function is bounded below (above) but unbounded above (below).

If there is *no discounting* or *upcounting* of the future so that $\Delta(t) \geqq 1$ a.e., then by the second inequality in (7.19) we must have $\beta < 0$, so that u is bounded above. The first inequality in (7.19) then requires that $\underline{\gamma}(t)$ (and hence $\gamma(t)$) grow at least at a certain rate, so as to force the integral to converge. In this case (7.20) will not give existence.

If γ (and hence $\underline{\gamma}$) is bounded,

$$0 < \underline{\gamma}(t) \leqq \gamma(t) \leqq b \quad \text{a.e.},$$

then (7.19) and (7.20) reduce to

$$(7.21) \qquad \int_I \Delta(\tau)\, d\tau < \infty.$$

If $\beta = 1$, then u is *homogeneous of degree 1*. Two common examples are obtained by letting h in (7.15) be given by

$$h(\chi, \zeta) = a\chi + b\zeta, \quad (a,b) > 0,$$

$$h(\chi, \zeta) = \left(\prod_{i=1}^{n+k} x_i^{a_i}\right)\left(\prod_{i=1}^{n} \zeta_i^{b_i}\right), \quad (a,b) > 0, \quad \sum_{i=1}^{n+k} a_i + \sum_{i=1}^{n} b_i = 1.$$

In this case the existence conditions (7.19) reduce to the single condition

$$(7.22) \qquad \int_I \gamma(\tau)\Delta(\tau)\, d\tau < \infty.$$

In the case where the functions in (7.19) and (7.20) are constant exponential functions

$$\underline{\gamma}(t) = \gamma(t) = e^{\rho t}, \quad \rho \in (-\infty, \infty), \quad \Delta(t) = e^{-\delta t}, \quad \delta \in (-\infty, \infty),$$

$$k = e,$$

the existence conditions (7.19) and (7.20) reduce to the single condition

$$(7.23) \qquad \delta > \rho\beta, \quad \beta \in (-\infty, 1].$$

If we let $\beta = 1 - \eta$, then (7.23) becomes

$$(7.24) \qquad \delta + \eta\rho > \rho.$$

The basic existence condition in this case reduces to the following: *the rate of impatience on a path of maximal growth of consumption must exceed the maximal rate of growth of output.*[4]

[4]The existence condition (7.24) is closely related to D. Bernoulli's solution of the St. Petersburg Paradox [**1**, **6**], for the condition requires that the marginal utility of consumption fall off sufficiently fast and the discount rate be sufficiently great relative to the rate at which the maximal consumption stream increases.

8. DECREASING RETURNS

In this section I shall consider a simple example of an economy whose program space is $\mathfrak{L}_{\infty}^{2n}(\lambda)$. I do not strive for generality. Rather, my objective is to show the way in which the theorem of the preceding section can be applied.

To focus attention on the essential ideas I consider the simplest case where the preference function u and the technology set Γ are time independent. In (6.1) I take $\bar{y}(t) = y(t)$, in (6.2) I let $u(t) = u(y(t), c(t))$ where $u(\cdot, \cdot)$ is upper semicontinuous and concave and $C(t) = R_+^n \times R_+^n$, $\forall t \in I$. The stocks of durable consumption goods are assumed to be finite lived or more accurately to *depreciate exponentially*.

$$(8.1) \qquad \dot{y}(t) = -My(t) + c(t), \quad t \in I, \quad y(0) = y_0 > 0,$$

where $M > 0$ is a diagonal matrix. (8.1) implies that (6.4) becomes

$$(8.2) \qquad y(t) = e^{-Mt}y_0 + \int_0^t e^{-M(t-\tau)}c(\tau)\,d\tau, \quad t \in I.$$

In (6.8) I take $\Delta(t) = e^{-\delta t}$, so that (6.12) reduces to

$$(8.3) \qquad U(w) = \int_I u(y(\tau), c(\tau))e^{-\delta\tau}\,d\tau.$$

On the production side the vector of nonproduced resources and the vector indicating the current state of technical change, are taken as fixed, $\xi(t) = \xi_0$ $\forall t \in I$. Thus I may take $\bar{z}(t) = z(t)$ in (4.1). The technology set is a fixed compact convex subset of $R_+^{2n} \times R^n$ (to allow disinvestment), $\Gamma(t) = \Gamma$ $\forall t \in I$. Depreciation of capital goods which, in conjunction with diminishing returns, accounts for the boundedness of Γ, is built into the technology set so that

$$(8.4) \qquad \dot{z}(t) = v(t), \quad t \in I, \quad z(0) = z_0 > 0,$$

and (4.4) reduces to

$$(8.5) \qquad z(t) = z_0 + \int_0^t v(\tau)\,d\tau, \quad t \in I.$$

Assumptions 1–5 are thus satisfied and the expansion function γ in Assumption 3 is bounded. To ensure that Assumption 6 is satisfied I add the following *productivity condition*.

ASSUMPTION 7: *There exists* $\underline{y}_0 > 0$ *and* $(\underline{z}_0, \underline{c}_0, \underline{v}_0) \in \Gamma$, $(\underline{y}_0, \underline{z}_0) \leqq (y_0, z_0)$ *such that* $(\underline{c}_0, \underline{v}_0) \geqq (M\underline{y}_0, 0)$. *Stocks of capital (consumption) goods are freely disposable.*

THEOREM 8.1: *If the preferences and technology satisfy the conditions of this section, including Assumption 7, and if* $\delta > 0$ *then there exists an optimal infinite horizon program.*

PROOF: It only remains to establish (6.9)–(6.11). Since Γ is compact, for any feasible program there exists $b > 0$ such that $\|c(t)\| \leqq b$ a.e. By (8.2) $\|y(t)\| \leqq e^{-\mu t}\|y_0\| + (b/\mu)(1 - e^{-\mu t}) \leqq \|y_0\| + (b/\mu)$, $\mu > 0$. Since $u(\cdot,\cdot)$ is upper semi-continuous there exists $\bar{u}$ such that $u(y(t), c(t)) \leqq \bar{u} < \infty$ a.e. for any feasible program w. Assumption 7 implies $(y(t), c(t)) = (\underline{y}_0, \underline{c}_0) > 0$ $\forall t \in I$ is feasible. Thus $-\infty < u(\underline{y}_0, \underline{c}_0) = \underline{u}$. Since $\delta > 0$, (6.9)–(6.11) are satisfied with $\underline{\alpha}(t) = \underline{u}$, $\bar{\alpha}(t) = \bar{u}$, $\Delta(t) = e^{-\delta t}$. *Q.E.D.*

EXAMPLE 8.2 (Koopmans [**29**]): Suppose $n = 1$ and that the consumption good is not durable so that (8.2) is absent. (8.3) reduces to

$$U(w) = \int_I u(c(\tau))e^{-\delta\tau}\, d\tau,$$

$C = \operatorname{dom} u = [0, \infty)$. (8.4) and (8.5) are unchanged. Let the *marginal product of capital* be a *positive, nonincreasing* function $g(z)$ which is *integrable* on any finite subinterval of $[0, \infty)$ and let $\lambda > 0$ denote the *depreciation rate* of the capital good. Assumption 7 is replaced by the following assumption:

ASSUMPTION 7′: (i) $\lambda < g(0)$ *and there exists* $0 < z' < \infty$ *such that* $g(z') < \lambda$; (ii) *the capital good is freely disposable.*

Assumption 7′ (i) implies that there exists $\bar{z}$ such that

$$(8.6) \qquad \int_0^{\bar{z}} (g(\zeta) - \lambda)\, d\zeta = 0, \quad 0 < \bar{z} < \infty.$$

As stressed by Koopmans [**29**, p. 237] there is no loss of generality in assuming $0 < z_0 \leqq \bar{z}$. I therefore introduce the function

$$(8.7) \qquad f(z) = \begin{cases} \int_0^{z} g(\zeta)\, d\zeta, & z \in [0, \bar{z}], \\ -\infty, & z \notin [0, \bar{z}], \end{cases}$$

and let the technology set Γ be given by

$$\Gamma = \{(z, c, v) \mid c + v \leqq f(z) - \lambda z, (z, c, v) \geqq (0, 0, -\lambda z)\}.$$

By the definition of $g(z)$, Assumption 7′, (8.6) and (8.7), Γ is a compact convex subset of $R_+^2 \times R$, which satisfies Assumption 7. By Theorem 8.1, if $\delta > 0$ there is an optimal infinite horizon program.

REMARK 8.3: The assumptions of Example 8.2 are those of Koopmans [**29**, p. 261, 275], with two exceptions. First, I make no *differentiability* assumption on $u(c)$ or $f(z)$. Second, I avoid the assumption

$$\delta < g(0) - \lambda$$

needed in Koopmans' analysis to ensure the existence of a *positive* steady state solution of the Euler equations. Neither assumption has a good economic justification. An attempt to relax either of these assumptions leads to complications in the analysis of Koopmans' differential equations (65) [**29**, p. 276]. *These difficulties are inherent in the differential equation approach to existence.* The first difficulty is more of a nuisance—since *subgradients* can replace gradients. The second presents a more serious difficulty since $u'(c) \to \infty$ as $c \to 0$ leads to a *singularity* in the differential equations (65).

9. CONSTANT RETURNS

In this section I establish the existence of an optimal infinite horizon program in a model of an expanding economy that generalizes the classical model of Von Neumann [**53**]. The current utility function u and the technology set Γ are again taken to be time-independent. In addition the utility function u exhibits a regularity property akin to asymptotic homogeneity, while the technology set Γ exhibits constant returns to scale. For this latter property to be meaningful it must be assumed that *nonproduced resources* are always available in whatever amounts are required by the production sector.

Stocks of durable consumption goods live indefinitely so that (6.4) becomes

$$(9.1) \qquad y(t) = y_0 + \int_0^t c(\tau)\, d\tau, \quad y_0 > 0,$$

while the current utility function (6.2) satisfies the following conditions.

ASSUMPTION U.1: *$u(\omega, t): R^{2n} \times I \to \bar{R}$ satisfies* (i) $u(\omega, t) = u(\omega)$ *a.e.*; (ii) $u(\cdot)$ *is upper semicontinuous and concave*; (iii) $C = dom\, u = R_+^{2n}$; (iv) $\omega' > \omega$, *with* $\omega \in R_+^{2n}$, *implies* $u(\omega') > u(\omega)$.

DEFINITION 9.1: If $u(\omega)$ is bounded above, the least upper bound may be taken to be zero. With this normalization, the function $\eta(\omega, \lambda)$ defined for all $(\omega, \lambda) \in R_+^{2n} \times R_+$ with $(\omega, \lambda) > (0, 1)$, by

$$(9.2) \qquad \eta(\omega,\lambda) = 1 - \log_\lambda |u(\lambda\omega)| \quad \textit{or equivalently} \quad |u(\lambda\omega)| = \lambda^{1-\eta(\omega,\lambda)}$$

is called the *exponent* of u at (ω, λ).

REMARK 9.2: It can be shown under Assumption U.1 that the following limits are independent of ω [**38**, p. 269]:

$$\bar{\eta} = \overline{\lim_{\lambda \to \infty}}\, \eta(\omega, \lambda), \quad \underline{\eta} = \underline{\lim}_{\lambda \to \infty}\, \eta(\omega, \lambda), \quad \omega > 0.$$

If in addition $\bar{\eta} = \underline{\eta} = \eta$, we say that u has an *asymptotic exponent* η. We impose the following important regularity condition on u.

ASSUMPTION U.2: *The utility function u has an asymptotic exponent η.*

REMARK 9.3: I shall say that u is *asymptotically homogeneous* if there exist functions $\phi(\cdot)$ and $h(\cdot)$ satisfying the conditions of Definition 7.7 such that

$$(9.3)\qquad |u(\omega)-\phi(h(\omega))|\to 0 \quad \text{as} \quad \|\omega\|\to\infty.$$

If u is asymptotically homogeneous and if its associated ϕ function has an asymptotic exponent β, then U.2 is satisfied and $1-\eta=\beta$.

REMARK 9.4: The following result is an immediate consequence of Definition 9.1. *Let the utility function u satisfy U.1 and U.2. For any $\omega>0$, $\epsilon>0$ there exists $\hat{\lambda}$ such that*

$$(9.4)\qquad \lambda^{\underline{\beta}}<|u(\lambda\omega)|<\lambda^{\bar{\beta}},\quad \forall\lambda>\hat{\lambda},$$

where $\underline{\beta}=1-n-\epsilon$, $\bar{\beta}=1-n+\epsilon$.

To complete the specification of preferences I let $\Delta(t)=e^{-\delta t}$ so that (6.12) reduces to

$$(9.5)\qquad U(w)=\int_I u(y(\tau),c(\tau))e^{-\delta\tau}\,d\tau.$$

On the production side the basic accumulation equation is

$$(9.6)\qquad z(t)=z_0+\int_0^t v(\tau)\,d\tau,\quad z_0>0,\quad t\in I.$$

DEFINITION 9.5: A *reduced process* is a pair of nonnegative vectors $(z(t),q(t))$ where $q(t)=c(t)+v(t)$ is the *current output* producible with the stocks $z(t)$. The set of all reduced processes available at time t is called the *production set* $\Pi(t)$.

ASSUMPTION T.1: $\Pi(t): I\to R_+^n\times R_+^n$ *is a closed, convex valued correspondence satisfying* (i) $\Pi(t)=\Pi$ *a.e.*; (ii) Π *is a cone*; (iii) $(\chi,\zeta)\in\Pi$ *with* $\chi=0$ *implies* $\zeta=0$; (iv) *there exists* $(\chi,\zeta)\in\Pi$ *with* $\zeta>0$; (v) $(\chi,\zeta)\in\Pi$, $\chi'\geqq\chi$, $0\leqq\zeta'\leqq\zeta$ *implies* $(\chi',\zeta')\in\Pi$.

EXAMPLE 9.6: The production set Π in Von Neumann's model [53] is a convex polyhedral cone

$$(9.7)\qquad \Pi=\{(\chi,\zeta)\,|\,(-\chi,\zeta)\leqq(-A,B)\xi,\ \xi\geqq 0,\ \zeta\geqq 0\}$$

where (A,B) is a pair of $n\times m$ matrices satisfying (i) a_{ij}, $b_{ij}\geqq 0$; (ii) for any j there exists i such that $a_{ij}>0$; (iii) for any i there exists j such that $b_{ij}>0$. (i)–(iii) imply (9.7) satisfies Assumption T.1.

EXAMPLE 9.7: Let $f(\chi): R_+^n\to R_+^n$, $f=(f^1,\ldots,f^n)$ satisfy (i) $f^i(\cdot)$ is upper semi-continuous and concave, $i=1,\ldots,n$; (ii) $f^i(\cdot)$ is homogeneous of degree 1, $i=1,\ldots,n$; (iii) $\chi'\geqq\chi$ implies $f(\chi')\geqq f(\chi)$ and $f^i(\chi')>f^i(\chi)$ for some

$i \notin \{j \mid \chi'_j > \chi_j\}$ if $\chi' \neq \chi$. (i)–(iii) imply that the production set

$$(9.8) \qquad \Pi = \{(\chi, \zeta) \mid \zeta \leqq f(\chi), (\chi, \zeta) \geqq 0\}$$

satisfies Assumption T.1. As a special case let $f(\chi) = A\chi$ where A is a nonnegative *indecomposable* $n \times n$ matrix. Example 9.7 is a generalization of [**50**].

To complete the specification of technology, I let Γ be defined in terms of the production set Π as follows:

$$(9.9) \qquad \Gamma = \{(\chi, \zeta, \kappa) \mid (\chi, \zeta + \kappa) \in \Pi, \quad (\zeta, \kappa) \geqq 0\}.$$

DEFINITION 9.8: For any $(\chi, \zeta) \in \Pi$,

$$\rho(\chi, \zeta) = \sup_{\rho \in R} \{\rho \mid \zeta \geqq \rho\chi\}$$

is called the *expansion rate* of the process (χ, ζ).

REMARK 9.9: Let $\chi \in R^n$. I let $\chi \geqslant 0$ denote $\chi_i \geqslant 0$ for $i = 1, \ldots, n$ and $\chi_j > 0$ for some j.

The following result was established by Von Neumann [**53**] in the case where the production set Π is given by (9.7) and was extended to the general production set Π of Assumption T.1 by Gale [**23**].

THEOREM 9.10 (Von Neumann): *If* Π *satisfies Assumption T.*1, *then there exists a vector of prices and an interest rate* (p^*, σ^*), *a commodity vector and an expansion rate* (χ^*, ρ^*) *such that*

$$\text{(i)} \qquad (\sigma^* p^*, p^*)(-\chi, \zeta) \leqq 0 \quad \textit{for all } (\chi, \zeta) \in \Pi;$$

$$\text{(ii)} \qquad \rho^* = \rho(\chi^*, \zeta^*) = \sup_{(\chi, \zeta) \in \Pi} \rho(\chi, \zeta), \quad \zeta^* = \rho^* \chi^*;$$

$$\text{(iii)} \qquad 0 < \rho^* = \sigma^* < \infty, \quad p^* \geqslant 0, \quad \chi^* \geqslant 0.$$

REMARK 9.11: Theorem 9.10 asserts the existence of a price supported process of proportional expansion, or more succinctly, a *proportional expansion equilibrium*, in which the *interest rate* coincides with the *expansion rate*.

DEFINITION 9.12: The production set Π is said to be *regular* if $(\chi, \zeta) \in \Pi$ with $\rho(\chi, \zeta) = \rho^*$ implies $\chi > 0$.

EXAMPLE 9.13: (i) It is easy to check that Π in Example 9.7 is regular. (ii) A sufficient condition for regularity of the Von Neumann model (Example 9.6) has been given by Gale [**23**, p. 295]. $I \subset N = \{1, \ldots, n\}$ is an *independent subset* of goods if there exists $J \subset M = \{1, \ldots, m\}$ such that for each $i \in M \setminus I$ and $j \in J$,

$a_{ij} = 0$, while for each $i \in I$, $b_{ij} > 0$ for some $j \in J$. The Von Neumann model (A, B) is *irreducible* if N has no nontrivial independent subsets. *If the Von Neumann model* (A, B) *is irreducible then it is regular.*

ASSUMPTION T.2: *The production set* Π *is regular.*

REMARK 9.14: To be able to use Theorem 9.10 in a meaningful way it is important that there be a minimal amount of interdependence in the way commodities are produced from one another, for otherwise the economy can break up into subsystems in which the maximal expansion rate ρ^* no longer has any real economic significance. The regularity Assumption T.2 prevents this kind of degeneracy.

THEOREM 9.15: *Let Assumptions* $(U.1, 2;\ T.1, 2)$ *be satisfied. If the preference ordering is represented by* (9.5) *and if*

$$\delta + \eta\rho^* > \rho^* \quad \textit{where} \quad \rho^* = \sup_{(\chi,\zeta)\in\Pi} \rho(\chi,\zeta), \tag{9.10}$$

then there exists an optimal infinite horizon program.

PROOF: Let $\pi_\epsilon(p;\chi,\zeta) = p\zeta - (\rho^* + \epsilon)p\chi$. Theorem 9.10 (i), (iii), and Assumption T.2 imply $\rho^*p^*\chi^* > 0$, $\pi_\epsilon(p^*;\chi,\zeta) \leqq \pi_\epsilon(p^*;\chi^*,\zeta^*) = \pi_\epsilon^* < 0$ $\forall(\chi,\zeta) \in \Pi$, $(\chi,\zeta) \neq 0$, for any $\epsilon > 0$. Let $h(p) = \sup_{(\chi,\zeta)\in\Pi\cap\Sigma}\pi_\epsilon(p;\chi,\zeta)$ where $\Sigma = \{(\chi,\zeta) \in R_+^{2n} \mid \sum_{i=1}^n(\chi_i + \zeta_i) = 1\}$. Since $\pi_\epsilon(\cdot\,;\cdot)$ is continuous on $R^n \times R^{2n}$ and since $\Pi \cap \Sigma$ is compact, $h(p)$ is upper semicontinuous [5, Theorem 2, p. 116]. Hence for any $\nu > 0$ such that $\pi_\epsilon^* + \nu < 0$ there exists $\mu > 0$ such that if $\eta \in R^n$, $\eta > 0$ satisfies $\|\eta\| < \mu$, then $h(p^* + \eta) < \pi_\epsilon^* + \nu < 0$. But then by T.1 (ii) $\pi_\epsilon(p^* + \eta;\chi,\zeta) < 0$ $\forall(\chi,\zeta) \in \Pi$, $(\chi,\zeta) \neq 0$, where $p^* + \eta > 0$. Thus if we let $\bar{p} = p^* + \eta$

$$\bar{p}\zeta < (\rho^* + \epsilon)\bar{p}\chi \quad (\chi,\zeta) \in \Pi, \quad (\chi,\zeta) \neq 0. \tag{9.11}$$

Consider a path of pure accumulation, $c(t) = 0$, a.e.

$$\left(z_0 + \int_0^t v(\tau)\,d\tau, v(t)\right) = \left(\bar{z}(t), \dot{\bar{z}}(t)\right) \in \Pi \quad \text{a.e.}$$

By (9.11)

$$\bar{p}\dot{\bar{z}}(t) < (\rho^* + \epsilon)\bar{p}\bar{z}(t) \quad \text{a.e.}$$

so that

$$\bar{p}\bar{z}(t) < (\bar{p}z_0)e^{(\rho^*+\epsilon)t} \quad \text{a.e.} \tag{9.12}$$

Thus if we consider an arbitrary feasible path with $c(t) \geqq 0$ a.e.,

$$\left(z_0 + \int_0^t v(\tau)\,d\tau, c(t) + v(t)\right) = (z(t), q(t)) \in \Pi \quad \text{a.e.,}$$

then by (9.11) and (9.12)

(9.13) $\quad \bar{p}q(t) < (\rho^* + \epsilon)\bar{p}\bar{z}(t) < (\rho^* + \epsilon)(\bar{p}z_0)e^{(\rho^*+\epsilon)t} \quad$ a.e.

Since $\bar{p} > 0$ there exists $\gamma_\epsilon \in R$, $\gamma_\epsilon > 0$ such that

(9.14) $\quad \|c(t), v(t)\| \leqq \gamma_\epsilon e^{(\rho^*+\epsilon)t} \quad$ a.e.

for any feasible path. Thus Assumption 3 is satisfied.

To obtain the upper and lower bounds $\bar{\alpha}$, $\underline{\alpha}$ of Assumption 6, we need to consider two cases. In case one, u is *unbounded* from above, so that $0 \leqslant \eta \leqslant 1$. In case two, $\eta > 1$, so that u is *bounded* from above. Consider first the upper bound $\bar{\alpha}$. (9.13) and (9.1) imply that there exist $\bar{a}_\epsilon$, $\bar{b}_\epsilon \in R^n$, $\bar{a}_\epsilon > 0$, $\bar{b}_\epsilon > 0$ such that for any feasible path

$$c(t) \leqq q(t) < \bar{a}_\epsilon e^{(\rho^*+\epsilon)t}, \quad y(t) = y_0 + \int_0^t c(\tau)\,d\tau < \bar{b}_\epsilon e^{(\rho^*+\epsilon)t}, \quad \text{a.e.}$$

In case one, if we apply U.1 (iv) and (9.4), then there exists $\tau_\epsilon < \infty$ such that

$$u(y(t), c(t)) < u\left(\bar{b}_\epsilon e^{(\rho^*+\epsilon)t}, \bar{a}_\epsilon e^{(\rho^*+\epsilon)t}\right) < e^{(\rho^*+\epsilon)\bar{\beta}t}, \quad \forall t > \tau_\epsilon.$$

Thus for any $\epsilon > 0$ there exists $\bar{\alpha}_\epsilon > 0$ such that

$$u(y(t), c(t)) < \bar{\alpha}_\epsilon e^{(\rho^*+\epsilon)\bar{\beta}t} \quad \text{a.e.}$$

Since $\psi(\epsilon) = \delta - (\rho^* + \epsilon)(1 - \eta + \epsilon)$ is continuous and $\psi(0) > 0$ by (9.10), there exists $\underline{\epsilon} > 0$ such that $\psi(\underline{\epsilon}) > 0$. Thus

(9.15) $\quad \bar{\alpha}(t) = \bar{\alpha}_{\underline{\epsilon}} e^{(\rho^*+\epsilon)(1-\eta+\underline{\epsilon})t} \quad$ a.e.

satisfies (6.9) and (6.10). In case two, since u is bounded from above by zero, it is immediate that $\bar{\alpha}(t) \equiv 0 \ \forall t \in I$ satisfies (6.9) and (6.10).

Consider the lower bound $\underline{\alpha}$. Since $z_0 > 0$ there exists $\theta > 0$ such that $\theta\chi^* \leqq z_0$. By Theorem 9.10 (ii) and Assumption T.1 (v), for any $0 < \epsilon < 1$, the path

$$\underline{v}(t) = \dot{z}(t) = (1 - \epsilon)\rho^* z(t), \quad \underline{c}(t) = \epsilon\rho^* z(t) \quad \text{a.e.}, \quad z(0) = \theta\chi^*,$$

is feasible. But then $\underline{c}(t) = \epsilon\rho^*\theta\chi^* e^{(1-\epsilon)\rho^* t}$, a.e. Thus for some $\underline{a}_\epsilon$, $\underline{b}_\epsilon \in R^n$, $\underline{a}_\epsilon > 0$, $\underline{b}_\epsilon > 0$,

(9.16) $\quad \underline{c}(t) > \underline{a}_\epsilon e^{(1-\epsilon)\rho^* t}, \quad \underline{y}(t) = y_0 + \int_0^t \underline{c}(\tau)\,d\tau > \underline{b}_\epsilon e^{(1-\epsilon)\rho^* t}, \quad$ a.e.

Since (9.16) implies $(\underline{y}(t), \underline{c}(t))$ is positive and increasing, it follows from U.1 (iii) and (iv) that

(9.17) $\quad u(\underline{y}(t), \underline{c}(t)) > u(\underline{y}(0), \underline{c}(0)) = u_0 > -\infty, \quad$ a.e.

In case one, if we apply U.1 (iv) and (9.4) to (9.15), then there exists $\tau_\epsilon < \infty$ such that

$$(9.18)\qquad u(\underline{y}(t),\underline{c}(t)) > u(\underline{b}_\epsilon e^{(1-\epsilon)\rho^* t},\underline{a}_\epsilon e^{(1-\epsilon)\rho^* t}) > e^{(1-\epsilon)\rho^* \underline{\beta} t} > 0, \quad \forall t > \tau_\epsilon.$$

Thus in case one, (9.17) and (9.18) imply that

$$(9.19)\qquad \underline{\alpha}(t) = \begin{cases} u_0, & t \in [0,\tau_\epsilon], \\ 0, & t \in (\tau_\epsilon,\infty), \end{cases}$$

satisfies (6.9) and (6.11). In case two, if we apply U.1 (iv) and (9.4) to (9.16), then there exists $\tau_\epsilon < \infty$ such that

$$(9.20)\quad u(\underline{y}(t),\underline{c}(t)) > u(\underline{b}_\epsilon e^{(1-\epsilon)\rho^* t},\underline{a}_\epsilon e^{(1-\epsilon)\rho^* t}) > -e^{(1-\epsilon)\rho^* \bar{\beta} t} = \alpha_\epsilon(t), \quad \forall t > \tau_\epsilon.$$

Since $\phi(\epsilon) = \delta - (1-\epsilon)\rho^*(1-\eta+\epsilon)$ is continuous and $\phi(0) > 0$ by (9.10), there exists $\epsilon' > 0$ such that $\phi(\epsilon') > 0$. Thus in case two, (9.16) and (9.20) imply that

$$(9.21)\qquad \underline{\alpha}(t) = \begin{cases} u_0, & t \in [0,\tau_{\epsilon'}], \\ \alpha_{\epsilon'}(t), & t \in (\tau_{\epsilon'},\infty), \end{cases}$$

satisfies (6.9) and (6.11). Applying Theorem 7.6 completes the proof. *Q.E.D.*

10. TECHNICAL CHANGE

In this section I consider a simple canonical example of a one commodity economy with technical change. I allow the current utility function to depend not only on current consumption, but also on past and future consumption. To direct attention to this interdependence I make the conventional assumption that output devoted to consumption is a *perishable* commodity so that no stocks of the consumption good are accumulated.

In accordance with (6.1) and (6.4) the current *state of tastes* at time t is given by a function $\sigma = (\sigma_1,\sigma_2) \in \mathfrak{M}^2$ where

$$(10.1)\qquad \begin{aligned} \sigma_1(t) &= e^{-\mu_1 t}\sigma_1(0) + \int_0^t e^{-\mu_1(t-\tau)}c(\tau)\,d\tau, \quad \sigma_1(0) > 0, \quad t \in I, \\ \sigma_2(t) &= \int_t^\infty e^{-\mu_2(\tau-t)}c(\tau)\,d\tau, \quad t \in I, \end{aligned}$$

and where $\mu_1 > 0$, $\mu_2 > 0$ satisfy (6.3), so that $\sigma_1(t)$ is today's weighted average of *past* consumption, while $\sigma_2(t)$ is today's weighted average of prospective *future* consumption. The preference ordering is represented by the integral functional

$$(10.2)\qquad U(w) = \int_I u(\sigma(\tau),c(\tau))e^{-\delta\tau}\,d\tau,$$

where u satisfies Assumptions U.1, U.2 of Section 9.

On the production side output devoted to investment leads to an accumulation of capital, according to (9.6). The production set $\Pi(t)$ satisfies the following assumptions.

ASSUMPTION T.1′: *The production correspondence* $\Pi(t): I \to R_+ \times R_+$ *is given by*

$$\Pi(t) = \{(\chi, q) \mid q \leqq \xi(t) f(\chi), \quad (\chi, q) \geqq 0\} \quad a.e.$$

where (i) $f(\chi): R \to \bar{R}$, *dom* $f = R_+$; *(ii)* $f(\cdot)$ *is upper semicontinuous and concave*; *(iii)* $f(0) = 0$; *(iv)* $\chi' > \chi$ *with* $\chi \geqq 0$ *implies* $f(\chi') > f(\chi)$; *(v)* $\xi(t) = e^{\lambda t}$, $\lambda > 0$, *a.e.*

ASSUMPTION T.2′: *The production function f has an asymptotic exponent* ν *with* $0 < \nu < 1$.

The technology set $\Gamma(t)$ is defined in terms of $\Pi(t)$:

$$\Gamma(t) = \{(\chi, \zeta, \kappa) \mid (\chi, \zeta + \kappa) \in \Pi(t), \quad (\zeta, \kappa) \geqq 0\} \quad \text{a.e.}$$

THEOREM 10.1: *Let Assumptions (U.1, 2; T.1′, 2′) be satisfied. If the preference ordering is represented by* (10.2) *and if*

$$(10.3) \qquad \mu_1 > 0, \quad \mu_2 > \rho, \quad \delta + \eta\rho > \rho \quad \textit{where} \quad \rho = \frac{\lambda}{1-\nu},$$

then there exists an optimal infinite horizon program.

PROOF: Consider the path of pure accumulation $(z(t), \dot{z}(t)) \in \Pi(t)$ a.e. or equivalently $\dot{z}(t) = e^{\lambda t} f(z(t))$, a.e., $z(0) = z_0 > 0$. In view of T.1′, $z(t) \to \infty$ as $t \to \infty$. By T.2′ and (9.4), for any $\nu < \bar{\nu} < 1$, there exists $\bar{\tau} < \infty$ such that $\dot{z}(t) < e^{\lambda t} z(t)^{\bar{\nu}}$, $\forall t > \bar{\tau}$. Let $z'(t)$ satisfy $\dot{z}'(t) = e^{\lambda t} z'(t)^{\bar{\nu}}$, then $z(t) < z'(t)$ for $t > \bar{\tau}$ if $z(\bar{\tau}) \leqq z'(\bar{\tau})$. Let $\bar{\rho} = \lambda/(1 - \bar{\nu})$ and let $z^*(t) = z'(t) e^{-\bar{\rho} t}$; then $\dot{z}^*(t) = z^*(t)^{\bar{\nu}} - \bar{\rho} z^*(t)$ so that $z^*(t) \to \bar{z}^* = (1/\bar{\rho})^{1/(1-\bar{\nu})}$ as $t \to \infty$. But then $z'(t) \to e^{\bar{\rho} t} \bar{z}^*$ as $t \to \infty$. Thus for any $\epsilon > 0$, if we let $\bar{\rho} = \rho + \epsilon$, there exist $\bar{\tau}_\epsilon < \infty$, $\bar{z}_\epsilon > 0$ such that $z(t) < \bar{z}_\epsilon e^{\bar{\rho} t}$, $\forall t > \bar{\tau}_\epsilon$. But then

$$(10.4) \qquad q(t) \leqq e^{\lambda t} f(z(t)) < e^{\lambda t} f(\bar{z}_\epsilon e^{\bar{\rho} t}) < e^{\lambda t} (\bar{z}_\epsilon e^{\bar{\rho} t})^{\bar{\nu}} = \bar{z}_\epsilon^{\bar{\nu}} e^{\bar{\rho} t}, \quad \forall t > \bar{\tau}_\epsilon.$$

Since $c(t) + v(t) = q(t)$ and $(c(t), v(t)) \geqq 0$ a.e. there exists $\gamma_\epsilon > 0$ such that for any feasible path

$$\|c(t), v(t)\| \leqq \gamma_\epsilon e^{(\rho+\epsilon)t} \quad \text{a.e.}$$

so that Assumption 3 is satisfied. By (10.3) there exists $\epsilon > 0$ such that $\mu_2 > \rho + \epsilon$. By (10.1) and (10.4) there exist $\bar{a}_\epsilon > 0$, $\bar{b}_\epsilon = (b_\epsilon^1, b_\epsilon^2) > 0$ such that for any feasible path

$$(\sigma(t), c(t)) < (\bar{b}_\epsilon, \bar{a}_\epsilon) e^{(\rho+\epsilon)t} \quad \text{a.e.}$$

It follows, as in the proof of Theorem 9.15, that when $0 \leqq \eta \leqq 1$, $\bar{\alpha}(t)$ defined by (9.15) and when $\eta > 1$, $\bar{\alpha}(t) \equiv 0$ $\forall t \in I$, satisfy (6.9) and (6.10). To obtain the lower bound $\underline{\alpha}(t)$, we note that if $0 < \theta < 1$ the path

$$\underline{v}(t) = \dot{z}(t) = (1-\theta)e^{\lambda t}f(z(t)), \quad \underline{c}(t) = \theta e^{\lambda t}f(z(t)), \quad z(0) = z_0$$

is feasible. In view of T.1′, $z(t) \to \infty$ as $t \to \infty$. By T.2′ and (9.4) for any $0 < \underline{\nu} < \nu$, there exists $\underline{\tau} < \infty$ such that $\dot{z}(t) > (1-\theta)e^{\lambda t}z(t)^{\underline{\nu}}$, $\forall t > \underline{\tau}$. Let $z'(t)$ satisfy $\dot{z}'(t) = (1-\theta)e^{\lambda t}z'(t)^{\underline{\nu}}$, $\underline{\rho} = \lambda/(1-\underline{\nu})$, $z^*(t) = z'(t)e^{-\underline{\rho}t}$ so that $\dot{z}^*(t) = z^*(t)^{\underline{\nu}} - \underline{\rho} z^*(t)$, implying $z^*(t) \to \underline{z}^* = ((1-\theta)/\underline{\rho})^{1/(1-\underline{\nu})}$ and hence $z'(t) \to e^{\underline{\rho}t}\underline{z}^*$ as $t \to \infty$. Thus for any $\epsilon > 0$, if we let $\underline{\rho} = \rho - \epsilon$, there exist $\underline{\tau}_\epsilon < \infty$, $\underline{z}_\epsilon > 0$ such that $z(t) > \underline{z}_\epsilon e^{\underline{\rho}t}$, $\forall t > \underline{\tau}_\epsilon$. But then

$$\underline{c}(t) = \theta e^{\lambda t}f(z(t)) > \theta e^{\lambda t}f(\underline{z}_\epsilon e^{\underline{\rho}t}) > \theta e^{\lambda t}(\underline{z}_\epsilon e^{\underline{\rho}t})^{\underline{\nu}} = \theta \underline{z}_\epsilon^{\underline{\nu}} e^{\underline{\rho}t}, \quad \forall t > \underline{\tau}_\epsilon.$$

Let

$$\underline{\sigma}_1(t) = e^{-\mu_1 t}\sigma_1(0) + \int_0^t e^{-\mu_1(t-\tau)}\underline{c}(\tau)\,d\tau,$$

$$\underline{\sigma}_2(t) = \int_t^\infty e^{-\mu_2(\tau-t)}\underline{c}(\tau)\,d\tau, \quad t \in I.$$

$(\underline{\sigma}(t), \underline{c}(t))$ is positive and increasing $\forall t \in I$. It follows, as in the proof of Theorem 9.15, that when $0 \leqq \eta \leqq 1$, $\underline{\alpha}(t)$ defined by (9.19) and when $\eta > 1$, $\underline{\alpha}(t)$ defined by (9.21) satisfy (6.9) and (6.11). The result follows by applying Theorem 7.6. *Q.E.D.*

REMARK 10.2: Consider the case where $u(\sigma, c) = c^\beta/\beta$, $0 \neq \beta \leqq 1$, $f(z) = z^\nu$, $0 < \nu < 1$. The change of variables

$$z^* = e^{-\rho t}z, \quad c^* = e^{-\rho t}c, \quad v^* = \dot{z}^*, \quad \rho = \frac{\lambda}{1-\nu}$$

leads to the preference function

$$U(w^*) = \int_I \frac{c^*(\tau)^\beta}{\beta} e^{-(\delta - \rho\beta)\tau}\,d\tau. \tag{10.5}$$

Let $\bar{z} = (1/\rho)^{1/(1-\nu)}$; then the technology set reduces to

$$\Gamma = \{(z^*, c^*, v^*) \mid c^* + v^* \leqq z^{*\nu} - \rho z^*, \quad (c^*, v^*) \geqq (0, -\rho z^*), 0 \leqq z^* \leqq \bar{z}\}.$$

By Koopmans' result [**29**, Theorem K, p. 252, pp. 539–545] *if*

$$\delta - \rho\beta < 0 \quad \textit{or equivalently} \quad \delta + \eta\rho < \rho,$$

where $\beta = 1 - \eta$, *then there is no optimal infinite horizon program.* When $\delta + \eta\rho < \rho$ the preference function (10.5) ceases to be upper semicontinuous. In this case *if we consider a sequence of finite horizon problems on intervals* $I_n = [0, n]$, $n = 1, 2, \ldots,$ *then the optimal finite horizon program* w_n *converges almost everywhere to a program* $\bar{w}$ *which is not optimal over the infinite horizon.* To establish this result more generally requires a rigorous analysis of the existence of supporting prices for an optimal infinite horizon program, a topic to which I shall return in a subsequent paper.

REMARK 10.3: It is of interest to note the relation between *nonexistence* and the problem of *overaccumulation* (see Cass [13]). The limit program $\bar{w} = (\bar{c}, \bar{v})$ fails to be optimal by being *inefficient*: there is another feasible path which provides at least as much consumption at every instant $t \in I$ and for $t \in A \subset I$ provides more consumption, where $\lambda(A) > 0$. Under the investment program $\bar{v}$, $z^*(t) \to \bar{z}^* = (\nu/\delta + \eta\rho)^{1/(1-\nu)} > (\nu/\rho)^{1/(1-\nu)} = \hat{z}^*$ where $\hat{z}^*$ is the *golden rule* capital stock. *When the asymptotic rate of impatience* $(\delta + \eta\rho)$ *is less than the asymptotic rate of growth of output* (ρ), *the limit of the finite horizon optimal programs is a path of overaccumulation.*[5]

11. IMPATIENCE

We have seen that under natural economic conditions it is possible to choose a program space $\mathcal{W}$ and a topology Υ such that the set $\mathcal{F}$ of feasible programs is Υ-compact. Given this technologically determined program space and topology $(\mathcal{W}, \Upsilon)$, the force of the existence conditions lies in the requirement that the preference ordering $\succsim$ be upper semicontinuous on $\mathcal{F}$ in the Υ-topology.

DEFINITION 11.1: Let χ_A denote the characteristic function of $A \subset \mathcal{I}$;

$$\chi_A(t) = \begin{cases} 1, & t \in A, \\ 0, & t \notin A. \end{cases}$$

Let $\mathcal{W} \subset \mathcal{M}$, $w \in \mathcal{W}$. The program $w_T = w \cdot \chi_{[T,\infty)}$ for any $T \in I$ is called the *T-period deferred annuity of w*.

DEFINITION 11.2: Let $(\mathcal{W}, \Upsilon)$ be a program space with topology Υ, $\mathcal{W} \subset \mathcal{M}$. Υ is a *topology of discounting* if for any $w \in \mathcal{W}$ the sequence of deferred annuities w_n, $n = 1, 2, \ldots$ converges to zero in the Υ-topology.

LEMMA 11.3: *The weak* topology on* $\mathcal{L}_\infty^{2n}(\lambda)$ *and the weak topology on* $\mathcal{L}_1^{2n}(\mu)$ *are topologies of discounting.*

PROOF: Consider the topology (3.3) and $y \in \mathcal{L}_\infty^{2n}(\lambda)$. Since $\|y_n\| \leqq \|y\|$, $n = 1, 2, \ldots$ by [4, Theorem 2, p. 123], it suffices to show that $\Phi_f(y_n) \to 0$ as $n \to \infty$ for all f in a fundamental[6] subset of $\mathcal{L}_1^{2n}(\lambda)$. The characteristic functions $\chi_{[0,t]}$, for all

[5] It can be argued that capital markets lead to an intertemporal allocation that is the limit of the finite horizon optimal programs. In such a framework, an economy with substantial technical change may be led to overaccumulate.

[6] Linear combinations of its elements are dense.

$t \in I$, $\{\chi_{[0,t]}, t \in I\}$, meet this requirement. It thus suffices to show

$$\int_I \chi_{[0,t]}^{(\tau)} y_n(\tau)\, d\tau = \int_0^t y_n(\tau)\, d\tau = J(t,n) \to 0 \quad \text{as } n \to \infty.$$

But this is immediate, since $J(t,n) = 0$ for $n > t$. Consider the topology (3.8) and $y \in \mathfrak{L}_1^{2n}(\mu)$. Since $\{y_n\}_{n=1}^{\infty}$ is uniformly integrable and $\int_0^t y_n(\tau)\, d\mu(\tau) = 0$ for $n > t$, the result follows by the theorem of Lebesgue [**4**, p. 8]. *Q.E.D.*

DEFINITION 11.4: $\mathcal{W} \subset \mathcal{M}$. A preference ordering $\succsim$ exhibits *impatience* on $\mathcal{F} \subset \mathcal{W}$ if for any w, $w' \in \mathcal{F}$ such that

$$w' \succ w$$

and for any $y \in \mathcal{W}$ such that $w + y \in \mathcal{F}$, if $y_T = y \cdot \chi_{[T,\infty)}$ there exists $\tau \in I$ such that

$$w' \succ w + y_T, \quad \forall T > \tau.$$

An immediate consequence of Definition 11.4 and Lemma 11.3 is the following *behavioral implication* of the requirement of upper semicontinuity (Assumption P.3).

PROPOSITION 11.5: *If the preference ordering $\succsim$ is upper semicontinuous in the weak* topology on $\mathfrak{L}_\infty^{2n}(\lambda)$ or the weak topology on $\mathfrak{L}_1^{2n}(\mu)$, then $\succsim$ exhibits impatience.*

REMARK 11.6: Since under Assumption 4 the preferred sets (2.1) for the preference ordering defined by (6.12) and (6.13) are *convex*, the weak and strong closures coincide [**19**, Theorem V.3.13, p. 422]. Thus $U(w)$ is upper semicontinuous in the $\sigma(\mathfrak{L}_p^{2n}, \mathfrak{L}_{p'}^{2n})$ topology if and only if $U(w)$ is upper semicontinuous in the Mackey topology $\tau(\mathfrak{L}_p^{2n}, \mathfrak{L}_{p'}^{2n})$. When $p = \infty$ this is part of the assumption made by Bewley [7, Theorem 1, (iii)]. When $p = 1$, $\tau(\mathfrak{L}_1^{2n}, \mathfrak{L}_\infty^{2n})$ is just the norm topology. This leads to a very direct form of Proposition 11.5 since the norm topology in $\mathfrak{L}_1^{2n}(\mu)$ is a "pure" topology of discounting.

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Manuscript received August, 1979; *revision received April*, 1980.

REFERENCES

[1] ARROW, K. J.: "Alternative Approaches to the Theory of Choice in Risk-Taking Situations," *Econometrica*, 19(1951), 404–437. Reprinted in K. J. Arrow, *Essays in the Theory of Risk-Bearing*. Chicago: Markham Publishing Company, 1971, pp. 1–43.

[2] ARROW, K. J., AND M. KURZ: *Public Investment, the Rate of Return, and Optimal Fiscal Policy*. Baltimore: The Johns Hopkins Press, 1970.

[3] AUMANN, R. J., AND M. PERLES: "A Variational Problem Arising in Economics," *Journal of Mathematical Analysis and Applications*, 11(1965), 488–503.

[4] BANACH, S.: *Théorie des Opérations Linéaires*. New York: Chelsea Publishing Company, 1932.
[5] BERGE, C.: *Topological Spaces*. Edinburgh: Oliver & Boyd, 1963.
[6] BERNOULLI, D.: "Specimen theoriae novae de mensura sortis," *Commentarii Academiae Scientiarum Imperiales Petropolitanae*, 5(1738), 175–192. Translated by L. Sommer as "Exposition of a New Theory on the Measurement of Risk," *Econometrica*, 22(1954), 23–36.
[7] BEWLEY, T. F.: "Existence of Equilibria in Economies with Infinitely Many Commodities," *Journal of Economic Theory*, 4(1972), 514–540.
[8] BROCK, W. A.: "On Existence of Weakly Maximal Programmes in a Multi-Sector Economy," *Review of Economic Studies*, 37(1970), 275–280.
[9] BROCK, W. A., AND D. GALE: "Optimal Growth under Factor Augmenting Progress," *Journal of Economic Theory*, 1(1969), 229–243.
[10] BROCK, W. A., AND A. HAURIE: "On Existence of Overtaking Optimal Trajectories over an Infinite Time Horizon," *Mathematics of Operations Research*, 1(1976), 337–346.
[11] BROCK, W. A., AND J. A. SCHEINKMAN: "Global Asymptotic Stability of Optimal Control Systems with Applications to the Theory of Economic Growth," *Journal of Economic Theory*, 12(1976), 164–190.
[12] BROWN, D. J., AND L. M. LEWIS: "Myopic Economic Agents," Cowles Foundation Discussion Paper No. 481, Yale University, 1978.
[13] CASS, D.: "On Capital Overaccumulation in the Aggregative, Neoclassical Model of Economic Growth: A Complete Characterization," *Journal of Economic Theory*, 4(1972), 200–223.
[14] CASS, D., AND K. SHELL: "The Structure and Stability of Competitive Dynamical Systems," *Journal of Economic Theory*, 12(1976), 31–70.
[15] CHAKRAVARTY, S.: "The Existence of an Optimum Savings Program," *Econometrica*, 30(1962), 178–187.
[16] CHICHILNISKY, G.: "Nonlinear Functional Analysis and Optimal Economic Growth," *Journal of Mathematical Analysis and Applications*, 61(1977), 504–520.
[17] DEBREU, G.: "Representation of a Preference Ordering by a Numerical Function," in *Decision Processes*, ed. by R. M. Thrall, C. H. Coombs, and R. L. Davis. New York: John Wiley & Sons, 1954, pp. 159–165.
[18] DEBREU, G.: "Topological Methods in Cardinal Utility Theory," in *Mathematical Methods in the Social Sciences, 1959*, ed. by K. J. Arrow, S. Karlin, and P. Suppes. Stanford: Stanford University Press, 1960, pp. 16–26.
[19] DUNFORD, N., AND J. T. SCHWARTZ: *Linear Operators, Vol. I, General Theory*. New York: Interscience Publishers, 1957.
[20] EKELAND, I., AND R. TEMAM: *Convex Analysis and Variational Problems*. Amsterdam: North-Holland, 1976.
[21] EVSTIGNEEV, I. V.: "Optimal Stochastic Programs and Their Stimulating Prices," in *Mathematical Models in Economics*, ed. by J. Los and M. W. Los. Amsterdam: North-Holland, 1974, pp. 219–252.
[22] FISHER, I.: *Theory of Interest*. New York: Augustus M. Kelley, 1965.
[23] GALE, D.: "The Closed Linear Model of Production," in *Linear Inequalities and Related Systems*, ed. by H. W. Kuhn and A. W. Tucker. Princeton: Princeton University Press, 1956, pp. 285–303.
[24] ———: "On Optimal Development in a Multi-Sector Economy," *Review of Economic Studies*, 34(1967), 1–19.
[25] HAMMOND, P. J., AND J. A. MIRRLEES: "Agreeable Plans," in *Models of Economic Growth*, ed. by J. A. Mirrlees and N. H. Stern. New York: John Wiley & Sons, 1973, pp. 283–299.
[26] KOOPMANS, T. C.: "Analysis of Production as an Efficient Combination of Activities," in *Activity Analysis of Production and Allocation*, ed. by T. C. Koopmans. New York: John Wiley & Sons, 1951, pp. 33–97.
[27] ———: "Objectives, Constraints, and Outcomes in Optimal Growth Models," *Econometrica*, 35(1967), 1–15. Reprinted in *Scientific Papers of Tjalling C. Koopmans*. New York: Springer-Verlag, 1970, pp. 548–562.
[28] ———: "On Flexibility of Future Preference," in *Human Judgements and Optimality*, ed. by M. W. Shelly, II and G. L. Bryan. New York: John Wiley and Sons, 1964, pp. 243–254. Reprinted in *Scientific Papers of Tjalling C. Koopmans*. New York: Springer-Verlag, 1970, pp. 469–480.
[29] ———: "On the Concept of Optimal Economic Growth," *Pontificiae Academiae Scientiarum Scripta Varia*, 28(1965), 225–300. Reprinted in *Scientific Papers of Tjalling C. Koopmans*. New York: Springer-Verlag, 1970, pp. 485–547.

[30] ———: "Stationary Ordinal Utility and Impatience," *Econometrica*, 28(1960), 287–309. Reprinted in *Scientific Papers of Tjalling C. Koopmans*. New York: Springer-Verlag, 1970, pp. 387–409.

[31] LEONTIEF, W.: "A Note on the Interrelation of Subsets of Independent Variables of a Continuous Function with Continuous First Derivatives," *Bulletin of the American Mathematical Society*, 53(1947), 343–350.

[32] MAGILL, M. J. P.: "Some New Results on the Local Stability of the Process of Capital Accumulation," *Journal of Economic Theory*, 15(1977), 174–210.

[33] ———: "Stability of Equilibrium," *International Economic Review*, 20(1979), 577–597.

[34] MAGILL, M. J. P., AND J. A. SCHEINKMAN: "The Stability of Regular Equilibria and the Correspondence Principle for Symmetric Variational Problems," *International Economic Review*, 20(1979), 297–315.

[35] MALINVAUD, E.: "Capital Accumulation and Efficient Allocation of Resources," *Econometrica*, 21(1953), 253–267.

[36] MCFADDEN, D.: "The Evaluation of Development Programmes," *Review of Economic Studies*, 34(1967), 25–50.

[37] ———: "On the Existence of Optimal Development Plans," in *Proceedings of the Princeton Symposium on Mathematical Programming*, ed. by H. W. Kuhn. Princeton: Princeton University Press, 1970, pp. 403–427.

[38] ———: "On the Existence of Optimal Development Programmes in Infinite-Horizon Economies," in *Models of Economic Growth*, ed. by J. A. Mirrlees and N. H. Stern. New York: John Wiley & Sons, 1973, pp. 260–282.

[39] MCKENZIE, L. W.: "Turnpike Theorems for a Generalized Leontief Model," *Econometrica*, 31(1963), 165–180.

[40] MEYER, R.: *Probability and Potentials*. Waltham, Massachusetts: Blaisdell, 1966.

[41] MIRRLEES, J. A.: "Optimum Growth when Technology is Changing," *Review of Economic Studies*, 34(1967), 95–124.

[42] MITRA, T.: "On Optimal Economic Growth with Variable Discount Rates: Existence and Stability Results," *International Economic Review*, 20(1979), 133–145.

[43] RADNER, R.: "Paths of Economic Growth that are Optimal with Regard only to Final States: A Turnpike Theorem," *Review of Economic Studies*, 28(1961), 98–104.

[44] RAMSEY, F. P.: "A Mathematical Theory of Saving," *Economic Journal*, 38(1928), 543–559.

[45] ROCKAFELLAR, R. T.: "Existence Theorems for General Control Problems of Bolza and Lagrange," *Advances in Mathematics*, 15(1975), 312–333.

[46] ———: "Integral Functionals, Normal Integrands and Measurable Selections," in *Nonlinear Operators and the Calculus of Variations*, ed. by J. P. Gossez, et al. New York: Springer-Verlag, 1976, pp. 157–207.

[47] ———: "Saddle Points of Hamiltonian Systems in Convex Lagrange Problems Having a Non-zero Discount Rate," *Journal of Economic Theory*, 12(1976), 71–113.

[48] ROYDEN, H. L.: *Real Analysis*. New York: Macmillan, 1968.

[49] RYDER, H. E., AND G. M. HEAL: "Optimal Growth with Intertemporally Dependent Preferences," *Review of Economic Studies*, 40(1973), 1–31.

[50] SAMUELSON, P. A., AND R. M. SOLOW: "Balanced Growth under Constant Returns to Scale," *Econometrica*, 21(1953), 412–424.

[51] TINBERGEN, J..: "Optimum Savings and Utility Maximization over Time," *Econometrica*, 28(1960), 481–489.

[52] TONELLI, L.: *Fondamenti di Calcolo delle Variazioni, Vol. I e II*. Bologna: Nicola Zanichelli, 1921–1923.

[53] VON NEUMANN, J.: "Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes," *Ergebnisse eins Mathematischen Kolloquiums*, 8(1939), 73–83. Translated by G. Morgenstern as "A Model of General Economic Equilibrium," *Review of Economic Studies*, 13 (1945), 1–9.

[54] VON WEISÄCKER, C. C.: "Existence of Optimal Programs of Accumulation for an Infinite Time Horizon," *Review of Economic Studies*, 32(1965), 85–104.

[55] YAARI, M. E.: "On the Existence of an Optimal Plan in a Continuous-Time Allocation Process," *Econometrica*, 32(1964), 576–590.

[56] YOUNG, L. C.: "Generalized Curves and the Existence of an Attained Absolute Minimum in the Calculus of Variations," *Comptes Rendus de la Sociéte des Sciences et des Lettres de Varsovie, classe III*, 30(1937), 212–234.