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INFINITE HORIZON INCOMPLETE MARKETS

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The model of general equilibrium with incomplete markets is a generalization of the Arrow-Debreu model which provides a rich framework for studying problems of macroeconomics. This paper shows how the model, which has so far been restricted to economies with a finite horizon, can be extended to the more natural setting of an open-ended future, thereby providing an extension of the finite horizon representative agent models of modern macroeconomics to economies with heterogeneous agents and incomplete markets.

There are two natural concepts of equilibrium over an infinite horizon which prevent agents from entering into Ponzi schemes, that is, from indefinitely postponing the repayment of their debts. The first is based on debt constraints which place bounds on debt at each date-event; the second is based on transversality conditions which limit the asymptotic rate of growth of debt. The concept of an *equilibrium with debt constraint* is a natural concept of equilibrium for macroeconomic analysis; however the concept of an *equilibrium with transversality condition* is more amenable to theoretical analysis since it permits the powerful techniques of Arrow-Debreu theory to be carried over to the setting of incomplete markets. In an economy in which agents are impatient (expressed by the Mackey continuity of their preference orderings) and have a degree of impatience at each date-event which is bounded below (a concept defined in the paper), we show that the equilibria of an economy with transversality condition coincide with the equilibria with debt constraints. An equilibrium with transversality condition is shown to exist: it follows that for each economy there is an explicit bound M such that an equilibrium with explicit debt constraint M exists, in which the constraint is never binding—this latter property ensuring that the debt constraint, whose objective is to prevent Ponzi schemes, does not in itself introduce a new imperfection into the model over and above the incompleteness of the markets.

KEYWORDS: Incomplete markets, infinite horizon, Ponzi schemes, debt constraints, transversality condition, Mackey continuity, degree of impatience, existence of equilibrium.

1. INTRODUCTION

THE ANALYSIS OF EQUILIBRIUM on a sequence of markets in which agents correctly anticipate future prices was first introduced in an abstract setting by Radner (1972). This model has recently evolved into the model of *general equilibrium with incomplete markets* (GEI for short). The analysis of the GEI model (which is surveyed in Magill-Shafer (1991)) suggests that it may provide a valuable framework for discussing many issues in macroeconomics. To provide

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such a framework however the model, which has so far been restricted to economies with a finite horizon, needs to be extended to the more natural setting of an open-ended future.

There are two natural ways of extending the analysis of an economy to an infinite horizon. The first is to assume that there are a finite number of agents (families) who are infinitely lived; the second is to assume that all agents are finitely lived and are succeeded by their children in an indefinite sequence of overlapping generations. The models that arise from these two approaches have become the basic workhorses of modern macroeconomics (see Blanchard-Fischer (1989)). In this paper we explore the first type of extension: we study an exchange economy with a finite number of infinitely lived agents who use spot markets for the current exchange of goods and a limited array of financial markets for redistributing their income across time periods and uncertain events. Such a model provides an extension of the representative agent models (for example Lucas (1978)) to an economy with heterogeneous agents in which markets can be incomplete. When the markets are incomplete and agents are heterogeneous, many phenomena can arise which cannot occur in the representative agent (complete market) version of the model.

In a model with a sequence of markets over an infinite horizon a new problem arises which has no counterpart in a finite horizon model: if agents are permitted to borrow, they may seek to postpone the repayment of their debts by rolling them over indefinitely from one period to the next and if such Ponzi schemes are permitted there is no solution to an agent's decision problem. Broadly speaking two approaches are used in the macroeconomics literature to limit the rate at which agents can accumulate debt: the first is to use a *debt constraint* which places a uniform bound on debt at all dates; the second is to use a *transversality condition* which, in the deterministic case, requires that debt grow asymptotically slower than the rate of interest (see, for example, Kehoe (1989) and Blanchard-Fischer (1989, Chapter 2)).

The first approach, which is the most widely used in macroeconomics, has the merit of simplicity: in principle a monitoring agency could check that no agent's debt exceeds some specified bound M. The second approach has the merit of greater generality and of forming a precise connection between the infinite-dimensional version of Arrow-Debreu theory and the theory of sequence economies when markets are complete.

In this paper we extend both approaches to a sequence economy with incomplete markets. To focus attention on the difficulties created by the presence of an infinite horizon we restrict attention to the simplest class of financial assets—short-lived (i.e. one-period) securities which pay dividends in a numeraire good. Two types of budget sets are introduced: the first with a debt constraint, the second with a transversality condition. A debt constraint which asserts that an agent's debt cannot grow without bound (or more precisely that it lies in the space of bounded sequences l_∞) is called an *implicit debt constraint*. The more specific debt constraint which requires that an agent's debt at each date not exceed some prespecified bound M is called an *explicit debt constraint*.

The budget sets defined by these two types of constraints lead to the concept of an *equilibrium with implicit* (resp. *explicit*) *debt constraint*.

Finding the appropriate way of bounding the rate of growth of an agent's debt based on a transversality condition presents some difficulties when markets are incomplete. In this case there is not a unique, objective market based vector of present value prices for income in the future that can be used to express the transversality condition—namely that the average asymptotic present value of debt be zero. In the absence of a market based present value vector, one solution is to bound the rate of growth of each agent's debt by his own present value vector at the equilibrium. Using this growth condition leads to the abstract concept of equilibrium that we call an *equilibrium with transversality condition*. This concept seems to form a natural bridge between infinite dimensional Arrow-Debreu theory and the theory of incomplete markets for sequence economies—permitting the powerful techniques and concepts of Arrow-Debreu theory to be exploited while moving into the more general setting of incomplete markets. As an economic concept of equilibrium it may well be considered controversial (and by many perhaps, as unacceptable) since it calls for far too much rationality on the part of agents. However as an abstract concept it will be seen to be most natural since it leads to a relatively direct proof of existence of equilibrium, exploiting the elegant constructions introduced by Bewley (1972).

Two assumptions play a crucial role in establishing existence of an equilibrium with transversality condition. The first is the assumption, introduced by Bewley (1972), that agent's preference orderings are continuous in the *Mackey topology*: as shown by Brown-Lewis (1981), this is an abstract way of formalizing the idea that agents are impatient. This assumption permits equilibria of finite horizon economies to approximate the equilibria of an infinite horizon economy since consumption in the very distant future is unimportant. The second assumption is a strengthening of the assumption of Mackey continuity: it requires that at each node the proportion of his future consumption that an agent is prepared to give up in order to obtain one more unit of the numeraire commodity at that node (which measures the agent's *degree of impatience*) is bounded away from zero *uniformly across the nodes*. This prevents each agent from having a degree of impatience which vanishes asymptotically.

The assumption of a uniform lower bound on each agent's degree of impatience permits bounds to be established on the indebtedness of each agent in equilibrium. These bounds are not only useful for establishing existence of an equilibrium with transversality condition, they also lead to the following important qualitative property: in an economy in which there is a uniform bound on the impatience of agents the equilibria with transversality condition coincide with the equilibria with (implicit) debt constraints. Thus an equilibrium with debt constraints which is most natural as a concept of equilibrium in macroeconomics can be analyzed using the concept of an equilibrium with transversality condition which is more amenable to theoretical analysis. In particular, establishing existence of an equilibrium with transversality condition implies the existence of an equilibrium with implicit debt constraints: the existence of an equilibrium with an explicit debt constraint for which the constraint is *never binding* follows as a corollary. The fact that the debt constraint is not binding at the equilibrium is important since it ensures that the debt constraint (whose object is to prevent Ponzi schemes) does not in itself introduce a new imperfection into the model over and above the incompleteness of the markets.

2. CHARACTERISTICS OF THE ECONOMY

We use an event tree $\boldsymbol{D}$ to describe time, uncertainty, and the revelation of information over an infinite horizon. More precisely, let $\boldsymbol{T} = \{0, 1, \dots\}$ denote the set of time periods and let $\boldsymbol{S}$ be a set of states of nature. The revelation of information is described by a sequence of partitions of $\boldsymbol{S}$, $\mathbb{F} = (\mathbb{F}_0, \mathbb{F}_1, \dots, \mathbb{F}_t, \dots)$ where the number of subsets in $\mathbb{F}_t$ is *finite* and $\mathbb{F}_t$ is finer than the partition $\mathbb{F}_{t-1}$(i.e. $\sigma \in \mathbb{F}_t, \sigma' \in \mathbb{F}_{t-1} \Rightarrow \sigma \subset \sigma'$ or $\sigma \cap \sigma' = \varnothing$) for all $t \geqq 1$. At date 0 we assume that there is no information so that $\mathbb{F}_0 = \boldsymbol{S}$. The information available at time t (for $t \in \boldsymbol{T}$) is assumed to be the same for all agents in the economy (*symmetric information*) and is described by the subset σ of the partition $\mathbb{F}_t$ in which the state of nature lies. A pair $\xi = (t, \sigma)$ with $t \in \boldsymbol{T}$ and $\sigma \in \mathbb{F}_t$ is called a *date-event* or *node* and $t(\xi) = t$ is the *date* of node ξ. The set $\boldsymbol{D}$ consisting of all date-events (or nodes) is called the *event-tree* induced by $\mathbb{F}$, $\boldsymbol{D} = \bigcup_{t \in \boldsymbol{T}, \sigma \in \mathbb{F}_t} (t, \sigma)$.

A node $\xi' = (t', \sigma')$ is said to succeed (strictly) node $\xi = (t, \sigma)$ if $t' \geqslant t$ $(t' > t)$ and $\sigma' \subset \sigma$; we write $\xi' \geqslant \xi$ $(\xi' > \xi)$. The set of nodes which succeed a node $\xi \in \boldsymbol{D}$ is called the subtree $\boldsymbol{D}(\xi)$ and $\boldsymbol{D}^+(\xi) = \{\xi' \in \boldsymbol{D}(\xi) | \xi' > \xi\}$ is the set of strict successors of ξ. The subset of nodes of $\boldsymbol{D}(\xi)$ at date T is denoted by $\boldsymbol{D}_T(\xi)$ and the subset of nodes between dates $t(\xi)$ and T by $\boldsymbol{D}^T(\xi)$:

$$\boldsymbol{D}_T(\xi) = \{\xi' \in \boldsymbol{D}(\xi) | t(\xi') = T\},$$

$$\boldsymbol{D}^T(\xi) = \{\xi' \in \boldsymbol{D}(\xi) | t(\xi) \leqslant t(\xi') \leqslant T\}.$$

When ξ is the initial node the notation is simplified to $\boldsymbol{D}^+, \boldsymbol{D}_T, \boldsymbol{D}^T$.

$\xi^+ = \{\xi' \in \boldsymbol{D}(\xi) | t(\xi') = t(\xi) + 1\}$ is the set of immediate successors of ξ. The number of elements of ξ^+ is finite and is called the branching number $b(\xi)$ at $\xi (b(\xi) = \#\xi^+)$. If $\xi = (t, \sigma)$ with $t \geqslant 1$, the unique node $\xi^- = (t-1, \sigma')$ with $\sigma \subset \sigma'$ is called the predecessor of ξ.

The economy consists of a finite collection of infinitely lived consumers (families) $\boldsymbol{I} = \{1, \dots, I\}$ who purchase commodities on spot markets and trade securities at every node in the event-tree $\boldsymbol{D}$ described above. There is a set $\boldsymbol{L} = \{1, \dots, L\}$ of commodities at each node: the set consisting of all commodities indexed over the event-tree is thus

$$\boldsymbol{D} \times \boldsymbol{L} = \{(\xi, \ell) | \xi \in \boldsymbol{D}, \ell \in \boldsymbol{L}\}.$$

Let $\mathbb{R}^{\boldsymbol{D} \times \boldsymbol{L}}$ denote the vector space of all maps $x: \boldsymbol{D} \times \boldsymbol{L} \to \mathbb{R}$ and let $\ell_\infty(\boldsymbol{D} \times \boldsymbol{L})$ denote the subspace of $\mathbb{R}^{\boldsymbol{D} \times \boldsymbol{L}}$ consisting of all *bounded* maps (sequences)

$$\ell_\infty(\boldsymbol{D} \times \boldsymbol{L}) = \left\{x \in \mathbb{R}^{\boldsymbol{D} \times \boldsymbol{L}} | \sup_{(\xi, \ell) \in \boldsymbol{D} \times \boldsymbol{L}} |x(\xi, \ell)| < \infty\right\}.$$

The norm $\|\cdot\|_\infty$ of $\ell_\infty(\boldsymbol{D}\times\boldsymbol{L})$ is defined by $\|x\|_\infty = \sup_{(\xi,\ell)\in \boldsymbol{D}\times\boldsymbol{L}}|x(\xi,\ell)|$. As in Bewley (1972) we take the commodity space to be $\ell_\infty(\boldsymbol{D}\times\boldsymbol{L})$. Each agent $i\in\boldsymbol{I}$ has an *initial endowment* process given by $\omega^i=(\omega^i(\xi,\ell),(\xi,\ell)\in\boldsymbol{D}\times\boldsymbol{L})$ which is assumed to lie in the nonnegative orthant $\ell_\infty^+(\boldsymbol{D}\times\boldsymbol{L})$. Let $\omega^i(\xi)=(\omega^i(\xi,\ell), \ell\in\boldsymbol{L})\in\mathbb{R}^L$ denote the agent's endowment of the L goods at node ξ. Agent i chooses a *consumption process* $x^i=(x^i(\xi,\ell),(\xi,\ell)\in\boldsymbol{D}\times\boldsymbol{L})$ which must lie in his consumption set $X^i=\ell_\infty^+(\boldsymbol{D}\times\boldsymbol{L})$; $x^i(\xi)=(x^i(\xi,\ell),\ell\in\boldsymbol{L})\in\mathbb{R}_+^L$ denotes the agent's consumption at node ξ. Note that this description of the commodity space assumes that each good is perfectly divisible and is perishable (no storable or durable goods) and that the supply of goods does not grow without bound. The agent's preference among consumption processes in X^i is expressed by a *preference ordering* $\succsim_i$.

At each date-event there are spot markets on which the L commodities are traded. Let

$$p=(p(\xi,\ell),(\xi,\ell)\in\boldsymbol{D}\times\boldsymbol{L})\in\mathbb{R}^{\boldsymbol{D}\times\boldsymbol{L}}$$

denote the *spot price process* and let $p(\xi)=(p(\xi,\ell),\ell\in\boldsymbol{L})$ denote the vector of spot prices for the L goods at node ξ. At each node ξ, good 1 plays the role of numeraire good

$$(2.1)\qquad p(\xi,1)=1,\qquad \forall\xi\in\boldsymbol{D}$$

so that all payments are denominated in units of good 1.

To focus attention directly on the difficulties created by the presence of an infinite horizon, we consider only the simplest class of financial assets—the so-called *short-lived numeraire* securities. Such a security pays dividends only at the immediate successors of its node of issue and its dividends are amounts of the numeraire good. A much more general class of securities is studied in the companion paper (Magill-Quinzii (1993)).

Let $J(\xi)$ denote the set of short-lived securities issued at node $\xi\in\boldsymbol{D}$ where $j(\xi)=\#J(\xi)<\infty$ is the number of these securities. For a security $j\in J(\xi)$, $A(\xi',j)$ denotes the dividend (in units of good 1) at an immediate successor ξ' of ξ and

$$A(\xi')=(A(\xi',j),j\in J(\xi)),\qquad \xi'\in\xi^+,$$

is the (row) vector of dividends at ξ' of the securities issued at node ξ. The *process of security payoffs* is denoted by A:

$$A=(A(\xi'),\xi'\in\boldsymbol{D}^+)\in\prod_{\xi\in\boldsymbol{D}}\mathbb{R}^{b(\xi)\times j(\xi)}.$$

Let $q(\xi)=(q(\xi,j),j\in J(\xi))$ be the (row) vector of prices of the securities issued at node ξ and let $q=(q(\xi),\xi\in\boldsymbol{D})$ denote the *security price* process which belongs to the *security price space* $Q=\prod_{\xi\in\boldsymbol{D}}\mathbb{R}^{J(\xi)}$. Agent i chooses a *portfolio process* $z^i=(z^i(\xi),\xi\in\boldsymbol{D})$ where $z^i(\xi)=(z^i(\xi,j),j\in J(\xi))$ is the (column) vector of security holdings at node ξ. z^i belongs to the *portfolio space* $Z=\prod_{\xi\in\boldsymbol{D}}\mathbb{R}^{J(\xi)}$.

If $\succsim = (\succsim_1, \ldots, \succsim_I), \omega = (\omega^1, \ldots, \omega^I)$ denote the profiles of preference orderings and endowments of the I agents and A is the security payoff process, then $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, A)$ denotes the associated economy over the event-tree $\boldsymbol{D}$.

3. ASSUMPTIONS

This section describes the assumptions imposed on the characteristics of the economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega)$. The crucial assumption required to establish existence of an equilibrium in an infinite horizon economy is the choice of a topology in which agents' preference orderings are continuous. Let $\ell_1(\boldsymbol{D} \times \boldsymbol{L})$ denote the subspace of $\mathbb{R}^{\boldsymbol{D} \times \boldsymbol{L}}$ consisting of all *summable* sequences

$$\ell_1(\boldsymbol{D} \times \boldsymbol{L}) = \left\{ P \in \mathbb{R}^{\boldsymbol{D} \times \boldsymbol{L}} \middle| \sum_{(\xi, \ell) \in \boldsymbol{D} \times \boldsymbol{L}} |P(\xi, \ell)| < \infty \right\}.$$

For $P \in \ell_1(\boldsymbol{D} \times \boldsymbol{L})$ and $x \in \ell_\infty(\boldsymbol{D} \times \boldsymbol{L})$ the scalar product is defined by

$$Px = \sum_{(\xi, \ell) \in \boldsymbol{D} \times \boldsymbol{L}} P(\xi, \ell) x(\xi, \ell).$$

The *Mackey topology* on $\ell_\infty(\boldsymbol{D} \times \boldsymbol{L})$ is the strongest locally convex topology such that the dual of $\ell_\infty(\boldsymbol{D} \times \boldsymbol{L})$ under this topology is $\ell_1(\boldsymbol{D} \times \boldsymbol{L})$. For a discussion of this topology see Bewley (1972) and Mas-Colell-Zame (1991).

A1 (Event-tree): *The branching number* $b(\xi) = \#\xi^+$ *is finite at each node* $\xi \in \boldsymbol{D}$.

A2 (Endowments): *There exists scalars* m, m' *with* $0 < m < m'$ *such that* $\forall (\xi, \ell) \in \boldsymbol{D} \times \boldsymbol{L}, \omega^i(\xi, \ell) > m, \forall i \in \boldsymbol{I}$ *and* $\sum_{i \in \boldsymbol{I}} \omega^i(\xi, \ell) < m'$.

Let $w = \sum_{i \in \boldsymbol{I}} \omega^i$; then A2 implies $\|w\|_\infty < m'$. Thus a feasible consumption process x^i for agent i must lie in the set

$$F = \{ y \in \ell_\infty^+(\boldsymbol{D} \times \boldsymbol{L}) \mid \|y\|_\infty \leqq m' \}.$$

A3 (Preferences): *For* $i \in \boldsymbol{I}$, $\succsim_i$ *is a transitive, reflexive, complete preference ordering on* $X^i = \ell_\infty^+(\boldsymbol{D} \times \boldsymbol{L})$ *which is convex and continuous in the Mackey topology (i.e. for all* $\tilde{x}^i \in X^i, \{x^i \in X^i \mid x^i \succsim_i \tilde{x}^i\}$ *is convex and closed in the Mackey topology and* $\{x^i \in X^i \mid x^i \succ_i \tilde{x}^i\}$ *is open in the Mackey topology).* $\succsim_i$ *is monotone and strictly monotone in good* 1 *in the sense that for each* $x^i \in X^i$ *and for each* $y \in \ell_\infty^+(\boldsymbol{D} \times \boldsymbol{L}), x^i + y \succsim_i x^i$ *with strict preference if* $y(\xi, 1) > 0$ *for some* ξ.

Let $E \subset \boldsymbol{D}$ be a subset of nodes and let χ_E denote the characteristic function of E:

$$\chi_E(\xi) = \begin{cases} 1 & \text{if } \xi \in E, \\ 0 & \text{if } \xi \notin E. \end{cases}$$

For $x \in \ell_\infty(\boldsymbol{D} \times \boldsymbol{L})$ define $x\chi_E = (x(\xi, \ell)\chi_E(\xi), (\xi, \ell) \in \boldsymbol{D} \times \boldsymbol{L})$. Let $e_\ell^\xi \in \ell_\infty(\boldsymbol{D} \times \boldsymbol{L})$ denote the process which has all components 0 except for the component of good ℓ at node ξ which is 1:

$$e_\ell^\xi(\xi', \ell') = \begin{cases} 1 & \text{if } (\xi', \ell') = (\xi, \ell), \\ 0 & \text{if } (\xi', \ell') \neq (\xi, \ell). \end{cases}$$

A4 (Uniform lower bound on impatience): *There exists* $\beta < 1$ *such that, for all* $i \in \boldsymbol{I}$,

$$x^i\chi_{\boldsymbol{D}\setminus \boldsymbol{D}^+(\xi)} + \beta x^i \chi_{\boldsymbol{D}^+(\xi)} + e_1^\xi \succ_i x^i \quad \forall x^i \in F, \forall \xi \in \boldsymbol{D}.$$

A5 (Securities): *Every security is a short-lived numeraire security and at each node* $\xi \in \boldsymbol{D}$ *the number of securities* $j(\xi)$ *is finite.*

A6 (Riskless bond): *For each* $\xi \in \boldsymbol{D}$ *there exists* $j_\xi \in J(\xi)$ *such that*

$$A(\xi', j_\xi) = 1, \qquad \forall \xi' \in \xi^+.$$

Remark: We have repeated A1 for completeness: it is essential that at each node ξ there are only a finite number of immediate successors. Assumption A2 asserts that the aggregate endowment process $w = \sum_{i \in \boldsymbol{I}} \omega^i$ is bounded above and hence that each individual endowment process ω^i is bounded above; in addition each agent has an endowment of each good which is uniformly positive across all nodes. Good 1 plays the role of a numeraire: by Assumption A3 it is strictly desired by all agents at all nodes and thus has a positive price at each node.

Assumption A3 is classical since the paper of Bewley (1972). Mackey continuity of the preference ordering $\succsim_i$ expresses the idea that agent $\boldsymbol{i}$ prefers early to more distant consumption: it is a precise abstract way of formalizing Irving Fisher's notion of impatience. Araujo (1985) has shown that all agents must have Mackey continuous preferences if an Arrow-Debreu equilibrium is to exist for a general class of economies. This assumption permits the infinite horizon economy to be approximated by truncated finite horizon economies since consumption in the very distant future is unimportant. The role of this assumption becomes clear when we establish the existence of an equilibrium for the economy $\mathscr{E}_\infty$ by taking limits of equilibria of truncated economies $\mathscr{E}_T$: roughly speaking impatience is a way of coping with the open-endedness of the future and ending up with an economy which in many respects behaves like a finite horizon economy (but with no pre-specified terminal date).

As Bewley (1972) has shown, $\succsim_i$ is Mackey continuous if it is represented by an additively separable utility function,

$$(3.1) \qquad u^i(x^i) = \sum_{\xi \in \boldsymbol{D}} \rho(\xi)\delta_i^{t(\xi)} v^i(x^i(\xi)),$$

where $\rho(\xi)$ is the probability of ξ (induced by a probability measure ρ on the measureable subsets of S), $\delta_i \in (0,1)$ is a discount factor and $v^i: \mathbb{R}_+^L \to \mathbb{R}$ is a continuous, increasing concave function with $v^i(0) = 0$.

To explain Assumption A4 it is useful to introduce the concept of the degree of impatience of an agent at a node $\xi \in \boldsymbol{D}$. To this end, pick any feasible consumption plan for agent i, say $x^i \in F$, and add one unit of commodity 1 at node ξ to this bundle. Since commodity 1 is desired, agent i will strictly prefer this new consumption plan, $x^i + e_1^\xi \succ_i x^i$. By the Mackey continuity of $\succsim_i$ there exist $\beta_\xi < 1$ such that agent i still prefers the new consumption plan even if it is scaled down by the factor β_ξ for all nodes that strictly succeed ξ:

$$x^i \chi_{D \setminus D^+(\xi)} + \beta_\xi x^i \chi_{D^+(\xi)} + e_1^\xi \succsim_i x^i. \tag{3.2}$$

(3.2) can also be expressed by saying that agent i is ready to give up the proportion $1 - \beta_\xi > 0$ of his future consumption plan in order to have one more unit of good 1 at node ξ. Let $\underline{\beta}_\xi^i(x^i)$ denote the smallest (infimum) of the numbers β_ξ satisfying (3.2). Then for agent i with consumption plan $x^i, 1 - \underline{\beta}_\xi^i(x^i)$ is a local measure of impatience that we may call the *degree of impatience* of agent i (with consumption plan x^i) at node ξ. It can be shown that when the consumption plans x^i belong to a bounded set F, Mackey continuity of $\succsim_i$ implies that the degree of impatience $1 - \underline{\beta}_\xi^i(x^i)$ of agent i at node ξ can be bounded below by a positive number independent of x^i. The new requirement in A4 over and above Mackey continuity of $\succsim_i$ is that the degree of impatience of agent i be bounded away from zero *uniformly across the nodes*. There is thus a *uniform lower bound on the degree of impatience* of agent i: at each node he is prepared to give up at least the positive proportion $1 - \beta$ of his future consumption in exchange for one additional unit of commodity 1 at that node. Since there is a finite number of agents, β can be chosen independent of the agents. A4 is the only new assumption on preferences and endowments that we add to the assumptions made by Bewley (1972) in order to obtain existence of an equilibrium with incomplete markets. Note that A4 is satisfied by a preference ordering represented by (3.1).

A5 and A6 are assumptions on the securities available to agents on the financial markets and are thus specific to the GEI model. Since by A1 only a finite amount of uncertainty ($b(\xi) < \infty$) is resolved at each node ξ it seems reasonable to assume that only a finite number ($j(\xi) < \infty$) of securities are available for trading at each node. If $j(\xi) \geqq b(\xi)$ and the payoffs of $b(\xi)$ securities are linearly independent for all $\xi \in \boldsymbol{D}$, then markets are *complete*; if $j(\xi) < b(\xi)$ for some node $\xi \in \boldsymbol{D}$, then the financial markets are *incomplete*. The existence of a portfolio at each node which gives positive returns at each of the immediate successors is a classical assumption in the analysis of financial markets: assuming the existence of a riskless (numeraire) bond at each node (A6) is a convenient way of ensuring that this condition is satisfied.

4. DEBT CONSTRAINT, TRANSVERSALITY CONDITION, AND EQUILIBRIUM

In this section we explain the new conceptual issues involved in extending a concept of competitive equilibrium to an infinite horizon in a sequence economy with incomplete markets. The problem is to define a consistent concept of equilibrium based on agents' (perceptions of their) trading opportunities on the markets. Two elements are involved in the construction of an agent's budget set; the first is the usual condition which asserts than an agent's net expenditure on the spot markets must not exceed the income earned on the financial markets at each node; the second is a new element introduced by the sequential nature of trade combined with the open-endedness of the future.

Let $z^i(\xi)$ be the portfolio chosen by agent i at node ξ. At the (unique) predecessor ξ^- the agent has chosen the portfolio $z^i(\xi^-)$ which, given the normalization $p(\xi,1)=1$ yields the payoff $A(\xi)z^i(\xi^-)$. The agent's budget constraint at node ξ is thus given by

$$p(\xi)x^i(\xi) \leqq p(\xi)\omega^i(\xi) + A(\xi)z^i(\xi^-) - q(\xi)z^i(\xi). \tag{4.1}$$

Note that $z^i(\xi_0^-)$ is not a choice variable for the agent: we assume $z^i(\xi_0^-)=0$ so that agents do not inherit financial commitments from the past. The consumption-portfolio process (x^i, z^i) which is chosen must satisfy (4.1) at every node. (Since we have assumed that each agent's preference ordering is strictly monotone with respect to the first commodity at each node we may replace the inequality in (4.1) by an equality.)

If the agent is to have a solution to his consumption-portfolio choice problem, then the prices (p, q) must not offer arbitrage opportunities at any node $\xi \in \boldsymbol{D}$; i.e., there must not exist a portfolio $z^i(\xi)$ such that

$$\begin{aligned} -q(\xi)z^i(\xi) &\geqq 0, \\ A(\xi')z^i(\xi) &\geqq 0, \qquad \forall \xi' \in \xi^+, \end{aligned}$$

with at least one strict inequality. This condition has been extensively discussed in the finite horizon incomplete markets literature and in the theory of finance (see, for example, Magill-Shafer (1991)) and is equivalent to the existence of a process $\pi = (\pi(\xi), \xi \in \boldsymbol{D})$ of positive *node (present value) prices* such that

$$\pi(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \pi(\xi')A(\xi'), \qquad \forall \xi \in \boldsymbol{D}. \tag{4.2}$$

In view of the open-endedness of the future, even if the prices (p, q) do not offer arbitrage opportunities there will not be a solution to the agent's choice problem if a further restriction is not placed on the portfolio processes which the agent is permitted to consider. For any no-arbitrage prices (p, q) with $q \neq 0$ the agent can change any given portfolio $(z^i \to z^i + \Delta z^i)$ so as to obtain one more unit of income at node ξ and can roll over his debt ad infinitum thereafter. More formally, a change Δz^i in the portfolio such that

$$\begin{aligned} 1 &= -q(\xi)\Delta z^i(\xi), \\ 0 &= A(\xi')\Delta z^i(\xi'^-) - q(\xi')\Delta z^i(\xi'), \qquad \forall \xi' \in \boldsymbol{D}^+(\xi), \end{aligned} \tag{4.3}$$

is always feasible from any chosen portfolio $z^i \in Z$ and is preferred by agent i if his preference for consumption goods is monotone. A portfolio Δz^i satisfying these conditions is called a Ponzi scheme (for a discussion of this, see Levine

(1989, Section 3) and Blanchard-Fischer (1989, p. 49). Thus some form of borrowing constraint which limits the amount of debt that an agent can plan to incur is necessary if his consumption-portfolio choice probem is to have a solution.

Debt Constraints

One approach often adopted in macroeconomics for limiting the indebtedness of agents is to impose a debt constraint of the form

$$(4.4) \qquad q(\xi)z^i(\xi) \geqq -M, \qquad \forall \xi \in \boldsymbol{D},$$

for some positive number M. This leads to a budget set with *explicit debt constraint* M:

$$\mathscr{B}_\infty^M(p,q,\omega^i,A) = \left\{ x^i \in \ell_\infty^+(\boldsymbol{D}\times\boldsymbol{L}) \;\middle|\; \begin{array}{l} \exists z^i \in Z, \text{ such that } \forall \xi \in \boldsymbol{D} \\ q(\xi)z^i(\xi) \geqq -M, \\ p(\xi)(x^i(\xi)-\omega^i(\xi)) \\ \qquad = A(\xi)z^i(\xi^-) - q(\xi)z^i(\xi) \end{array} \right\}.$$

In an economy with impatient agents, the presence of the bound M puts an end to Ponzi schemes. For any process of present value prices π satisfying (4.2) and any date $T \geqq t(\xi)$, multiplying each equation in (4.3) by the appropriate node price and summing up to date T gives

$$(4.5) \qquad \pi(\xi) = -\sum_{\xi' \in \boldsymbol{D}_T(\xi)} \pi(\xi')q(\xi')\Delta z^i(\xi').$$

If there exists a present value process π satisfying (4.2) with $\pi \in \ell_1(\boldsymbol{D})$—so that the present value of income in the distant future goes to zero sufficiently fast—then the equality (4.6) can be satisfied only if $-q(\xi')\Delta z^i(\xi')$ grows without bound so that sooner or later $-q(\xi')(z^i(\xi') + \Delta^i(\xi))$ will exceed M.

Debt constraints can be introduced for a variety of reasons (for example to model capital market imperfections in addition to those introduced by the incompleteness of markets), but if the only role that we want the debt constraints to perform is to discourage agents from entering into Ponzi schemes then they should not per se introduce imperfections into the model. However, if M is chosen arbitrarily—that is, independently of the characteristics of the economy—then there will always be economies $\mathscr{E}(\boldsymbol{D}, \succsim, \omega, A)$ for which M is too small so that the debt constraints are *binding* and prevent agents from borrowing amounts which would otherwise be justified given their anticipated future income. What we want to establish is that for each economy $\mathscr{E}(\boldsymbol{D}, \succsim, \omega, A)$ there is a bound M which is well adapted to the economy in the sense that it prevents Ponzi schemes but is sufficiently large to permit all justified transfers of income—in short which is never binding.

One way to establish such a result is to leave the bound unspecified in the budget sets of the agents, imposing instead an implicit debt constraint of the form

$$(4.6) \qquad (qz^i) = (q(\xi) z^i(\xi), \xi \in \boldsymbol{D}) \in \ell_\infty(\boldsymbol{D})$$

which prevents each agent from considering trading strategies that lead to debt which grows without bound. This leads to a budget set with *implicit debt constraint*

$$\mathscr{B}_\infty^{DC}(p, q, \omega^i, A)$$

$$= \left\{ x^i \in \ell_\infty^+(\boldsymbol{D} \times \boldsymbol{L}) \,\middle|\, \begin{array}{l} \exists z^i \in Z \text{ with } (qz^i) \in \ell_\infty(\boldsymbol{D}) \text{ such that } \forall\ \xi \in \boldsymbol{D} \\ p(\xi)(x^i(\xi) - \omega^i(\xi)) \\ \qquad = A(\xi) z^i(\xi^-) - q(\xi) z^i(\xi) \end{array} \right\}.$$

If (x^i, z^i) is a consumption-portfolio plan for agent i satisfying the constraints in the budget set $\mathscr{B}_\infty^*(p, q, \omega^i, A)$ (where $*$ takes the place of the superscript indicating the type of constraint involved in the budget set), then z^i is said to *finance* x^i and with a slight abuse of notation we write

$$(x^i; z^i) \in \mathscr{B}_\infty^*(p, q, \omega^i, A).$$

$(\bar{x}^i; \bar{z}^i)$ is said to be $\succsim_i$ *maximal* in $\mathscr{B}_\infty^*(p, q, \omega^i, A)$ if $\bar{z}^i$ finances $\bar{x}^i$ and $\bar{x}^i \succsim_i x^i$ for all $(x^i; z^i) \in \mathscr{B}_\infty^*(p, q, \omega^i, A)$.

The budget sets $\mathscr{B}_\infty^{DC}$ and $\mathscr{B}_\infty^{M}$ lead to the following two concepts of equilibrium for an infinite horizon sequence economy.

DEFINITION 4.1: An *equilibrium with implicit debt constraint* (resp. with *explicit debt constraint M*) of the economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, A)$ is a pair

$$((\bar{x}, \bar{z}), (\bar{p}, \bar{q})) \in \ell_\infty^+(\boldsymbol{D} \times \boldsymbol{L} \times \boldsymbol{I}) \times Z^I \times \mathbb{R}^{\boldsymbol{D} \times \boldsymbol{L}} \times Q$$

where $(\bar{x}, \bar{z}) = (\bar{x}^1, \ldots, \bar{x}^I, \bar{z}^1, \ldots, \bar{z}^I)$, such that:

(i) $(\bar{x}^i; \bar{z}^i)$ is $\succsim_i$ maximal in $\mathscr{B}_\infty^{DC}(\bar{p}, \bar{q}, \omega^i, A)$ (resp. in $\mathscr{B}_\infty^{M}(\bar{p}, \bar{q}, \omega^i, A)$), for each $i \in \boldsymbol{I}$;

(ii) $\sum_{i \in \boldsymbol{I}} (\bar{x}^i - \omega^i) = 0$;

(iii) $\sum_{i \in \boldsymbol{I}} \bar{z}^i = 0$.

REMARK: Establishing existence of an equilibrium with implicit debt constraint shows that a solution to an agent's maximum problem exists as soon as trading strategies are eliminated which lead to debt that grows without bound. This property can be given either a *subjective* (self-monitoring) or an *objective* (market based) interpretation. In the first interpretation each agent conceives that there is a bound beyond which he will not be able to finance further debt in the market and he restricts his trading strategies accordingly: the mere fact that agents perceive that there is a limit to indebtedness leads to an equilibrium. The fact that existence of an equilibrium with implicit debt constraint implies existence of an equilibrium with an explicit bound M which is never binding leads to the second interpretation: a monitoring agency can announce an explicit bound M on the debts which are permissible for the agents in the economy (M is any number larger than $-\bar{q}(\xi)\bar{z}^i(\xi)$ for all $i \in \boldsymbol{I}$ and all $\xi \in \boldsymbol{D}$). The mere fact that this bound is announced is sufficient to discourage agents from attempting to indefinitely postpone their debts and the bound never needs to be enforced. The latter interpretation is close in spirit to the equilibrium concept considered by Levine-Zame (1992) who show that there is a system of debt constraints which satisfy certain consistency conditions and in addition are nonbinding which are sufficient to ensure the existence of an equilibrium. Corollary 5.3 will show that under the assumptions given in Section 3, the constraints can take the simple form of a uniform bound on the debts of the agents.

REMARK: In view of the normalizations of the spot prices $p(\xi,1)=1, \forall \xi \in \boldsymbol{D}$ adopted above, the constraints (4.4) or (4.6) imply that the real value of debt is bounded. If the price level grows over time (for example as a result of a continually growing supply of outside money) since the securities are real their prices q will grow in the same way and these constraints would not make sense: they would need to be replaced by

$$\frac{q(\xi)z^i(\xi)}{p(\xi,1)} \geqq -M, \quad \forall \xi \in \boldsymbol{D} \quad \text{and} \quad \left(\frac{qz^i}{p(\cdot,1)}\right) \in \ell_\infty(\boldsymbol{D})$$

respectively (for (4.4) and (4.6)), since it is the *real* and not the nominal value of debt which must be bounded. Note that $p(\xi,1)$ in the denominator can be replaced by any price index (for example $p(\xi)\sum_{i \in \boldsymbol{I}}\omega^i(\xi)$) which represents the level of prices at each node.

Transversality Condition

The other approach which is used in the literature to prevent Ponzi schemes is based on the use of a transversality condition. This more abstract approach has behind it both the tradition of Arrow-Debreu equilibrium theory and much of modern capital theory. Its use until now has been restricted to economies with complete markets. The idea which motivates it can be expressed as follows. If (p,q) is a no-arbitrage system of prices for a financial structure with complete markets then there exists a unique (up to normalization) vector of present value prices π such that (4.2) is satisfied at every node. If the trading activity (x^i, z^i) of an agent satisfies the budget equation (4.1) at each node ξ, then multiplying this equation by $\pi(\xi)$ and summing over all nodes to date T gives

$$\sum_{\xi \in \boldsymbol{D}^T} \pi(\xi)p(\xi)(x^i(\xi)-\omega^i(\xi)) = -\sum_{\xi' \in \boldsymbol{D}_T} \pi(\xi')q(\xi')z^i(\xi').$$

If the transversality condition

$$(4.7)\qquad \lim_{T\to\infty} \sum_{\xi'\in \boldsymbol{D}_T} \pi(\xi')q(\xi')z^i(\xi') = 0$$

holds, then the consumption plan x^i of agent i satisfies the Arrow-Debreu budget constraint

$$\sum_{\xi\in \boldsymbol{D}} \pi(\xi)p(\xi)(x^i(\xi) - \omega^i(\xi)) = 0 \Leftrightarrow \sum_{\xi\in \boldsymbol{D}} P(\xi)(x^i(\xi) - \omega^i(\xi)) = 0$$

where $P(\xi) = \pi(\xi)p(\xi)$ is the vector of Arrow-Debreu present value prices of the goods (deliverable) at node ξ. It is clear that the transversality condition (4.7) forms the bridge between equilibria of sequence economies with complete markets and the standard Arrow-Debreu equilibria.

Our object is to show that there is an appropriate extension of the transversality condition (4.7) to economies with incomplete markets which can be used to place a bound on the rate at which agents accumulate debt. This leads to a more abstract concept of equilibrium which has closer links with Arrow-Debreu theory than the concept of a debt constrained equilibrium and which can be exploited to obtain a relatively direct proof of existence of an equilibrium. Existence of a debt constrained equilibrium follows as a corollary.

The heuristic argument behind the transversality condition that we propose can be explained as follows. If agent i has an optimal consumption-portfolio plan (x^i, z^i) subject to the budget equation (4.1) and an appropriate growth condition on his debt, then there is associated with this plan a present value vector $\pi^i = (\pi^i(\xi), \xi \in \boldsymbol{D})$ where $\pi^i(\xi)$ is the multiplier (dual variable) induced by the budget equation (4.1) at node ξ. The vector π^i describes how agent i translates (discounts) a stream of income in the future to date 0. If the plan (x^i, z^i) is optimal, then π^i must satisfy first order conditions (4.2) which express the fact that for agent i the marginal cost of each security at each node is equal to the marginal benefit of its return at the following nodes. In addition at each node $\xi \in \boldsymbol{D}, (\pi^i, z^i)$ must satisfy the condition

$$(4.8)\qquad \limsup_{T\to\infty} \sum_{\xi'\in \boldsymbol{D}_T(\xi)} \pi^i(\xi')q(\xi')z^i(\xi') \leqq 0$$

which asserts that an optimal portfolio does not leave value (make agent i a lender) at infinity. If (4.8) were not satisfied, agent i could find a preferred consumption stream by decreasing his lending (which is always possible even if markets are incomplete), thereby increasing earlier consumption.

This is the first part of the argument—on which there is no disagreement. The next step, which seeks to use (4.8) and the rationality of agents to obtain a self-imposed restriction on the indebtedness of agents, is more controversial.

Even if trade is anonymous and agents do not know more about the characteristics of other traders than that they are rational, impatient and prefer more, no agent should count on finding lenders on the markets who would finance a portfolio z^i that permits him to be a borrower at infinity:

$$\liminf_{T\to\infty} \sum_{\xi' \in D_T(\xi)} \pi^i(\xi')q(\xi')z^i(\xi') < 0 \tag{4.9}$$

since this would oblige some other traders to be lenders at infinity. Strictly speaking, since markets are incomplete, agents on the other side of the transaction will evaluate their lending with different present value vectors, for when markets are incomplete the no-arbitrage equations (4.2) admit many solutions. But if markets are large and anonymous agent i cannot be expected to know the present value vectors of all other agents. In such circumstances it has become usual in the incomplete markets literature to make the assumption of *competitive perceptions* introduced by Grossman-Hart (1979): an agent uses his own present value vector to fill in the information regarding valuations which cannot be deduced from observed or anticipated prices. Using this convention, agent i will not attempt to finance a portfolio satisfying (4.9). Thus (recalling that (4.8) must be satisfied) the candidate growth (transversality) condition[2] on an agent i's debt is given by

$$\lim_{T\to\infty} \sum_{\xi' \in D_T(\xi)} \pi^i(\xi')q(\xi')z^i(\xi') = 0, \quad \forall \xi \in \boldsymbol{D}. \tag{4.10}$$

This leads us to define the budget set with *transversality condition* for agent i

$$\mathscr{B}_\infty^{TC}(p, q, \pi^i, \omega^i, A) = \left\{ x^i \in \ell_\infty^+(\boldsymbol{D}\times \boldsymbol{L}) \;\middle|\; \begin{array}{l} \exists z^i \in Z, \text{ such that } \forall \xi \in \boldsymbol{D} \\ \lim_{T\to\infty} \sum_{\xi' \in D_T(\xi)} \pi^i(\xi')q(\xi')z^i(\xi') = 0, \\ p(\xi)(x^i(\xi) - \omega^i(\xi)) \\ \qquad = A(\xi)z^i(\xi^-) - q(\xi)z^i(\xi) \end{array} \right\}$$

where π^i is the present value vector of agent i defined by condition (ii) below. The budget set $\mathscr{B}_\infty^{TC}$ leads to the following concept of equilibrium.

DEFINITION 4.2: An *equilibrium with transversality condition* of the economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, A)$ is a pair

$$(\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i\in \boldsymbol{I}}) \in \ell_\infty^+(\boldsymbol{D}\times\boldsymbol{L}\times\boldsymbol{I}) \times Z^I \times \mathbb{R}^{D\times L} \times Q \times \ell_1^+(\boldsymbol{D}\times\boldsymbol{I})$$

[2] In an earlier draft we imposed condition (4.10) solely at the initial node; in this case (as was pointed out by a perceptive referee), even if markets are incomplete, any Arrow-Debreu equilibrium can be achieved as a sequence equilibrium. It was by establishing necessary and sufficient conditions for an Arrow-Debreu equilibrium to be achievable as a sequence equilibrium only if markets are complete that we were led to condition (4.10); this result has been omitted in this paper. Note that when markets are complete (4.7) is sufficient to ensure that (4.10) is satisfied.

where $(\bar{x}, \bar{z}) = (\bar{x}^1, \ldots, \bar{x}^I, \bar{z}^1, \ldots, \bar{z}^I)$, such that:

(i) $(\bar{x}; \bar{z}^i)$ is $\succsim_i$ maximal in $\mathscr{B}_\infty^{TC}(\bar{p}, \bar{q}, \bar{\pi}^i, \omega^i, A)$, for each $i \in \boldsymbol{I}$;

(ii) for each $i \in \boldsymbol{I}$, (a) $\bar{\pi}^i(\xi) > 0$, $\forall \xi \in \boldsymbol{D}$ and $\bar{P}^i \in \ell_1^+(\boldsymbol{D} \times \boldsymbol{L})$ where $\bar{P}^i = (\bar{P}^i(\xi), \xi \in \boldsymbol{D}) = (\bar{\pi}^i(\xi)\bar{p}(\xi), \xi \in \boldsymbol{D})$; (b) $\bar{x}^i$ is $\succsim_i$ maximal in $B_\infty(\bar{P}^i, \omega^i) = \{x^i \in \ell_\infty^+(\boldsymbol{D} \times \boldsymbol{L}) | \bar{P}^i(x^i - \omega^i) \leqq 0\}$; (c) $\bar{\pi}^i(\xi)\bar{q}(\xi, j) = \sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi')A(\xi', j)$, $\forall j \in J(\xi)$, $\forall \xi \in \boldsymbol{D}$;

(iii) $\sum_{i \in \boldsymbol{I}} (\bar{x}^i - \omega^i) = 0$;

(iv) $\sum_{i \in \boldsymbol{I}} z^i = 0$.

Remark: Condition (ii) characterizes the equilibrium present value vector $\bar{\pi}^i$ of agent i. (b) and (c) express the fact that the first order conditions for the maximum problem of agent i are satisfied at $(\bar{x}^i, \bar{z}^i)$ if $\bar{\pi}^i(\xi)$ is the multiplier associated with the budget constraint at node ξ, for each $\xi \in \boldsymbol{D}$. (b) is simply a way of expressing the first order conditions with respect to x^i: more abstractly it asserts that the discounted price vector $\bar{P}^i$ supports the preferred set of agent i at $\bar{x}^i$ or equivalently that $\bar{x}^i$ is the agent's most preferred bundle in his induced Arrow-Debreu budget set $\mathscr{B}_\infty(\bar{P}^i, \omega^i)$. Mackey continuity of the agent's preference ordering will be shown to imply that $\bar{P}^i$ lies in $\ell_1(\boldsymbol{D} \times \boldsymbol{L})$, the condition required in (a). With the normalization (2.1) this implies that $\bar{\pi}^i$ lies in $\ell_1(\boldsymbol{D})$. (c) expresses the first order conditions with respect to z^i.

Remark: If the preferences of the agents are represented by additively separable utility functions of the form given in (3.1), then the transversality condition (4.10) can be written in standard stochastic process notation as

$$E\left(\delta_i^T v_i'(x_T^i) q_T z_T^i | \mathscr{F}_t\right) \to 0 \quad \text{when } T \to \infty, \quad \forall t \in \boldsymbol{T}.$$

Remark: In a model with incomplete markets over an open-ended future in which there are no objective present value prices for future income, an agent has no market-based way of assessing his ability to borrow from the market. In the absence of such prices to guide the actions of the agent, the idea of competitive behavior is blocked. It is at this step that we follow Grossman-Hart (1979) and use their concept of competitive price perceptions—the agent's own present value vector $\bar{\pi}^i$ being used as the agent's stand-in to calculate what he expects to be able to borrow from the market. The nonobservability of the agent's $\bar{\pi}^i$ vector implies that the transversality condition must be self-imposed since it cannot be objectively monitored by an agency (auctioneer)—and it is this step that some readers may find hard to accept. For such readers the concept of equilibrium in Definition 4.2 should be considered as an abstract construct which is used for proving existence of a debt constrained equilibrium. The full power of this abstract approach becomes clear when the proof of existence is extended to the case of infinite-lived securities (such as equity of firms): obtaining meaningful prices for such securities must be based on establishing the summability of the present value vectors in terms of which their fundamental values are calculated (see Magill-Quinzii (1993)).

5. EXISTENCE OF EQUILIBRIUM

Our object is to study the two concepts of equilibrium introduced in the previous section. We have argued that from a theoretical point of view an equilibrium with transversality condition is attractive because it permits concepts and techniques of infinite dimensional Arrow-Debreu theory to be carried over to a model with incomplete markets: this will become clear in the proof of existence of an equilibrium with transversality condition. On the other hand an equilibrium with debt constraint is simpler and probably provides a more plausible concept of equilibrium from the perspective of applied macroeconomics. In this section we show that for an economy satisfying Assumptions A1–A6 these two concepts lead to the same equilibria: every equilibrium with transversality condition is an equilibrium with debt constraints and conversely. Thus establishing existence of an equilibrium with transversality condition implies the existence of an equilibrium with debt constraints.

THEOREM 5.1: *Each economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, A)$ satisfying Assumptions A1–A6 has an equilibrium with transversality condition.*

THEOREM 5.2: *If $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, A)$ is an economy satisfying Assumptions A1–A6, then $((\bar{x}, \bar{z}),(\bar{p}, \bar{q}))$ is an equilibrium with implicit debt constraints if and only if there exist present value vectors $(\bar{\pi}^i)_{i\in \boldsymbol{I}}$ such that $((\bar{x}, \bar{z}),(\bar{p}, \bar{q},(\bar{\pi}^i)_{i\in \boldsymbol{I}}))$ is an equilibrium with transversality condition.*

COROLLARY 5.3: *For each economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, A)$ satisfying A1–A6 there is a bound $M > 0$ such that the economy has an equilibrium with explicit debt constraint M which is never binding.*

PROOF: For the given economy let $((\bar{x}, \bar{z}),(\bar{p}, \bar{q}))$ be an equilibrium with implicit debt constraint and let $M^* = \max_{i\in \boldsymbol{I}} \sup_{\xi\in \boldsymbol{D}} |\bar{q}(\xi)\bar{z}^i(\xi)|$. Choose $M > M^*$. *Q.E.D.*

REMARK: When markets are complete an equilibrium with transversality condition exists without Assumption A4. This follows from Bewley's theorem (1972, Theorem 2) asserting that an Arrow-Debreu equilibrium exists under Assumptions A1–A3. In fact when the security payoff process A is complete, Arrow-Debreu equilibrium allocations coincide with the equilibria with transversality condition. However there exist economies for which an equilibrium with transversality condition is not an equilibrium with debt constraint as shown by the following example.

EXAMPLE 5.4: Consider a one good economy ($L = 1$) with no uncertainty ($\boldsymbol{D} = \boldsymbol{Z}^+$, the nonnegative integers) and two (types of) agents with utility functions

$$u^1(x) = x_0, \qquad u^2(x) = \sum_{t=1}^{\infty} \frac{2}{t^2}\sqrt{x_t}$$

and endowments

$$\omega^1 = (0,1,1,\ldots,1,\ldots), \qquad \omega^2 = \left(\sum_{t=1}^{\infty} \frac{1}{t^2}, 0, 0, \ldots, 0, \ldots\right).$$

The Arrow-Debreu equilibrium is given by

$$\bar{x}^1 = \omega^2, \qquad \bar{x}^2 = \omega^1, \qquad \bar{\pi}_0 = 1, \qquad \bar{\pi}_t = \frac{1}{t^2}, \qquad t \geqq 1.$$

If there is a one-period bond at each date, then the sequence equilibrium with transversality condition has the same allocation $(\bar{x}^1, \bar{x}^2)$ and

$$\bar{z}_t^1 = -(t+1)^2 \sum_{\theta=t+1}^{\infty} \frac{1}{\theta^2}, \qquad \bar{q}_0 = 1, \qquad \bar{q}_t = \frac{t^2}{(t+1)^2}, \qquad t \geqq 1$$

so that

$$\bar{\pi}_T \bar{q}_T \bar{z}_T^1 = -\sum_{\theta=1}^{\infty} \frac{1}{(T+\theta)^2} \to 0$$

and

$$\bar{q}_T \bar{z}_T^1 = -\sum_{\theta=1}^{\infty} \left(\frac{T}{T+\theta}\right)^2 \to -\infty, \qquad \text{as } T \to \infty.$$

The Arrow-Debreu prices (after date 1) are determined by the marginal utility of agent 2. Since agent 2 becomes progressively more patient as time evolves, the present value at date T of agent 1's annuity $\sum_{\theta=1}^{\infty} \bar{\pi}_{T+\theta}/\bar{\pi}_T$ tends to infinity. It is the fact that the present value of agent 1's future income keeps growing that gives him the right to go progressively deeper into debt.

Assumption A4 (uniform lower bound on impatience) prevents agents from becoming progressively more patient and cuts out the phenomenon of agents going progressively deeper into debt. In fact Theorem 5.2 shows that *A4 is precisely the assumption which makes the equilibria with transversality condition coincide with the equilibria with (implicit) debt constraint for any economy* $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, A)$.

Proof of Theorem 5.1: With short-lived numeraire securities, redundant securities can be removed without changing the exchange opportunities of the agents. We may thus assume, without loss of generality, that the returns on the securities $j \in J(\xi)$ are linearly independent, i.e.,

$$\text{(5.1)} \qquad \operatorname{rank}\left[A(\xi', j)\right]_{\substack{\xi' \in \xi^+ \\ j \in J(\xi)}} = j(\xi) \leqq b(\xi).$$

The proof of existence of an equilibrium for the infinite horizon economy is obtained by taking limits of equilibria in truncated economies in which trade stops at some finite date. Let $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, A)$ be an infinite horizon economy. The associated T-truncated economy $\mathscr{E}_T(\boldsymbol{D}, \succsim, \omega, A)$ is the economy with the same characteristics as $\mathscr{E}_\infty$ in which agents are constrained to stop trading at date T. If $(p_T, q_T) \in \mathbb{R}^{D \times L} \times Q$ is a commodity and security price process, then the budget set of agent i in the truncated economy $\mathscr{E}_T$ is defined by

$$\mathscr{B}_T(p_T, q_T, \omega^i, A) = \left\{ x^i \in \ell_\infty^+(\boldsymbol{D} \times \boldsymbol{L}) \;\middle|\; \begin{array}{l} \exists z^i \in Z,\, z^i(\xi) = 0 \text{ if } t(\xi) \geqq T, \\ p_T(\xi)(x^i(\xi) - \omega^i(\xi)) \\ \qquad = A(\xi) z^i(\xi^-) - q_T(\xi) z^i(\xi) \text{ if } t(\xi) \leqq T, \\ x^i(\xi) = \omega^i(\xi) \text{ if } t(\xi) > t \end{array} \right\}.$$

Even though the consumption-portfolio process of an agent is defined over the whole event-tree, a T-truncated economy is essentially a finite horizon economy with $T+1$ periods since the consumption-portfolio process of an agent is fixed after date T.

DEFINITION 5.5: A GEI *equilibrium* of the truncated economy $\mathscr{E}_T(\boldsymbol{D}, \succsim, \omega, A)$ is a pair

$$((\bar{x}_T, \bar{z}_T), (\bar{p}_T, \bar{q}_T)) \in \ell_\infty^+((\boldsymbol{D} \times \boldsymbol{L} \times \boldsymbol{I})) \times Z^I \times \mathbb{R}^{D \times L} \times Q$$

such that:
(i) $(\bar{x}_T^i; \bar{z}_T^i)$ is $\succsim_i$ maximal in $\mathscr{B}_T(\bar{p}_T, \bar{q}_T, \omega^i, A)$, $\forall i \in \boldsymbol{I}$;
(ii) $\sum_{i \in \boldsymbol{I}} (\bar{x}_T^i - \omega^i) = 0$;
(iii) $\sum_{i \in \boldsymbol{I}} \bar{z}_T^i = 0$;
(iv) $\bar{p}_T(\xi) = 0$ if $t(\xi) > T$, $\bar{q}_T(\xi) = 0$ if $t(\xi) \geqq T$.

Since only the prices of the commodities and securities which are traded in $\mathscr{E}_T$ are well-determined, (iv) is a natural way of extending these prices to the whole event-tree. Since in an equilibrium of the truncated economy the terminal condition $z_T^i(\xi) = 0$ for all ξ with $t(\xi) \geqq T$ replaces the transversality condition (4.10), the present value vectors of the agents do not appear explicitly in Definition 5.5. Each agent has, however, a well-defined present value vector in an equilibrium of $\mathscr{E}_T$ which is characterized as follows.

LEMMA 5.6: *Under Assumptions A1–A3, if* $((\bar{x}_T, \bar{z}_T), (\bar{p}_T, \bar{q}_T))$ *is a GEI equilibrium of* $\mathscr{E}_T(\boldsymbol{D}, \succsim, \omega, A)$ *then each agent* $i \in \boldsymbol{I}$ *has a present value vector* $\bar{\pi}_T^i \in \mathbb{R}^D$ *satisfying*:
(a) $\bar{\pi}_T^i(\xi) > 0$ *if* $t(\xi) \leqq T$, $\bar{\pi}_T^i(\xi) = 0$ *if* $t(\xi) > T$;
(b) $\bar{x}_T^i$ *is* $\succsim_i$ *maximal in*

$$B_T(\bar{P}_T^i, \omega^i) = \{ x^i \in \ell_\infty^+(\boldsymbol{D} \times \boldsymbol{L}) \mid \bar{P}_T^i(x^i - \omega^i) \leqq 0;\ x^i(\xi) = \omega^i(\xi) \text{ if } t(\xi) > T \}$$

where $\bar{P}_T^i = (\bar{P}_T^i(\xi), \xi \in \boldsymbol{D}) = (\bar{\pi}_T^i(\xi) \bar{p}_T(\xi), \xi \in \boldsymbol{D})$;
(c) $\bar{\pi}_T^i(\xi) \bar{q}_T(\xi, j) = \sum_{\xi' \in \xi^+} \bar{\pi}_T^i(\xi') A(\xi', j)$, $\forall j \in J(\xi)$, $t(\xi) \leqq T - 1$.

PROOF: See Appendix.

Since all securities are short-lived numeraire securities, under A1–A6 every truncated economy has an equilibrium (see Geanakoplos-Polemarchakis (1986)). This existence result will be used to prove existence of an equilibrium for the infinite economy. The proof can be decomposed into three steps. The first consists in establishing uniform bounds (in T) on the truncated equililbria. The second is to take an appropriate limit of these equilibria. The final step is to show that this limit is an equilibrium for $\mathscr{E}_\infty$.

Step 1—Uniform Bounds: For every $T \in \boldsymbol{T}$ there exists a GEI equilibrium $((\bar{x}_T, \bar{z}_T), (\bar{p}_T, \bar{q}_T))$ for the truncated economy $\mathscr{E}_T$. Recall that the spot prices have been normalized by setting $p_T(\xi, 1) = 1$, $\forall \xi \in \boldsymbol{D}^T$. For each $i \in \boldsymbol{I}$ let $(\bar{\pi}_T^i, \bar{P}_T^i)$ denote the present value vector and the vector of discounted prices of agent i defined in Lemma 5.6. Since the relations satisfied by $(\bar{\pi}_T^i, \bar{P}_T^i)$ are homogeneous we may normalize $\bar{P}_T^i$ by setting

$$(5.2) \qquad \bar{P}_T^i \mathbb{1} = \sum_{(\xi, \ell) \in \boldsymbol{D}^T \times \boldsymbol{L}} \bar{P}_T^i(\xi, \ell) = 1, \qquad \forall i \in \boldsymbol{I}, \qquad \forall T \in \boldsymbol{T},$$

where $\mathbb{1} = (1, \ldots, 1, \ldots) \in \ell_\infty(\boldsymbol{D} \times \boldsymbol{L})$ denotes the vector whose components are all equal to 1. We shall now find bounds independent of T for $(\bar{x}_T^i(\xi, \ell), \bar{z}_T^i(\xi, j), \bar{\pi}_T^i(\xi))$ and $(\bar{p}_T(\xi, \ell), \bar{q}_T(\xi, j))$, $\forall i \in \boldsymbol{I}, \forall (\xi, \ell, j) \in \boldsymbol{D} \times \boldsymbol{L} \times \boldsymbol{J}$.

(i) *Bounds on* $\bar{x}_T^i(\xi, \ell)$: Since $\bar{x}_T^i(\xi, \ell) \geqq 0$ and $\sum_{i \in I} \bar{x}_T^i(\xi, \ell) = \sum_{i \in I} \omega^i(\xi, \ell) \leqq m'$, $0 \leqslant \bar{x}_T^i(\xi, \ell) \leqq m'$, $\forall\, (\xi, \ell) \in \boldsymbol{D} \times \boldsymbol{L}$, $\forall i \in \boldsymbol{I}$.

(ii) *Bounds on* $\bar{\pi}_T^i(\xi)$ *and* $\bar{p}_T(\xi, \ell)$: It suffices to consider $T \geqq t(\xi)$ since for $T < t(\xi)$, $\bar{\pi}_T^i(\xi) = 0$, $\forall i \in \boldsymbol{I}$ and $\bar{p}_T(\xi, \ell) = 0$, $\forall \ell \in \boldsymbol{L}$. Since $\bar{P}_T^i(\xi, 1) = \bar{\pi}_T^i(\xi)$ and $\sum_{(\xi, \ell) \in \boldsymbol{D}^T \times \boldsymbol{L}} \bar{P}_T^i(\xi, \ell) = 1$, $0 \leqq \bar{\pi}_T^i(\xi) \leqq 1$. Let us show that $\bar{\pi}_T^i(\xi)$ is uniformly positive for $T \geqq t(\xi)$. This is a consequence of the continuity of the agents' preferences and their strict monotonicity with respect to good 1, which imply the following property.

LEMMA 5.7: *For each* $\xi \in \boldsymbol{D}$ *there exists* $\alpha_\xi < 1$ *such that* $\forall i \in \boldsymbol{I}$

$$\alpha_\xi x^i + e_1^\xi \succ_i x^i, \qquad \forall x^i \in F.$$

PROOF: See Appendix.

By Lemma 5.6, $\bar{x}_T^i$ is $\succsim_i$ maximal in $B_T(\bar{P}_T^i, \omega^i)$. Consider scaling down agent i's consumption up to date T to $\alpha_\xi \bar{x}_T^i$. This would free the income $(1 - \alpha_\xi)\bar{P}_T^i \bar{x}_T^i = (1 - \alpha_\xi)\bar{P}_T^i \omega^i \geqq (1 - \alpha_\xi)m$ which could be converted into $(1 - \alpha_\xi)m / \bar{P}_T^i(\xi, 1)$ units of good 1 at node ξ. By Lemma 5.7 we must have for $T \geqq t(\xi)$

$$(5.3) \qquad \frac{(1 - \alpha_\xi)m}{\bar{P}_T^i(\xi, 1)} \leqq 1 \Leftrightarrow \bar{P}_T^i(\xi, 1) \geqq (1 - \alpha_\xi)m \Leftrightarrow \bar{\pi}_T^i(\xi) \geqq (1 - \alpha_\xi)m$$

since otherwise the new consumption would be preferred to $\bar{x}_T^i$, contradicting the optimality of $\bar{x}_T^i$ in $B_T(\bar{P}_T^i, \omega^i)$.

Since $0 \leqq \bar{P}^i_T(\xi, \ell) = \bar{\pi}^i_T(\xi)\bar{p}_T(\xi, \ell) \leqq 1$, (5.3) implies

$$(5.4) \qquad 0 \leqq \bar{p}_T(\xi, \ell) \leqq \frac{1}{(1-\alpha_\xi)m}, \qquad \forall \ell \in \boldsymbol{L}.$$

(iii) *Bounds on* $\bar{q}_T(\xi, j)$: Since $\bar{q}_T(\xi) = 0$ for $T \leqq t(\xi)$ it suffices to consider $T > t(\xi)$. Since $\sum_{i \in \boldsymbol{I}} \bar{z}^i_T(\xi) = 0 \Rightarrow \sum_{i \in \boldsymbol{I}} \bar{q}_T(\xi)\bar{z}^i_T(\xi) = 0$ there exists at least one agent $i \in \boldsymbol{I}$ with $\bar{q}_T(\xi)\bar{z}^i_T(\xi) \geqq 0$. Consider the following change in the portfolio of this agent: he scales down the portfolio $\bar{z}^i_T$ from node ξ onwards:

$$(5.5) \qquad \bar{z}^i_T \longrightarrow \begin{cases} \bar{z}^i_T(\xi') & \forall \xi' \notin \boldsymbol{D}(\xi), \\ \beta \bar{z}^i_T(\xi') & \forall \xi' \in \boldsymbol{D}(\xi), \end{cases}$$

where $\beta < 1$ is the factor defined by A4. Agent i can still consume $\omega^i(\xi')$ if $t(\xi') > T$, $\bar{x}^i_T(\xi')$ if $\xi' \in \boldsymbol{D} \backslash \boldsymbol{D}^+(\xi)$, and $\beta \bar{x}^i_T(\xi')$ if $\xi' \in \boldsymbol{D}^+(\xi)$ with $t(\xi') \leqslant T$ since

$$\bar{p}_T(\xi)\omega^i(\xi) + A(\xi)\bar{z}^i_T(\xi^-) - \beta \bar{q}_T(\xi)\bar{z}^i_T(\xi) \geqq \bar{p}_T(\xi)\bar{x}^i_T(\xi)$$

and for all $\xi' \in \boldsymbol{D}^+(\xi)$ with $t(\xi') \leqq T$

$$\begin{aligned} &\bar{p}_T(\xi')\omega^i(\xi') + A(\xi')\beta \bar{z}^i_T(\xi'^-) - \bar{q}_T(\xi')\beta \bar{z}^i_T(\xi') \\ &\quad = \beta \bar{p}_T(\xi')\bar{x}^i_T(\xi') + (1-\beta)\bar{p}_T(\xi')\omega^i(\xi'). \end{aligned}$$

This change frees the income

$$(1-\beta)\bar{p}_T(\xi')\omega^i(\xi') \geqq (1-\beta)m$$

at each node $\xi' \in \boldsymbol{D}^+(\xi)$ with $t(\xi') \leqq T$ and in particular at each successor $\xi' \in \xi^+$. By going short (i.e. borrowing) $(1-\beta)m$ units of the riskless bond j_ξ (which exists at each node ξ by A6) agent i can then increase his consumption of good 1 at node ξ by at least $\bar{q}_T(\xi, j_\xi)(1-\beta)m$. By A4 we must have

$$(5.6) \qquad \bar{q}_T(\xi, j_\xi)(1-\beta)m \leqq 1 \Leftrightarrow \bar{q}_T(\xi, j_\xi) \leqq \frac{1}{(1-\beta)m},$$

since otherwise the new consumption would be preferred to $\bar{x}^i_T$, contradicting the optimality of $\bar{x}^i_T$ in $\mathscr{B}_T(\bar{p}_T, \bar{q}_T, \omega^i, A)$. (5.6) then implies (with (c) in Lemma 5.6)

$$\bar{q}_T(\xi, j_\xi) = \frac{\sum_{\xi' \in \xi^+} \bar{\pi}^i_T(\xi')}{\bar{\pi}^i_T(\xi)} \leqq \frac{1}{(1-\beta)m}.$$

For an arbitrary security $j \in J(\xi)$, (5.6) implies

$$(5.7) \qquad |\bar{q}_T(\xi, j)| \leqq \sum_{\xi' \in \xi^+} \frac{\bar{\pi}^i_T(\xi')}{\bar{\pi}^i_T(\xi)} |A(\xi', j)| \leqq \frac{\delta(\xi)}{(1-\beta)m}$$

where

$$\delta(\xi) = \max\{|A(\xi', j)|, \xi' \in \xi^+, j \in J(\xi)\}.$$

(iv) *Bounds on* $\bar{q}_T(\xi)\bar{z}_T^i(\xi)$: As before, let $T > t(\xi)$. Consider an agent who is a net lender at node ξ, i.e., $\bar{q}_T(\xi)\bar{z}_T^i(\xi) \geqq 0$. This agent can consider scaling down his portfolio as in (5.5); he can then consume at least $\beta\bar{x}_T^i(\xi')$ for $\xi' \in \boldsymbol{D}^+(\xi)$ and increase his consumption of good 1 at node ξ by $(1-\beta))\bar{q}_T(\xi)\bar{z}_T^i(\xi)$. By A4 this increase must be less than 1 so that

$$\bar{q}_T(\xi)\bar{z}_T^i(\xi) \geqq 0 \Rightarrow \bar{q}_T(\xi)\bar{z}_T^i(\xi) \leqq \frac{1}{1-\beta}.$$

Since $\sum_{i \in I}\bar{q}_T(\xi)\bar{z}_T^i(\xi) = 0$ agents who are net borrowers must find net lenders. Thus

$$\bar{q}_T(\xi)\bar{z}_T^i(\xi) \leqq 0 \Rightarrow -\left(\frac{I-1}{1-\beta}\right) \leqq \bar{q}_T(\xi)\bar{z}_T^i(\xi)$$

so that

$$(5.8) \qquad -\left(\frac{I-1}{1-\beta}\right) \leqq \bar{q}_T(\xi)\bar{z}_T^i(\xi) \leqq \frac{1}{1-\beta}, \qquad \forall i \in \boldsymbol{I}.$$

Note that these bounds do not depend on ξ.

(v) *Bounds on* $\bar{z}_T^i(\xi, j)$: Let $T > t(\xi)$. For all $\xi' \in \xi^+$

$$A(\xi')\bar{z}_T^i(\xi) = \bar{p}_T(\xi')\bar{x}_T^i(\xi') - \bar{p}_T(\xi')\omega^i(\xi') + \bar{q}_T(\xi')\bar{z}_T^i(\xi').$$

(5.4) implies the inequalities

$$0 \leqq \bar{p}_T(\xi')\bar{x}_T^i(\xi') \leqq \frac{Lm'}{(1-\alpha_{\xi'})m}, \qquad 0 \leqq \bar{p}_T(\xi')\omega^i(\xi') \leqq \frac{Lm'}{(1-\alpha_{\xi'})m},$$

which with (5.8) imply for each $\xi' \in \xi^+$

$$(5.9) \qquad -\frac{Lm'}{(1-\alpha_{\xi'})m} - \frac{(I-1)}{1-\beta} \leqq A(\xi')\bar{z}_T^i(\xi) \leqq \frac{Lm'}{(1-\alpha_{\xi'})m} + \frac{1}{1-\beta}.$$

Since by (5.1) there are no redundant securities, (5.9) bounds $\bar{z}_T^i(\xi)$. This can be seen as follows. Let $\bar{u} \in \mathbb{R}^{b(\xi)}$ be defined by $\bar{u}(\xi') = A(\xi')\bar{z}_T^i(\xi)$. Consider the system of equations

$$\sum_{j \in J(\xi)} A(\xi', j)z(\xi, j) = \bar{u}(\xi'), \qquad \xi' \in \xi^+.$$

By (5.1) there exists a $j(\xi) \times j(\xi)$ determinant in $[A(\xi', j)]_{\xi' \in \xi^+, j \in J(\xi)}$ which is not zero. Keeping the corresponding equations and applying Cramer's formula to compute $(\bar{z}_T^i(\xi, j), j \in J(\xi))$ shows that there exists a bound $\gamma(\xi)$ such that

$$|\bar{z}_T^i(\xi, j)| < \gamma(\xi), \qquad j \in J(\xi).$$

Step 2—Limits: For a convenient summary of the notation and results from functional analysis that we use in the rest of the proof the reader is referred to the mathematical appendix of Bewley (1972). Let $ba(\boldsymbol{D}\times\boldsymbol{L})=l_\infty^*(\boldsymbol{D}\times\boldsymbol{L})$ denote the norm dual of $\ell_\infty(\boldsymbol{D}\times\boldsymbol{L})$ consisting of bounded finitely additive set functions on $\boldsymbol{D}\times\boldsymbol{L}$ and let $\|\cdot\|_{ba}$ denote the norm of $ba(\boldsymbol{D}\times\boldsymbol{L})$. The prices $(\bar{P}_T^i, T\in\boldsymbol{T})_{i\in\boldsymbol{I}}$ can be viewed as elements of $ba(\boldsymbol{D}\times\boldsymbol{L})$.

Let $\sigma(ba,\ell_\infty)$ denote the weak* topology of ba. Since $\bar{P}_T^i\mathbb{1}=\|\bar{P}_T^i\|_{ba}=1$, $\forall T\in\boldsymbol{T}$, $\forall i\in\boldsymbol{I}$ and since by Alaoglu's theorem the unit sphere in $ba(\boldsymbol{D}\times\boldsymbol{L})$ is $\sigma(ba,\ell_\infty)$ compact there exists a directed set $(\Lambda,\geqq)$, and a subnet $\{(\bar{P}_{T_\lambda}^i, i\in\boldsymbol{I}), \lambda\in(\Lambda,\geqq)\}$ such that $\bar{P}_{T_\lambda}^i$ converges to $\bar{P}^i$ in the $\sigma(ba,\ell_\infty)$ topology, $\forall i\in\boldsymbol{I}$.

Let $Y=\mathbb{R}^{D\times L\times I}\times Z^I\times\mathbb{R}^{D\times L}\times Q\times\mathbb{R}^{D\times I}$. In view of the bounds established in (i)–(v) of Step 1 it follows from Tychonov's theorem (Dunford-Schwartz (1966, p. 32)) that there exists a set $K\subset Y$ which is compact in the product topology on Y such that $((\bar{x}_T,\bar{z}_T,\bar{p}_T,\bar{q}_T,(\bar{\pi}_T^i)_{i\in\boldsymbol{I}})_{T\in\boldsymbol{T}}\subset K$. Thus by extracting an appropriate subnet, $((\bar{x}_{T_\lambda},\bar{z}_{T_\lambda},\bar{p}_{T_\lambda},\bar{q}_{T_\lambda},(\pi_{T_\lambda}^i)_{i\in\boldsymbol{I}})$ converges to $(\bar{x},\bar{z},\bar{p},\bar{q},(\bar{\pi}^i)_{i\in\boldsymbol{I}})$ in the product topology. By Theorem 9 (p. 292) and Theorem 1 (p. 430) of Dunford-Schwartz (1966) the Mackey topology and product topology coincide on bounded subsets of $\ell_\infty(\boldsymbol{D}\times\boldsymbol{L})$. Since $x_{T_\lambda}^i\in F$, $\forall T_\lambda$, $\forall i\in\boldsymbol{I}$ it follows that $\bar{x}_{T_\lambda}^i$ converges to $\bar{x}^i$ in the Mackey topology $\forall i\in\boldsymbol{I}$.

Step 3—Limit Is an Equilibrium: We begin by showing that $\bar{x}^i$ is $\succsim_i$ maximal in agent i's induced Arrow-Debreu budget set

$$B_\infty(\bar{P}^i,\omega^i)=\left\{x^i\in\ell_\infty^+(\boldsymbol{D}\times\boldsymbol{L})\,|\,\bar{P}^i(x^i-\omega^i)\leqq 0\right\}.$$

To this end we first show that

$$(5.10)\qquad x^i\in\ell_\infty^+(\boldsymbol{D}\times\boldsymbol{L}),\quad x^i\succsim_i\bar{x}^i\Rightarrow\bar{P}^ix^i\geqq\bar{P}^i\omega^i.$$

If $x^i\succsim_i\bar{x}^i$ then for any $\varepsilon>0$, $x^i+\varepsilon\mathbb{1}\succ_i\bar{x}^i$. Since $\chi_{\boldsymbol{D}\setminus\boldsymbol{D}^T}$ converges to zero in the Mackey topology,

$$(x^i+\varepsilon\mathbb{1})\chi_{\boldsymbol{D}^T}+\omega^i\chi_{\boldsymbol{D}\setminus\boldsymbol{D}^T}\to x^i+\varepsilon\mathbb{1}$$

in the Mackey topology. Since $\bar{x}_{T_\lambda}^i$ converges to $\bar{x}^i$ (Mackey) there exists $\bar{\lambda}\in\Lambda$ such that $\lambda>\bar{\lambda}$ implies

$$(x^i+\varepsilon\mathbb{1})\chi_{\boldsymbol{D}^{T_\lambda}}+\omega^i\chi_{\boldsymbol{D}\setminus\boldsymbol{D}^{T_\lambda}}\succ_i\bar{x}_{T_\lambda}^i.$$

The consumption vector on the left side of this relation could be considered by agent i in the economy $\mathscr{E}_{T_\lambda}$ since it coincides with ω^i after date T_λ. Since by Lemma 5.6 $\bar{x}_{T_\lambda}^i$ is $\succsim_i$ maximal in $B_{T_\lambda}(\bar{P}_{T_\lambda}^i,\omega^i)$ we must have

$$\bar{P}_{T_\lambda}^ix^i+\varepsilon>\bar{P}_{T_\lambda}^i\bar{x}_{T_\lambda}^i=\bar{P}_{T_\lambda}^i\omega^i,\qquad\forall\lambda>\bar{\lambda}.$$

Since $\bar{P}_{T_\lambda}^i$ converges to $\bar{P}^i$ in the $\sigma(ba,\ell_\infty)$ topology and since $x^i\in\ell_\infty^+(\boldsymbol{D}\times\boldsymbol{L})$,

$$\bar{P}^ix^i+\varepsilon\geqq\bar{P}^i\omega^i.$$

Letting $\varepsilon \to 0$ gives (5.10). It is now easy to show that

(5.11) $\quad x^i \in \ell_\infty^+(\boldsymbol{D}\times\boldsymbol{L}), \qquad x^i \succ_i \bar{x}^i \Rightarrow \bar{P}^i x^i > \bar{P}^i\omega^i.$

For suppose $x^i \succ_i \bar{x}^i$; then by continuity of $\succsim_i$ there exists $\alpha < 1$ such that $\alpha x^i \succ_i \bar{x}^i$. By (5.10), $\alpha \bar{P}^i x^i \geqq \bar{P}^i\omega^i \Rightarrow \bar{P}^i x^i > \bar{P}^i\omega^i$.

Since for each $\xi \in \boldsymbol{D}$

$$p_{T_\lambda}(\xi)\big(\bar{x}^i_{T_\lambda}(\xi) - \omega^i(\xi)\big) = A(\xi)\bar{z}^i_{T_\lambda}(\xi^-) - \bar{q}_{T_\lambda}(\xi)\bar{z}^i_{T_\lambda}(\xi)$$

involves only a finite number of terms, the equation is satisfied in the limit. Thus $(\bar{x}^i, \bar{z}^i)$ satisfy the budget equations:

(5.12) $\quad \bar{p}(\xi)(\bar{x}^i(\xi) - \omega^i(\xi)) = A(\xi)\bar{z}^i(\xi^-) - \bar{q}(\xi)\bar{z}^i(\xi), \qquad \forall \xi \in \boldsymbol{D}.$

For the same reason the first order conditions for agent i,

(5.13) $\quad \bar{\pi}^i(\xi)\bar{q}(\xi, j) = \sum_{\xi' \in \xi^+} \bar{\pi}^i(\xi')A(\xi', j), \qquad j \in J(\xi), \qquad \forall \xi \in \boldsymbol{D},$

are satisfied in the limit.

Since $\bar{P}^i \in ba(\boldsymbol{D}\times\boldsymbol{L})$ and $\bar{P}^i \geqq 0$, it follows from the Yosida-Hewitt theorem that there exists a unique decomposition $\bar{P}^i = \bar{P}^i_c + \bar{P}^i_f$ where $\bar{P}^i_c \in \ell_1^+(\boldsymbol{D}\times\boldsymbol{L})$ and $\bar{P}^i_f$ is a nonnegative purely finitely additive measure (sometimes called a pure charge). Furthermore $\bar{P}^i_f y = 0$ whenever $y \in \ell_\infty(\boldsymbol{D}\times\boldsymbol{L})$ has only a finite number of nonzero components. Since

$$\bar{P}^i_{T_\lambda} e^\xi_\ell = \bar{P}^i_{T_\lambda}(\xi, \ell) = \bar{\pi}^i_{T_\lambda}(\xi)\bar{p}_{T_\lambda}(\xi, \ell), \qquad \forall (\xi, \ell) \in \boldsymbol{D}\times\boldsymbol{L},$$

passing to the limit gives

$$\bar{P}^i e^\xi_\ell = \bar{P}^i_c(\xi, \ell) = \bar{\pi}^i(\xi)\bar{p}(\xi, \ell), \qquad \forall (\xi, \ell) \in \boldsymbol{D}\times\boldsymbol{L}.$$

Multiplying the budget equation (5.12) for node ξ by $\bar{P}^i_c(\xi, 1) = \bar{\pi}^i(\xi)$, adding the resulting equations for all nodes ξ with $t(\xi) \leqq T$ and using the equations (5.13) gives

(5.14) $\quad \sum_{\xi \in \boldsymbol{D}^T} \bar{P}^i_c(\xi)(\bar{x}^i(\xi) - \omega^i(\xi)) = -\sum_{\xi \in \boldsymbol{D}_T} \bar{\pi}^i(\xi)\bar{q}(\xi)\bar{z}^i(\xi).$

By (5.8), $\bar{q}_{T_\lambda}(\xi)\bar{z}^i_{T_\lambda}(\xi)$ is bounded uniformly in T_λ and ξ so that $(\bar{q}\bar{z}^i) = (\bar{q}(\xi)\bar{z}^i(\xi), \xi \in \boldsymbol{D}) \in \ell_\infty(\boldsymbol{D})$. Since

$$\Sigma_{\xi \in \boldsymbol{D}}\bar{\pi}^i(\xi) = \Sigma_{\xi \in \boldsymbol{D}}\bar{P}^i_c(\xi, 1) \leqq \Sigma_{(\xi, \ell) \in \boldsymbol{D}\times\boldsymbol{L}}\bar{P}^i_c(\xi, \ell) \leqq 1$$

implies $\bar{\pi}^i \in \ell_1(\boldsymbol{D})$, the term on the right side of (5.14) tends to zero in the limit and

(5.15) $\quad \bar{P}^i_c(\bar{x}^i - \omega^i) = 0.$

Suppose $\bar{P}^i_f > 0$. Then $\omega^i \geqq m\mathbb{1}$ implies $\bar{P}^i_f\omega^i > 0$. By (5.15) $\bar{P}^i_c\bar{x}^i = \bar{P}^i_c\omega^i < \bar{P}^i\omega^i$. By the strict monotonicity and Mackey continuity of $\succsim_i$ for all $\alpha > 0$ there exists $T > 0$ such that

(5.16) $\quad (\bar{x}^i + \alpha\mathbb{1})\chi_{\boldsymbol{D}^T} \succ_i \bar{x}^i.$

Choose $0 < \alpha \leqq \bar{P}^i_f \omega^i$; then

$$(5.17) \qquad \bar{P}^i(\bar{x}^i + \alpha \mathbb{1})\chi_{D^T} \leqq \bar{P}^i_c(\bar{x}^i + \alpha \mathbb{1}) \leqq \bar{P}^i_c \bar{x}^i + \alpha \leqq \bar{P}^i \omega^i.$$

(5.16) and (5.17) contradict (5.11). Thus $\bar{P}^i_f = 0$ so that $\bar{P}^i = \bar{P}^i_c$ and by (5.15), $\bar{P}^i(\bar{x}^i - \omega^i) = 0$. By (5.11) $\bar{x}^i$ is $\succsim_i$ maximal in the agent's induced Arrow-Debreu budget set $B_\infty(\bar{P}^i, \omega^i)$ and $(\bar{\pi}^i, \bar{P}^i)$ satisfy (a)–(c) in Definition 4.2(ii).

Let us show that $(\bar{x}^i; \bar{z}^i)$ is $\succsim_i$ maximal in the budget set $\mathscr{B}^{TC}_\infty(\bar{p}, \bar{q}, \bar{\pi}^i, \omega^i, A)$. $(\bar{q}\bar{z}^i) \in \ell_\infty(\boldsymbol{D})$ and $\bar{\pi}^i \in \ell_1(\boldsymbol{D})$ implies $\lim_{T \to \infty} \sum_{\xi' \in \boldsymbol{D}_T(\xi)} \bar{\pi}^i(\xi')\bar{q}(\xi')\bar{z}^i(\xi') = 0$, $\forall \xi \in \boldsymbol{D}$. Since the budget equations (5.12) are satisfied, $\bar{z}^i$ finances $\bar{x}^i$ and $\bar{x}^i \in \mathscr{B}^{TC}_\infty(\bar{p}, \bar{q}, \bar{\pi}^i, \omega^i, A)$. Since for any $(x^i; z^i) \in \mathscr{B}^{TC}_\infty(\bar{p}, \bar{q}, \bar{\pi}^i, \omega^i, A)$, $\lim_{T \to \infty} \sum_{\xi \in \boldsymbol{D}_T} \bar{\pi}^i(\xi)\bar{q}(\xi)z^i(\xi) = 0$ replacing $(\bar{x}^i, \bar{z}^i)$ in (5.14) by (x^i, z^i) gives $\bar{P}^i(x^i - \omega^i) = 0$ so that

$$\mathscr{B}^{TC}_\infty(\bar{p}, \bar{q}, \bar{\pi}^i, \omega^i, A) \subset B_\infty(\bar{P}^i, \omega^i), \qquad \forall i \in \boldsymbol{I}.$$

Thus $\bar{x}^i$ $\succsim_i$ maximal in $B_\infty(\bar{P}^i, \omega^i)$ is also $\succsim_i$ maximal in $\mathscr{B}^{TC}_\infty(\bar{p}, \bar{q}, \bar{\pi}^i, \omega^i, A)$, $\forall i \in \boldsymbol{I}$.

Since in the limit

$$\sum_{i \in \boldsymbol{I}} (\bar{x}^i - \omega^i) = 0 \quad \text{and} \quad \sum_{i \in \boldsymbol{I}} \bar{z}^i = 0$$

the limit $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in \boldsymbol{I}})$ is an equilibrium with transversality condition of the economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, A))$ and the proof is complete. *Q.E.D.*

Proof of Theorem 5.2: $(\Leftarrow)$ Let $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in \boldsymbol{I}}))$ be an equilibrium with transversality condition of the economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, A)$. The argument in (iv) of Step 1 in the proof of Theorem 5.1 can be applied to this equilibrium, since an agent who is a net lender at a node ξ can scale down his portfolio on $\boldsymbol{D}(\xi)$ without violating the transversality conditions (4.10). This leads to the inequality

$$-\left(\frac{I-1}{1-\beta}\right) \leqq \bar{q}(\xi)\bar{z}^i(\xi) \leqq \frac{1}{1-\beta}, \qquad \forall \xi \in \boldsymbol{D}, \qquad \forall i \in \boldsymbol{I},$$

which implies that $(\bar{q}\bar{z}^i) \in \ell_\infty(\boldsymbol{D})$ for each agent $i \in \boldsymbol{I}$. Thus for all $i \in \boldsymbol{I}$, $(\bar{x}^i; \bar{z}^i) \in \mathscr{B}^{DC}_\infty(\bar{p}, \bar{q}, \omega^i, A) \subset \mathscr{B}^{TC}_\infty(\bar{p}, \bar{q}, \bar{\pi}^i, \omega^i, A)$ and since $(\bar{x}^i; \bar{z}^i)$ is $\succsim_i$ maximal in the larger budget set $\mathscr{B}^{TC}_\infty$, it is $\succsim_i$ maximal in $\mathscr{B}^{DC}_\infty$. Thus $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}))$ is an equilibrium with implicit debt constraint.

$(\Rightarrow)$ Let $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}))$ be an equilibrium with implicit debt constraint of the economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, A)$. To show that this is an equilibrium with transversality condition we need to recover the agents' present value vectors and this can be done by extending the separation argument in Lemma 5.6 to the infinite horizon.

Lemma 5.8: *If $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}))$ is an equilibrium with implicit debt constraint of an economy $\mathscr{E}_\infty(\boldsymbol{D}, \succsim, \omega, A)$ satisfying $A1$–$A3$, then there exists a present value* *vector* $\bar{\pi}^i \in \ell_1(\boldsymbol{D})$ *satisfying conditions* (*a*)–(*c*) *in Definition* 4.2 (*ii*), *for each* $i \in \boldsymbol{I}$.

PROOF: See Appendix.

Since for each $i \in \boldsymbol{I}$, $\mathscr{B}_\infty^{DC}(\bar{p}, \bar{q}, \omega^i, A) \subset \mathscr{B}_\infty^{TC}(\bar{p}, \bar{q}, \omega^i, \bar{\pi}^i, A) \subset B(\bar{P}^i, \omega^i)$ and since $(\bar{x}^i; \bar{z}^i) \in \mathscr{B}_\infty^{DC}$ is $\succsim_i$ maximal in the larger budget set $B(\bar{P}^i, \omega^i)$, $(\bar{x}^i; \bar{z}^i)$ is $\succsim_i$ maximal in $\mathscr{B}_\infty^{TC}$. Thus $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in \boldsymbol{I}}))$ is an equilibrium with transversality condition. *Q.E.D.*

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APPENDIX

PROOF OF LEMMA 5.6: For an infinite dimensional vector $x^i \in \mathbb{R}^{\boldsymbol{D} \times \boldsymbol{L}}$, let $\hat{x}^i \in \mathbb{R}^{\boldsymbol{D}^T \times \boldsymbol{L}}$ denote the date T truncation of x^i, i.e., the components of x^i up to data T. Since the two convex subsets of $\mathbb{R}^{\boldsymbol{D}^T \times \boldsymbol{L}}$,

$$\mathscr{U}_T^i = \left\{ \hat{x}^i \in \mathbb{R}_+^{\boldsymbol{D}^T \times \boldsymbol{L}} \mid \hat{x}^i \chi_{\boldsymbol{D}^T} + \omega^i \chi_{\boldsymbol{D} \setminus \boldsymbol{D}^T} \succ_i \bar{x}_T^i \right\},$$

$$\hat{\mathscr{B}}_T^i = \left\{ \hat{x}^i \in \mathbb{R}_+^{\boldsymbol{D}^T \times \boldsymbol{L}} \mid \hat{x}^i \chi_{\boldsymbol{D}^T} + \omega^i \chi_{\boldsymbol{D} \setminus \boldsymbol{D}^T} \in \mathscr{B}_T\left(\bar{p}_T, \bar{q}_T, \omega^i, A \right) \right\},$$

satisfy $\mathscr{U}_T^i \cap \hat{\mathscr{B}}_T^i = \varnothing$, by the standard separation theorem there exists a vector $P_T^i \in \mathbb{R}^{\boldsymbol{D} \times \boldsymbol{L}}$ whose truncation $\hat{P}_T^i \in \mathbb{R}^{\boldsymbol{D}^T \times \boldsymbol{L}}$ is nonzero, such that

$$\sup_{\hat{x}^i \in \hat{\mathscr{B}}_T^i} \hat{P}_T^i \hat{x}^i \leqq \inf_{\hat{x}^i \in \mathscr{U}_T^i} \hat{P}_T^i \hat{x}^i.$$

Since $\hat{\bar{x}}_T^i \in \bar{\mathscr{U}}_T^i \cap \hat{\mathscr{B}}_T^i$ (where $\bar{\mathscr{U}}_T^i$ denotes the closure of $\mathscr{U}_T^i$),

$$\text{(A1)} \qquad \hat{P}_T^i \hat{x}^i \geqq \hat{P}_T^i \hat{\bar{x}}_T^i, \qquad \forall \hat{x}^i \in \mathscr{U}_T^i,$$

$$\text{(A2)} \qquad \hat{P}_T^i \hat{x}^i \leqq \hat{P}_T^i \hat{\bar{x}}_T^i, \qquad \forall \hat{x}^i \in \hat{\mathscr{B}}_T^i.$$

Monotonicity of $\succsim_i$ implies $P_T^i(\xi) \geqq 0$, $\forall \xi \in \boldsymbol{D}^T$. By (A2), when the system of linear inequalities

$$\text{(A3)} \qquad \bar{p}_T(\xi)\left(x^i(\xi) - \omega^i(\xi) \right) - A(\xi) z^i(\xi^-) + \bar{q}_T(\xi) z^i(\xi) \leqq 0, \qquad \forall \xi \in \boldsymbol{D}^T,$$

is satisfied by a vector $(\hat{x}^i, \hat{z}^i)$ such that $x^i(\xi) \geqq 0$, $\forall \xi \in \boldsymbol{D}^T$, $z^i(\xi_0) = 0 = z^i(\xi)$ if $t(\xi) = T$, then the inequality

$$\sum_{\xi \in \boldsymbol{D}^T} P_T^i(\xi)\left(x^i(\xi) - \bar{x}_T^i(\xi) \right) \leqq 0$$

holds. It follows from Theorem 21.2 in Rockafellar (1970) that there exists a nonnegative vector $(\bar{\pi}^i_T(\xi), \xi \in \boldsymbol{D}^T) \in \mathbb{R}^{\boldsymbol{D}^T}$ such that

(A4) $$P^i_T(\xi) \leqq \bar{\pi}^i_T(\xi)\bar{p}_T(\xi), \qquad \forall \xi \in \boldsymbol{D}^T,$$

(A5) $$\bar{\pi}^i_T(\xi)\bar{q}_T(\xi) - \sum_{\xi' \in \xi^+} \bar{\pi}^i_T(\xi')A(\xi') = 0, \qquad \forall \xi \in \boldsymbol{D}^{T-1},$$

(A6) $$\sum_{\xi \in \boldsymbol{D}^T} \bar{\pi}^i_T(\xi)\bar{p}_T(\xi)\omega^i(\xi) \leqq \sum_{\xi \in \boldsymbol{D}^T} P^i_T(\xi)\bar{x}^i_T(\xi).$$

Since $(\hat{\bar{x}}^i_T, \hat{\bar{z}}^i_T)$ satisfy (A3), by a standard summation argument (A5) implies

(A7) $$\sum_{\xi \in \boldsymbol{D}^T} \bar{\pi}^i_T(\xi)\bar{p}_T(\xi)\bar{x}^i_T(\xi) \leqq \sum_{\xi \in \boldsymbol{D}^T} \bar{\pi}^i_T(\xi)\bar{p}_T(\xi)\omega^i(\xi) \leqq \sum_{\xi \in \boldsymbol{D}^T} P^i_T(\xi)\bar{x}^i_T(\xi)$$

where the last inequality comes from (A6). It follows from (A7) that the inequality in (A4) can be strict only if $\bar{x}^i(\xi, \ell) = 0$ for some good ℓ. Thus if we define the new vector of discounted prices $\bar{P}^i_T$ by

$$\bar{P}^i_T(\xi) = \begin{cases} \bar{\pi}^i_T(\xi)\bar{p}_T(\xi), & \xi \in \boldsymbol{D}^T, \\ 0, & \xi \notin \boldsymbol{D}^T, \end{cases}$$

then $\hat{\bar{P}}^i_T \hat{\bar{x}}^i_T = \hat{\bar{P}}^i_T \omega^i = \hat{P}^i_T \hat{\bar{x}}^i_T > 0$ where the positivity follows from $0 \leqq \hat{P}^i_T \leqq \hat{\bar{P}}^i_T$, $\hat{P}^i_T \neq 0$, and $\omega^i(\xi, \ell) > m$, $\forall \xi \in \boldsymbol{D}^T$, $\forall \ell \in \boldsymbol{L}$. Thus if $\hat{x}^i \in \bar{\mathscr{U}}^i_T$, then $\hat{\bar{P}}^i_T \hat{x}^i \geq \hat{P}^i_T \hat{x}^i \geq \hat{P}^i_T \hat{\bar{x}}^i_T = \hat{\bar{P}}^i_T \hat{\bar{x}}^i_T = \hat{\bar{P}}^i_T \hat{\omega}^i$. Strict monotonicity with respect to good 1 and $\bar{P}^i_T \bar{x}^i_T > 0$ imply $\bar{P}_T(\xi, 1) > 0$ and hence $\bar{\pi}^i_T(\xi) > 0$ for all $\xi \in \boldsymbol{D}^T$. $\hat{\bar{P}}^i_T \hat{\bar{x}}^i_T > 0$ also implies that if $\hat{x}^i \in \mathscr{U}^i_T$, then $\hat{\bar{P}}^i_T \hat{x}^i > \hat{\bar{P}}^i_T \hat{\bar{x}}^i = \hat{\bar{P}}^i_T \hat{\omega}^i$. Thus $\hat{\bar{x}}^i_T$ is $\succsim_i$ maximal in $B_T(\bar{P}^i_T, \omega^i)$ and the proof is complete. *Q.E.D.*

Proof of Lemma 5.7: Let $\alpha_\xi(x^i)$ be defined by

$$\alpha_\xi(x^i) = \inf\left\{\alpha \in \mathbb{R} \,|\, 0 \leqslant \alpha \leqslant 1,\ \alpha x^i + e^\xi_1 \succ_i x^i\right\}.$$

By Assumption A3, $\alpha_\xi(x^i) < 1$. $F \subset \ell^+_\infty(\boldsymbol{D} \times \boldsymbol{L})$ is compact in the product topology. Let us show that $x^i \mapsto \alpha_\xi(x^i)$ is upper semi-continuous on F in the product topology. Let (x^i_ν) be a sequence of F converging to $\bar{x}^i \in F$ componentwise and suppose that for $\varepsilon > 0$, there exists a subsequence (which without loss of generality we call (x^i_ν)) such that

$$\alpha_\xi\left(x^i_\nu\right) > \alpha_\xi(\bar{x}^i) + \varepsilon.$$

Then, by definition of $\alpha_\xi(x^i_\nu)$

$$x^i_\nu \succsim_i \left(\alpha_\xi(\bar{x}^i) + \varepsilon\right)x^i_\nu + e^\xi_1.$$

Since on bounded sets the product topology and the Mackey topology coincide, x^i_ν converges to $\bar{x}^i$ in the Mackey topology and by Assumption A3

$$\bar{x}^i \succ_i \left(\alpha_\xi(\bar{x}^i) + \varepsilon\right)\bar{x}^i + e^\xi_1.$$

But this contradicts the definition of $\alpha_\xi(\bar{x}^i)$ since by monotonicity of the preferences if $\alpha \in (\alpha_\xi(\bar{x}^i), 1]$, then $\alpha\bar{x}^i + e^\xi_1 \succ_i \bar{x}^i$.

Thus α_ξ is upper semicontinuous for the product topology on F and attains its maximum on the compact set F. Then $\max_{i \in \boldsymbol{I}} \max_{x^i \in F}\{\alpha_\xi(x^i)\} < 1$ and there exists $\alpha_\xi < 1$ as asserted in Lemma 5.7. *Q.E.D.*

Proof of Lemma 5.8: Let $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}))$ be an equilibrium with implicit debt constraint and let

$$\mathscr{U}^i = \left\{x^i \in \ell^+_\infty(\boldsymbol{D} \times \boldsymbol{L}) \,|\, x^i \succ_i \bar{x}^i\right\}$$

be the strictly preferred set of agent i at $\bar{x}^i$. Since

$$\mathscr{U}^i \cap \mathscr{B}^{DC}_\infty\left(\bar{p}, \bar{q}, \omega^i, A\right) = \varnothing$$

and since both sets are convex and by the monotonicity of $\succsim_i$ (Assumption A3), $\mathscr{U}^i$ has a nonempty interior in the norm topology of $\ell_\infty(\boldsymbol{D}\times\boldsymbol{L})$, it follows from the separation theorem (Kelley-Namioka (1963, Theorem 14.2, p. 118)) that there exists a continuous linear functional $P^i \in ba(\boldsymbol{D}\times\boldsymbol{L})$, $P^i \neq 0$ which separates the two sets. Since $\bar{x}^i$ belongs to the closure of $\mathscr{U}^i$ in the norm topology it follows that

$$\text{(A8)} \qquad P^i x^i \geqq P^i \bar{x}^i, \qquad \forall\, x^i \in \mathscr{U}^i,$$

$$\text{(A9)} \qquad P^i x^i \leqq P^i \bar{x}^i, \qquad \forall\, x^i \in \mathscr{B}_\infty^{DC}.$$

Monotonicity of $\succsim_i$ implies $P^i \geqq 0$. Mackey continuity of $\succsim_i$ implies that P^i can be chosen in $\ell_1^+(\boldsymbol{D}\times\boldsymbol{L})$. This can be seen as follows. By the Yosida-Hewitt theorem there exists a unique decomposition $P^i = P_c^i + P_f^i$ where $P_c^i \in \ell_1^+(\boldsymbol{D}\times\boldsymbol{L})$ and P_f^i is a nonnegative purely finitely additive measure. Suppose $P_f^i \bar{x}^i = \varepsilon > 0$. Choose $\alpha > 0$ such that $\alpha P_c^i \mathbb{1} < \varepsilon$. By the monotonicity and Mackey continuity of $\succsim_i$ there exists $T > 0$ such that $(\bar{x}^i + \alpha \mathbb{1})\chi_{D^T} \succ_i \bar{x}^i$ and

$$P^i(x^i + \alpha \mathbb{1})\chi_{D^T} \leqq P_c^i(\bar{x}^i + \alpha \mathbb{1}) < P^i \bar{x}^i,$$

contradicting (A8). Thus $P_f^i \bar{x}^i = 0$. It follows that (A9) holds with P_c^i, since $x^i \in \mathscr{B}_\infty^{DC}$ implies $P_c^i x^i \leqq P^i x^i \leqq P^i \bar{x}^i = P_c^i \bar{x}^i$. Suppose (A8) does not hold with P_c^i; then there exists $x^i \in \mathscr{U}^i$ such that $P_c^i x^i < P_c^i \bar{x}^i = P^i \bar{x}^i$. By Mackey continuity of $\succsim_i$ there exists T sufficiently large such that $x^i \chi_{D^T} \succ_i \bar{x}^i$ and $P^i x^i \chi_{D^T} \leqq P_c^i x^i < P^i \bar{x}^i$, contradicting (A8). Thus (A8) and (A9) hold with $P^i \in \ell_1^+(\boldsymbol{D}\times\boldsymbol{L})$.

For each $T \in \boldsymbol{T}$, consider the consumption-portfolio plans (x^i, z^i) such that $z^i(\xi_0) = 0$, $z^i(\xi) = \bar{z}^i(\xi)$ if $t(\xi) \geqq T$, $x^i(\xi) = \bar{x}^i(\xi)$ if $t(\xi) > T$. By (A9), whenever such plans satisfy the linear inequalities

$$\bar{p}(\xi)\big(x^i(\xi) - \omega^i(\xi)\big) - A(\xi) z^i(\xi^-) + \bar{q}(\xi) z^i(\xi) \leqq 0, \qquad \forall\, \xi \in \boldsymbol{D}^T,$$

then the inequality

$$\sum_{\xi \in \boldsymbol{D}^T} P^i(\xi)\big(x^i(\xi) - \bar{x}^i(\xi)\big) \leqq 0$$

is satisfied. By the same arguments as in the proof of Lemma 5.6 there exists $\bar{\pi}_T^i \in \mathbb{R}_+^{\boldsymbol{D}}$ (with $\bar{\pi}_T^i(\xi) = 0$ if $t(\xi) > T$) such that

$$\text{(A10)} \qquad P^i(\xi, \ell) \leqq \bar{\pi}_T^i(\xi)\bar{p}(\xi, \ell) \quad \text{with equality whenever} \quad \bar{x}^i(\xi, \ell) > 0, \qquad \forall\, \xi \in \boldsymbol{D}^T,$$

$$\text{(A11)} \qquad \bar{\pi}_T^i(\xi)\bar{q}(\xi) - \sum_{\xi' \in \xi^+} \bar{\pi}_T^i(\xi') A(\xi') = 0, \qquad \forall\, \xi \in \boldsymbol{D}^{T-1},$$

$$\text{(A12)} \qquad \sum_{\xi \in \boldsymbol{D}^T} \bar{\pi}_T^i(\xi)\bar{p}(\xi)\omega^i(\xi) \leqq \sum_{\xi \in \boldsymbol{D}^T} \bar{\pi}_T^i(\xi)\bar{p}(\xi)\bar{x}^i(\xi) \leqq P^i \bar{x}^i.$$

By (A12), since $\bar{p}(\xi, 1) = 1$ and $\omega^i(\xi, 1) > 0$, the sequence $\{\bar{\pi}_T^i(\xi)\}_{T \in \boldsymbol{T}}$ is bounded for each $\xi \in \boldsymbol{D}$. By Tychonov's theorem there exists a subsequence $\{\bar{\pi}_{T_n}^i\}_{n=1}^\infty$ such that $\bar{\pi}_{T_n}^i$ converges to $\bar{\pi}^i \in \mathbb{R}^{\boldsymbol{D}}$ in the product topology. By Fatou's lemma

$$\sum_{\xi \in \boldsymbol{D}} \bar{\pi}^i(\xi)\bar{p}(\xi)\omega^i(\xi) \leqq P^i \bar{x}^i.$$

Thus if $\bar{P}^i$ is defined by $\bar{P}^i(\xi) = \bar{\pi}^i(\xi)\bar{p}(\xi)$, since $\omega^i(\xi, \ell) > m$, $\forall\,(\xi, \ell) \in \boldsymbol{D}\times\boldsymbol{L}$ it follows that

$$0 \leqq m \sum_{(\xi, \ell) \in \boldsymbol{D}\times\boldsymbol{L}} \bar{P}^i(\xi, \ell) \leqq \sum_{\xi \in \boldsymbol{D}} \bar{\pi}^i(\xi)\bar{p}(\xi)\omega^i(\xi)$$

so that $\bar{P}^i \in \ell_1(\boldsymbol{D}\times\boldsymbol{L})$.

Since (A10) and (A11) hold for the limit $\bar{\pi}^i$ for all $\xi \in \boldsymbol{D}$, arguments similar to those used in the proof of Lemma 5.6 show that $\bar{x}^i$ is $\succsim_i$ maximal in $B_\infty(\bar{P}^i, \omega^i)$. Thus $\bar{\pi}^i$ and $\bar{P}^i$ satisfy (a)–(c) in Definition 4.2(ii) and the proof is complete. *Q.E.D.*

REFERENCES

ARAUJO, A. (1985): "Lack of Pareto Optimal Allocations in Economies with Infinitely Many Commodities: The Need for Impatience," *Econometrica*, 53, 455–461.

BEWLEY, T. (1972): "Existence of Equilibria in Economies with Infinitely Many Commodities," *Journal of Economic Theory*, 4, 514–540.

BLANCHARD, O., AND S. FISCHER (1989): *Lectures on Macroeconomics*. Cambridge: MIT Press.

BROWN, D., AND L. LEWIS (1981): "Myopic Economic Agents," *Econometrica*, 49, 359–368.

DUNFORD, N., AND J. SCHWARTZ (1966): *Linear Operators, Part I*. New York: Interscience.

GEANAKOPLOS, J., AND H. POLEMARCHAKIS (1986): "Existence, Regularity and Constrained Suboptimality of Competitive Allocations when Markets are Incomplete," in *Uncertainty, Information and Communication, Essays in Honor of Kenneth Arrow, Vol. 3*, ed. by W. P. Heller, R. M. Ross, and D. A. Starrett. Cambridge: Cambridge University Press.

GROSSMAN, S. J., AND O. D. HART (1979): "A Theory of Competitive Equilibrium in Stock Market Economies," *Econometrica*, 47, 293–330.

KEHOE, T. J. (1989): "Intertemporal General Equilibrium Models," in *The Economic of Missing Markets, Information and Games*, ed. by F. Hahn. Oxford: Oxford University Press.

KELLEY, J. L., AND I. NAMIOKA (1963): *Linear Topological Spaces*. New York: Springer-Verlag.

LEVINE, D. (1989): "Infinite Horizon Equilibrium with Incomplete Markets," *Journal of Mathematical Economics*, 18, 357–376.

LEVINE, D., AND W. ZAME (1992): "Debt Constraints and Equilibrium in Infinite Horizon Economies with Incomplete Markets," Discussion Paper, University of California, Los Angeles.

LUCAS, R. (1978): "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429–1445.

MAGILL, M., AND W. SHAFER (1991): "Incomplete Markets," in *Handbook of Mathematical Economics, Vol. IV*, ed. by W. Hildenbrand and H. Sonnenschein. Amsterdam: North Holland.

MAGILL, M. J. P., AND M. QUINZII (1993): "Incomplete Markets Over an Infinite Horizon: Long-Lived Securities and Speculative Bubbles," University of Southern California Discussion Paper, forthcoming in *Journal of Mathematical Economics*.

MAS-COLELL, A., AND W. ZAME (1991): "Equilibrium Theory in Infinite Dimensional Spaces" in *Handbook of Mathematical Economics, Vol. IV*, ed. by W. Hildenbrand and H. Sonnenschein. Amsterdam: North Holland.

RADNER, R. (1972): "Existence of Equilibrium of Plans, Prices and Price Expectations in a Sequence of Markets," *Econometrica*, 40, 289–303.

ROCKAFELLAR, R. T. (1970): *Convex Analysis*. Princeton: Princeton University Press.