Economic Theory **21**, 435–454 (2003)

Economic Theory

Indeterminacy of equilibrium in stochastic OLG models*

Michael Magill[1] **and Martine Quinzii**[2]

[1] Department of Economics, University of Southern California, Los Angeles, CA 90089-0253, USA (e-mail: magill@usc.edu)
[2] Department of Economics, University of California, Davis, CA 95616-8578, USA (e-mail: mmquinzii@ucdavis.edu)

Received: November 19, 2001; revised version: March 22, 2002

Summary. This paper studies the equilibria of a stochastic OLG exchange economies consisting of identical agents living for two periods, and having the opportunity to trade a single infinitely-lived asset in constant supply. The agents have uncertain endowments and the stochastic process determining the endowments is Markovian. For such economies, the literature has focused on studying strongly stationary equilibria in which quantities and prices are functions of the exogenous states of nature which describe the uncertainty: such equilibria are generalizations of deterministic steady states, and this paper investigates if they have the same special status as asymptotic limits of other equilibrium paths. The difficulty in extending the analysis of equilibria beyond the class of strongly stationary equilibria comes from the presence of indeterminacy: we propose a procedure for overcoming this difficulty which can be decomposed into two steps. First backward induction arguments are used to restrict the domain of possible prices; then if some indeterminacy is left, expectation functions are introduced to make the forward equilibrium equations determinate. The properties of the resulting trajectories, in particular their asymptotic properties, can then be studied. For the class of models that we study this procedure provides a justification for focusing on strongly stationary equilibria. For the model with positive dividends (equity or land) the justification is complete, since we show that the strongly stationary equilibrium is the unique equilibrium. For the model with zero dividends (money) there is a continuum of self-fulfilling expectation functions resulting in a continuum of equilibrium paths starting from any admissible initial condition: under conditions given in the paper, these equilib-

* We are grateful for the stimulating environment and research support provided by the Cowles Foundation at Yale University during the Fall 2000 when this paper was first conceived. We are also grateful to the participants of the SITE Workshop at Stanford University and the Incomplete Markets Workshop at SUNY Stony Brook during the summer 2001 for helpful discussions.

Correspondence to: M. Magill

rium paths converge almost surely to one of the strongly stationary equilibria-either autarchy or the stochastic analogue of the Golden Rule.

Keywords and Phrases: Stochastic overlapping generations model, Stationary rational expectations equilibrium, Indeterminacy, Expectation functions, Martingale convergence theorem.

JEL Classification Numbers: D50, D84, C62.

1 Introduction

The overlapping generations model has proved to be a useful model for exploring properties of equilibria over time. It draws on the fact that one of the simplest sources of heterogeneity of agents comes from the fact they are born at different dates and the resulting intergenerational trade reflects the differences in agents' needs over their life cycle. While the model can be studied at various levels of generality (see Geanakoplos and Polemarchakis, 1991, for a survey), many of its basic properties and insights can be derived within the simplest class consisting of identical agents living for two periods, and having the opportunity to trade a single infinitely-lived asset in constant supply. The deterministic model was analyzed with great clarity by Gale (1973). For stochastic economies the theoretical literature has focused on studying strongly stationary equilibria in which quantities and prices are functions of the exogenous states of nature which describe the uncertainty. Existence of non-trivial equilibria of this type has been proved by Cass et al. (1992) and Gottardi (1996), while their normative properties, which are similar to those of the Golden Rule in the deterministic case, have been analyzed by Peled (1984), Aiyagari and Peled (1991), Demange and Laroque (1999) and Chattopadhyay and Gottardi (1999).

In the deterministic model, steady-state equilibria have a special status not only because they are simple, but also because in standard cases they are the asymptotic limits of all other equilibrium paths. Strongly stationary equilibria of the stochastic model are the generalizations of deterministic steady states, and the goal of this paper is to investigate if they have the same special status as asymptotic limits of other equilibrium paths.

The difficulty in extending the analysis of equilibria beyond the class of strongly stationary equilibria comes from the presence of indeterminacy. If uncertainty is modeled by the occurrence of S possible shocks at each date, then at each node of the associated event-tree the equilibrium equation is a first-order condition relating the price of the asset at this node to the S prices at the immediate successors: if the equilibrium equation is read forward then there is an $(S-1)$-dimensional manifold of "candidate prices" for the next period. If the model takes place over a finite horizon T and if there is a natural terminal condition (typically that the price is zero) then the equations can be read (solved) backwards to obtain a (typically) determinate equilibrium. If the horizon is infinite and there is no natural terminal condition, this way of obtaining determinacy cannot be applied. However backward induction can still be useful for finding restrictions on the equilibrium prices: at each date T in the future the prices have to satisfy certain inequalities (e.g. be positive and affordable by young agents) and drawing the consequences of these inequalities by backward induction can substantially restrict the indeterminacy of the prices. In the case where the asset pays a positive dividend (the asset is "land" or "equity") we show that this procedure has a rather dramatic outcome, for it eliminates all price sequences except the strongly stationary equilibrium, which, under our assumptions, is thus the unique equilibrium. However if the asset pays zero dividends (i.e is "money"), then this procedure only eliminates prices which lie above the stationary equilibrium prices, thus leaving an indeterminacy of dimension $S-1$ at each node.

The hypothesis of rational expectations requires that agents correctly anticipate both the support and the probabilities of future prices. In a series of papers, Kurz (1997) has argued that the latter hypothesis is unreasonable because agents can not be expected to know the stochastic process driving the uncertainty: learning about frequencies from past data still leaves room for differences in beliefs about the nature of the stochastic process.[1] However in the examples of equilibria with rational beliefs, Kurz still retains the assumption that agents anticipate the same prices, which is natural given the strong stationarity of the equilibria. In the stochastic model with money in which there is a continuum of prices which can be self fulfilling, it is the former assumption which might seem more restrictive, namely that agents anticipate the same future prices.

To resolve this difficulty we assume that agents co-ordinate their expectations through an expectation function, and the set of expectation functions parametrizes the $(S-1)$-dimensions of next period prices at each node. Since we retain a form of stationarity – the expectations are stationary in that they depend only on the current price and the current shock – it is perhaps not unreasonable to assume that agents could learn such an expectation function. Once introduced, the expectation function leads to a determinate stochastic difference equation for the equilibrium prices: the price equation at each node can be read forward as in the deterministic case, and the asymptotic properties of the equilibrium paths can again be studied. Using a martingale convergence argument we prove, under conditions spelled out in Section 2, that the equilibrium paths converge almost surely to one of the strongly stationary equilibria–either autarchy or the stochastic analogue of the Golden Rule.

Thus the general procedure that we propose for extending the analysis beyond the class of strongly stationary equilibria can be decomposed into two steps. First backward induction arguments are used to restrict the domain of prices. Then if some indeterminacy is left, expectation functions are introduced to make the forward

[1] More precisely Kurz introduces the concept of a "rational belief equilibrium" in which agents have differing conditional probabilities for prices at the immediate successors, even though they agree on the probabilities of tail (or asymptotic or long-run) events. Keynes (1930, ch.15) has emphasized the importance of differences in investors' opinions for a proper understanding of the functioning of financial markets–in the language of Wall Street, it is the changing proportion of the population of investors between bulls (optimists) and bears (pessimists) which accounts for much of the volatility of asset prices. The theory of rational beliefs is a way of formalizing this view of the functioning of financial markets based on short-run differences in beliefs, which are however compatible with the long-run behavior of prices.

equations determinate: the properties of the resulting trajectories, in particular their asymptotic properties, can then be studied. For the class of models that we are studying this procedure provides a justification for focusing on strongly stationary equilibria. For the model with positive dividends (equity or land) the justification is complete in that the the strongly stationary equilibrium is the only equilibrium; for the model with zero dividends (money) the justification is less complete–indeed the analysis suggests that it is something of an act of faith to focus attention on the stationary Golden Rule since many (and for some expectation functions, all) equilibrium paths converge to autarchy.

2 The model with money

Consider a one-good overlapping generations exchange economy in which agents live for two periods and have random endowments. The uncertainty is modeled by the occurrence of one of a finite number of shocks at each date, $s \in \boldsymbol{S} = \{1, \dots, S\}$, s_0 being the initial shock. Let $\sigma_t = (s_0, \dots, s_t)$ denote the history of the shocks from date 0 to date t: let $\Sigma_t = \boldsymbol{S} \times \cdots \times \boldsymbol{S}$ denote the set of all such histories up to date t and let $\Sigma = \cup_{t=0}^{\infty} \Sigma_t$ denote the collection of all such histories for all dates, $\sigma = (s_0, \dots, s_t, \dots)$ denoting a typical path of the event-tree Σ. We assume that the shocks follow a first-order Markov process and denote by P the induced probability on the event-tree Σ.

Assumption 1. (*Markov structure*): There exists a Markov transition matrix $\rho = [\rho_{ss'}]_{s,s' \in \boldsymbol{S}}$ with $\rho_{ss'} > 0$, $\forall s, s' \in \boldsymbol{S}$ such that $P(s_{t+1} = s' | s_t = s) = \rho_{ss'}$, $\forall s, s' \in \boldsymbol{S}$.

At each date-event $\sigma_t \in \Sigma$, n identical agents enter the economic stage: since we do not consider growth or fluctuations in the cohort size, we may set $n = 1$. The representative agent lives for two periods and has the random endowment stream $\omega(\sigma_t) = (\omega^1(\sigma_t), \omega^2(\sigma_t, s')_{s' \in \boldsymbol{S}})$ which depends only on the shock s_t realized when the agent is young, i.e. if $\sigma_t = (s_0, \dots, s_t)$, then $\omega(\sigma_t) = \left(\omega^1_{s_t}, (\omega^2_{s_t, s'})_{s' \in \boldsymbol{S}}\right)$.

Assumption 2. (*Positive endowments*): $\omega(s) = (\omega^1_s, (\omega^2_{ss'})_{s' \in \boldsymbol{S}}) \in \mathbb{R}^{S+1}_{++}$, $\forall s \in \boldsymbol{S}$.

All agents maximize the expected utility of their lifetime consumption streams, with the same utility indices. The representative agent born at node σ_t ranks the possible consumption streams $x(\sigma_t) = \left(x^1(\sigma_t), (x^2(\sigma_t, s'))_{s' \in \boldsymbol{S}}\right) \in \mathbb{R}^{S+1}_{+}$ according to a utility function $U_{\sigma_t} : \mathbb{R}^{S+1}_{+} \to \mathbb{R}$ satisfying:

Assumption 3. (*Preferences*): There exist increasing, concave, differentiable functions $u_1, u_2 : \mathbb{R}_{++} \to \mathbb{R}$ such that

$$U_{\sigma_t}(x(\sigma_t)) = u_1(x^1(\sigma_t)) + \sum_{s' \in \boldsymbol{S}} \rho_{s_t s'} u_2(x^2(\sigma_t, s'))$$

where for $i = 1, 2$, $\lim_{c \to 0} u_i'(c) = +\infty$ and u_2 has a coefficient of relative risk aversion less than or equal to 1: $\forall c > 0, -c\dfrac{u_2''(c)}{u_2'(c)} \leq 1$.

To study the consequences of intergenerational trade, we assume that there is an infinitely-lived asset available in positive supply, normalized to 1, which pays no dividends (usually called money). The asset is initially held by the representative old agent at date 0 and is then exchanged (if prices are non zero) at each date between the old and the young. Let $q(\sigma_t)$ denote the price of the asset at node σ_t. The young agent at node σ_t faces the budget constraints

$$x^1(\sigma_t) = \omega^1(\sigma_t) - q(\sigma_t)z, \; z \in \mathbf{R} \tag{1}$$

$$x^2(\sigma_t, s') = \omega^2(\sigma_t, s') + q(\sigma_t, s')z, \; s' \in \boldsymbol{S} \tag{2}$$

and chooses z to maximize $U_{\sigma_t}(x(\sigma_t))$. Under Assumption 3 the optimal choice of z is defined by the FOC for maximizing U_{σ_t} under the budget constraints (1), (2): since the equilibrium condition is $z(\sigma_t) = 1, \; \forall \sigma_t \in \Sigma$, the definition of an equilibrium takes the simple form:

Definition 1. $(q(\sigma_t))_{\sigma_t \in \Sigma}$ is an *equilibrium price process* if

$$u_1'(\omega^1_{s_t} - q(\sigma_t))q(\sigma_t) = \sum_{s' \in \boldsymbol{S}} \rho_{s_t s'} u_2'(\omega^2_{s_t s'} + q(\sigma_t, s'))q(\sigma_t, s'), \; \forall \sigma_t \in \Sigma \; (\mathcal{E})$$

It will sometimes be convenient to write $(\mathcal{E})$ in stochastic process notation as

$$u_1'(\omega^1_t - q_t)q_t = E\left(u_2'(\omega^2_{t+1} + q_{t+1})q_{t+1} \,|\mathcal{F}_t\right), \; \forall t \geq 0 \tag{$\mathcal{E}'$}$$

where $\mathcal{F}_t$ is the information available at date t.

In the deterministic case the stochastic difference equation $(\mathcal{E}')$ reduces to a simple difference equation which, under Assumption 3, defines q_{t+1} as a function of q_t: an equilibrium is a solution which satisfies some initial condition $q_0 = \bar{q}_0$ and respects the non-negativity of consumption at every date. In the stochastic case $(\mathcal{E})$ gives a single equation at each node σ_t for determining the S prices $(q(\sigma_t, s')_{s' \in \boldsymbol{S}})$ at the immediate successors, and this suggests that the equilibria will be indeterminate – unless the assumption of rational expectations which requires that agents' expectations be fulfilled at all dates along a trajectory introduces further restrictions which eliminate this indeterminacy. Most of the current literature on stochastic OLG models sidesteps the indeterminacy problem by studying stationary equilibria which depend only on the current shock. Since we will be led to study equilibria which are stationary on a larger state space, we will refer to equilibria which depend only on the exogenous shocks as strongly stationary equilibria.

Definition 2. $(q^*_s)_{s \in \boldsymbol{S}}$ is a *strongly stationary* equilibrium (SE^*) price vector[2] if

$$u_1'(\omega^1_s - q^*_s)q^*_s = \sum_{s' \in \boldsymbol{S}} \rho_{ss'} u_2'(\omega^2_{ss'} + q^*_{s'})q^*_{s'}, \quad \forall s \in \boldsymbol{S} \tag{$\mathcal{E}^*$}$$

[2] We use the short hand SE^* for strongly stationary equilibria to avoid confusion with the stationary sunspot equilibria which are often referred to as SSE.

($\mathcal{E}^*$) is a system of S equations to determine the S unknowns $(q_s^*)_{s \in \boldsymbol{S}}$, which typically has a finite number of solutions. It clear that the trivial (no-trade) equilibrium $q_s^* = 0$ for all s is always a solution of ($\mathcal{E}^*$). Cass et al. (1992) and Gottardi (1996) have proved existence of non-trivial SE^* in a more general class of models in which there is intergenerational trade with the same infinitely-lived asset as that considered here and, in addition, the young at each node are heterogeneous and trade short-lived assets with members of the same cohort to share their risks. The model studied here is simplified to focus attention on the intergenerational trade of the long-lived asset. Cass et al. concentrate on the case in which there is a positive solution to the system of equations ($\mathcal{E}^*$), while Gottardi studies the general case in which there is either a positive or a negative solution, a negative solution corresponding to the case where the supply of the asset is negative. We follow the former authors and focus on the positive case – namely where agents want to transfer income forward and the welfare of all agents can be improved by trading an asset in positive supply. To express the conditions under which this occurs, consider the matrix of present-value vectors of the representative agents born in the S possible states, at their initial endowments

$$\Pi^0 = \left[\pi^0_{ss'}\right]_{s,s' \in \boldsymbol{S}} = \left[\frac{\rho_{ss'}\, u_2'(\omega^2_{ss'})}{u_1'(\omega^1_s)}\right]_{s,s' \in \boldsymbol{S}}$$

Since Π^0 is a matrix with positive coefficients, by the Frobenius theorem (Gantmacher, 1959; Takayama, 1974) it has a unique positive eigenvalue (its Frobenius root) associated with a positive eigenvector. Let $\lambda_f(\Pi^0)$ denote this eigenvalue.

Assumption 4. $\lambda_f(\Pi^0) > 1$.

When Assumption 4 is satisfied, by the Frobenius theorem, there exists[3] a vector of transfers $dx = (dx_s)_{s \in \boldsymbol{S}} \gg 0$ such that

$$\Pi^0 dx = \lambda_f(\Pi^0) dx \gg dx$$

which can be expressed as

$$-u_1'(\omega^1_s)dx_s + \sum_{s' \in \boldsymbol{S}} \rho_{ss'}\, u_2'(\omega^2_{ss'})dx_{s'} > 0,\ \forall s \in \boldsymbol{S}$$

Thus if an infinitely-lived benevolent planner were to transfer the amount dx_s from the young to the old at each date-event when the shock is s, ($s \in \boldsymbol{S}$), the welfare of all agents would be improved. In short when $\lambda_f(\Pi^0) > 1$, agents need to transfer income to their old age, and an asset in positive supply permits such transfers to occur.

Proposition 1. *Under Assumptions A1–A4, there exists a unique positive strongly stationary equilibrium.*

[3] We use the following notation for vector inequalities. For $x \in \mathrm{R}^S$, $x \geq 0$ implies $x_s \geq 0, \forall\, s$ (x non-negative); $x > 0$ implies $x_s \geq 0, \forall\, s$, and $x_{s'} > 0$ for some s' (x semi-positive); $x \gg 0$ implies $x_s > 0, \forall\, s$ (x positive).

Proof. For a price vector $q \in \mathbf{R}^S$ for which $\omega_s^1 - q_s > 0, \omega_{ss'}^2 + q_{s'} > 0, \forall\, s, s' \in \boldsymbol{S}$, define

$$\Pi(q) = [\pi_{ss'}(q)]_{s,s' \in \boldsymbol{S}} = \left[\frac{\rho_{ss'}\, u_2'(\omega_{ss'}^2 + q_{s'})}{u_1'(\omega_s^1 - q_s)}\right]_{s,s' \in \boldsymbol{S}}$$

Equation $(\mathcal{E}^*)$ for a SE^* can be written as

$$\Pi(q^*)q^* = q^* \tag{3}$$

which implies that the Frobenius root $\lambda_f(\Pi(q^*)) = 1$. It can be deduced from Gottardi (1996) that if $\lambda_f(\Pi^0) \neq 1$ there exists either a positive or a negative solution to (3). By concavity of u_1 and u_2, if $q^* \ll 0$, $\pi_{ss'}(q^*) > \pi_{ss'}^0$ so that, by the Frobenius theorem, $\lambda_f(\Pi(q^*)) > \lambda_f(\Pi^0)$. Thus if the initial endowments of the economy satisfy $\lambda_f(\Pi^0) > 1$, (3) cannot have a negative solution: it follows that it has at least one positive solution.

To prove uniqueness of this solution, construct by induction the following sequence of prices: let $q^{(0)} = \omega^1$ where $\omega^1 = (\omega_s^1)_{s \in \boldsymbol{S}}$. Note that, for any SE^* price vector q^*, $u'(\omega_s^1 - q_s^*)$ must be well defined so that $q^* \ll q^{(0)}$. Define the next price $q^{(1)}$ by

$$u_1'(\omega_s^1 - q_s^{(1)})q_s^{(1)} = \sum_{s' \in \boldsymbol{S}} \rho_{ss'} u_2'(\omega_{ss'}^2 + q_{s'}^{(0)})q_{s'}^{(0)},\ \forall s \in \boldsymbol{S}$$

By concavity of the function u_1, the function $y \to u_1'(\omega_s^1 - y)y$ is increasing. By the Inada condition it increases from 0 to $+\infty$ when y increases from 0 to ω_s^1: thus $q^{(1)}$ is well defined and, since $u_1'(\omega_s^1 - q_s^{(1)})$ is also well defined for all s, $q^{(1)} \ll q^{(0)}$. By Assumption 3, the functions $h_{ss'}$ defined by $h_{ss'}(y) = u_2'(\omega_{ss'}^2 + y)y$ are increasing for $y > 0$ since

$$\begin{aligned} h_{ss'}'(y) &= u_2'(\omega_{ss'}^2 + y)\left(1 + \frac{u_2''(\omega_{ss'}^2 + y)y}{u_2'(\omega_{ss'}^2 + y)}\right) \\ &> u_2'(\omega_{ss'}^2 + y)\left(1 + \frac{u_2''(\omega_{ss'}^2 + y)(\omega_{ss'}^2 + y)}{u_2'(\omega_{ss'}^2 + y)}\right) \geq 0 \end{aligned}$$

Since, for any SE^*, $(\mathcal{E}^*)$ holds, $q^* \ll q^{(0)}$ implies $q^* \ll q^{(1)}$. It is now easy to see that the sequence $(q^{(n)})_{n \geq 1}$ defined by

$$u_1'(\omega_s^1 - q_s^{(n)})q_s^{(n)} = \sum_{s' \in \boldsymbol{S}} \rho_{ss'}\, u_2'(\omega_{ss'}^2 + q_{s'}^{(n-1)})q_{s'}^{(n-1)},\ \forall s \in \boldsymbol{S} \tag{4}$$

is a decreasing sequence such that $q^* \ll q^{(n)}$ for all n. It converges to $\bar{q}$ which, since the functions in (4) are continuous, is a SE^*.

Suppose that there is a SE^* price $q^* < \bar{q}$. Then by the monotonicity properties shown above $\pi_{ss'}(q^*) \geq \pi_{ss'}(\bar{q})$, with at least one strict inequality. By the Frobenius theorem, $\lambda_f(\Pi(q^*)) > \lambda_f(\Pi(\bar{q}))$, which contradicts $\lambda_f(\Pi(q^*) = \lambda_f((\Pi(\bar{q})) = 1$. Thus $\bar{q}$ is the unique positive SE^*. □

Enlarging the class of equilibrium solutions. Our objective is to study a broader class of solutions of the equilibrium equations ($\mathcal{E}$) than the strongly stationary solutions defined by ($\mathcal{E}^*$). The equilibrium conditions ($\mathcal{E}$) assert that at any given node σ_t of the event-tree Σ, given a current price $y \in \mathrm{R}_+$, there is a-priori an $(S-1)$-manifold $\Phi_{\sigma_t}(y)$ of prices of the asset at the S immediate successors of σ_t

$$\Phi_{\sigma_t}(y) = \left\{ q' \in \mathrm{R}^S \;\middle|\; \sum_{s' \in \boldsymbol{S}} \rho_{s_t s'}\, u_2'(\omega^2_{s_t s'} + q'_{s'}) q'_{s'} = u_1'(\omega^1_{s_t} - y) y \right\}$$

which justify an agent paying y for the asset at node σ_t. There are thus two possible ways of studying a broader class of solutions of ($\mathcal{E}$): the first is to study the solutions of the system of stochastic inclusions $q(\sigma_t^+) \in \Phi_{\sigma_t}(q(\sigma_t)), \forall\, \sigma_t \in \Sigma$, where $q(\sigma_t^+)$ denotes the vector of prices at the S immediate successors of σ_t; the second is to pick selections $\phi_{\sigma_t}(q(\sigma_t)) \in \Phi_{\sigma_t}(q(\sigma_t))$ and then study the stochastic difference equation $q(\sigma_t^+) = \phi_{\sigma_t}(q(\sigma_t)), \forall\, \sigma_t \in \Sigma$. We follow the latter approach.

Given the Markov structure of the endowments, Φ only depends on the current price y and the current shock s_t: $\Phi_{\sigma_t}(y) = \Phi_{s_t}(y)$. Although the selection in $\Phi_{\sigma_t}(q(\sigma_t))$ could depend on properties of the trajectory before date t, we restrict attention to selections which depend only on the current variables: $\phi_{\sigma_t}(q(\sigma_t)) = \phi_{s_t}(q(\sigma_t))$. Under this stationarity requirement a selection of the correspondence Φ is a family of S functions $q_s \to \phi_s(q_s)$ such that $\phi_s(q_s) \in \Phi_s(q_s)$ for all $s \in \boldsymbol{S}$, and such a family of functions will be called an expectation function. This function must be restricted to prices which are feasible, i.e. it must be restricted to a domain of prices affordable by the young.

Definition 3. An *expectation function for state* s is a function $\phi_s : Q_s \to \mathrm{R}^S_+$, where $Q_s \subset [0, \omega^1_s)$, which associates with every current price q_s a vector of prices $(\phi_{s1}(q_s), \ldots, \phi_{sS}(q_s))$ which justifies the purchase of the asset at price q_s, i.e. which satisfies

$$u_1'(\omega^1_s - q_s) q_s = \sum_{s' \in \boldsymbol{S}} \rho_{ss'}\, u_2'(\omega^2_{ss'} + \phi_{ss'}(q_s))\phi_{ss'}(q_s),\ \forall\, q_s \in Q_s \tag{5}$$

If $(\phi_s)_{s \in \boldsymbol{S}}$ is a family of expectation functions for the S states then the function

$$\phi : Q \times \boldsymbol{S} \to \mathrm{R}^S_+ \quad \text{defined by} \quad \phi(q, s) = \phi_s(q_s), \quad \text{where } Q = \prod_{s \in \boldsymbol{S}} Q_s$$

is called an *expectation function* for the economy.

Definition 4. $(q(\sigma_t))_{\sigma_t \in \Sigma}$ is a *rational expectations equilibrium with expectation function* ϕ if at each node $\sigma_t \in \Sigma$, with $\sigma_t = (s_0, \ldots, s_t)$, the prices at the S successors satisfy

$$q(\sigma_t, s') = \phi_{s_t s'}(q(\sigma_t)), \quad \forall s' \in \boldsymbol{S} \tag{$\mathcal{E}_\phi$}$$

Note that the assumption of rational expectations requires that all agents have the same expectation function ϕ: since the function ϕ is stationary on the state space

$Q \times \boldsymbol{S}$ it is perhaps not unreasonable to think that agents could learn to coordinate on a function ϕ. To generate a solution to $(\mathcal{E}_\phi)$, at each node $\sigma_t \in \Sigma$ and for each of its successors $s' \in \boldsymbol{S}$, the expectation function $\phi_{s_t s'}(q(\sigma_t))$ must select a feasible price to which the function $\phi_{s'}$ can be applied to form the expectation for the next date, i.e. $\phi_{s_t s'}(q(\sigma_t)) \in Q_{s'}$, and so on indefinitely. Thus to show that a rational expectations equilibrium in the sense of Definition 4 exists, one must exhibit an expectation function which is defined on a *self-justified domain*[4] $\bar{Q}$ i.e. a domain such that $\phi(\bar{Q} \times \boldsymbol{S}) \subset \bar{Q}$.

For any $a,\ b \in \mathbf{R}^S$ such that $a \leq b$ let $[a,b] = \prod_{s \in \boldsymbol{S}} [a_s, b_s]$ and $(a,b) = \prod_{s \in \boldsymbol{S}} (a_s, b_s)$.

Proposition 2. *Let q^* denote the positive strongly stationary equilibrium of Proposition 1. (i) The maximal domain $\bar{Q}$ for which there exists an expectation function ϕ such that $\phi(\bar{Q} \times \boldsymbol{S}) \subset \bar{Q}$ is $\bar{Q} = [0, q^*]$. (ii) There is a continuum of candidate expectation functions $\phi : \bar{Q} \times \boldsymbol{S} \to \bar{Q}$, and they all satisfy $\phi_s(0) = 0,\ \phi_s(q^*_s) = q^*$ for all $s \in \boldsymbol{S}$.*

Proof. (i) Consider the sequence $q^{(0)}, q^{(1)}, \ldots, q^{(n)}, \ldots$ constructed in the proof of Proposition 1, which converges to the unique SE^* price vector q^*. The domain Q of an expectation function must satisfy $Q \subset [0, \omega^1]$. Since by Assumption 3 the functions $y \to u_2'(\omega^2_{ss'} + y)y$ are increasing, in order that the next period price expectations can be chosen in the domain Q, the price q_s observed in state s must be such that

$$\begin{aligned} u_1'(\omega^1_s - q_s)q_s &\leq \sum_{s' \in \boldsymbol{S}} \rho_{ss'}\, u_2'(\omega^2_{ss'} + \omega^1_{s'})\omega^1_{s'} \\ &= \sum_{s' \in \boldsymbol{S}} \rho_{ss'}\, u_2'(\omega^2_{ss'} + q^{(0)}_{s'})q^{(0)}_{s'},\ \forall s \in \boldsymbol{S} \end{aligned}$$

Thus $Q \subset [0, q^{(1)}]$. By the same reasoning, since $\phi_s(q_s)$ needs to be in $[0, q^{(1)}]$ for all s, q_s must be less than $q^{(2)}$. Thus

$$\phi(Q \times \boldsymbol{S}) \subset Q \Longrightarrow Q \subset \bigcap_{n \geq 0} [0, q^{(n)}] = [0, q^*]$$

To show that there is an expectation function defined on $\bar{Q} = [0, q^*]$ whose range is in $\bar{Q}$, define $\phi_{ss'}(q_s) = \lambda q^*_{s'}$ where λ is the solution to

$$u_1'(\omega^1_s - q_s)q_s = \sum_{s' \in \boldsymbol{S}} \rho_{ss'}\, u_2'(\omega^2_{ss'} + \lambda q^*_{s'})\lambda q^*_{s'} \tag{6}$$

The function $h_s(\lambda) = \sum_{s' \in \boldsymbol{S}} \rho_{ss'} u_2'(\omega^2_{ss'} + \lambda q^*_{s'})\lambda q^*_{s'}$ is increasing (by Assumption 3), and such that $h_s(0) = 0 \leq u_1'(\omega^1_s - q_s)q_s \leq u_1'(\omega^1_s - q^*_s)q^*_s = h_s(1)$ for $q_s \in [0, q^*_s]$. Thus for any $q_s \in [0, q^*_s]$, there is a solution $\lambda \in [0,1]$ and the function $\phi = (\phi_s)_{s \in \boldsymbol{S}}$ maps $\bar{Q}$ into itself.

[4] Using the terminology of Duffie et al. (1994).

(ii) Note that if $0 < q_s < q_s^*$, then the solution λ of (6) satisfies $0 < \lambda < 1$ and the intersection of $\Phi_s(q_s)$ and the S-dimensional open set $(0, q^*)$ is a $(S-1)$-dimensional manifold of possible expectations $q' \in \bar{Q}$. Thus for each $s \in \boldsymbol{S}$ and each $q_s \in (0, q_s^*)$ there is a continuum of possible choice for $\phi_s(q_s)$. However since $q' \leq q^*$ implies

$$\sum_{s' \in \boldsymbol{S}} \rho_{ss'}\, u_2'(\omega_{ss'}^2 + q_{s'}')q_{s'}' \leq \sum_{s' \in \boldsymbol{S}} \rho_{ss'}\, u_2'(\omega_{ss'}^2 + q_{s'}^*)q_{s'}^* = u_1'(\omega_s^1 - q_s^*)q_s^*$$

the only expectation in $\bar{Q}$ which justifies the price q_s^* in state s is q^*. Since it is clear that the only price expectation which justifies a zero price is zero, the properties (ii) hold. □

Thus the global requirement that agents must anticipate prices which can be continued to form an infinite horizon equilibrium forces the feasible initial conditions to lie in $[0, q_{s_0}^*]$ and constrains agents' expectations of next period prices to lie in $\bar{Q}$ at every date. This property leads to a minimal stability property for the positive SE^* : if at some node where the current shock is s, the observed price of the asset is q_s^*, then the only expectation compatible with equilibrium is q^*, and the equilibrium coincides with the strongly stationary equilibrium forever after. The same stability property clearly holds for the no-trade equilibrium. However if the initial condition lies in the intermediate range, $0 < q_0 < q_{s_0}^*$, then there is a continuum of possible price expectations for the next period which can be fulfilled in equilibrium, so that there is a continuum of equilibrium trajectories starting at q_0.

In many deterministic OLG models the equilibrium conditions do not determine the initial conditions (prices and consumption at date 0) and these initial conditions must be specified exogenously: in these models, the dimension of indeterminacy is equal to the number of initial conditions which must be exogenously specified. For the deterministic version of the model that we are studying the dimension of indeterminacy is 1, since there is an equilibrium trajectory starting from any initial price q_0 for money such that $0 \leq q_0 \leq q^*$, where q^* is the Golden Rule price. In the stochastic case, not only is there an equilibrium starting at $q_0 \in [0, q_{s_0}^*]$, but if $0 < q_0 < q_{s_0}^*$ there is a continuum of such equilibrium paths. Thus the indeterminacy associated with the choice of the expectation function is over and above the indeterminacy present in the deterministic model.

The indeterminacy studied here has interesting connexions with the indeterminacy created by sunspots. The literature on sunspots has analyzed the conditions under which an economy with deterministic fundamentals (preferences and endowments) admits stochastic equilibria in which agents co-ordinate their beliefs on an exogenous stochastic process (see e.g. Chiappori and Guesnerie, 1991, or Guesnerie and Woodford, 1992, for surveys). The exogenous state space (extrinsic uncertainty) is introduced as a device to show that when agents believe that exogenous events can influence the economic outcome, such beliefs can be self-fulfilling – as a result the exogenous events end up influencing the equilibrium. The sunspot literature has focused on conditions under which economies with deterministic fundamentals have strongly stationary sunspot equilibria in which the equilibrium variables are functions of the extrinsic state. Because of this strong stationarity requirement no explicit expectation function ϕ is needed to describe the equilibria.

Our motivation for introducing a state space is quite different – we study equilibria of economies in which the fundamentals are stochastic, and this brings with it the natural state space required to describe the uncertainty. If agents' endowments are stochastic, then agents' beliefs must depend on the state of nature and this "intrinsic uncertainty" creates a continuum of possible beliefs. However the argument in the proof of Proposition 2 does not rely on the property that agents' endowments are random. Thus in the case where endowments are non-random ($\omega_s^1 = a, \forall s$ and $\omega_{s,s'}^2 = b, \forall s, s'$) Proposition 2 shows that there is a continuum of sunspot equilibria, stationary on the state space $Q \times \boldsymbol{S}$. However there is no strongly stationary sunspot equilibrium i.e. one for which the price is a non-trivial function of the exogenous state.[5]

The choice of an expectation function ϕ is less crucial if the asymptotic properties of the equilibria do not depend too much on the particular choice of ϕ. To study these asymptotic properties we need to introduce some additional notation.

Any vector $q \in \bar{Q}$ can be written as $q = \lambda \circ q^*$ where $q^* = (q_1^*, \ldots, q_S^*)$ is the positive SE^* price vector, $\lambda = (\lambda_1, \ldots, \lambda_S) \in [0,1]^S$ is the vector of scale factors relative to this price vector, and $\circ$ denotes the component-wise multiplication: $\lambda \circ q^* = (\lambda_1 q_1^*, \ldots, \lambda_S q_S^*)$. An expectation function $\phi : \bar{Q} \times \boldsymbol{S} \to \bar{Q}$ has associated with it a unique function ψ which determines the scale factors expected for next period. We define the *scale function for state* s, $\psi_s : [0,1] \to [0,1]^S$, induced by the expectation function ϕ, by:

$$\psi_s(\lambda_s) \circ q^* = \phi_s(\lambda_s q_s^*) = \phi_s(q_s)$$

where $\psi_s(\lambda_s) = (\psi_{s1}(\lambda_s), \ldots, \psi_{sS}(\lambda_s))$. The family of functions $(\psi_s)_{s \in \boldsymbol{S}}$ can be summarized by the function $\psi : [0,1]^S \times \boldsymbol{S} \to [0,1]^S$ defined by:

$$\psi(\lambda, s) = \psi_s(\lambda_s), \quad \forall\, \lambda \in [0,1]^S,\ s \in \boldsymbol{S}$$

The requirement that ϕ satisfies the equilibrium condition (5) is equivalent to the requirement that ψ satisfies the condition

$$u_1'(\omega_s^1 - \lambda_s q_s^*)\lambda_s q_s^* = \sum_{s' \in \boldsymbol{S}} \rho_{ss'}\, u_2'(\omega_{ss'}^2 + \psi_{ss'}(\lambda_s) q_{s'}^*)\psi_{ss'}(\lambda_s) q_{s'}^*, \qquad (7)$$
$$\forall\, \lambda_s \in [0,1],\ \forall\, s \in \boldsymbol{S}$$

If, for $s, s' \in \boldsymbol{S}$, we define the function

$$\Gamma_{ss'}(x,y) = \frac{u_2'(\omega_{ss'}^2 + x q_{s'}^*) q_{s'}^*}{u_1'(\omega_s^1 - y q_s^*) q_s^*}, \quad \forall x, y \in [0,1]$$

[5] It is well known that under Assumption A3 there is no non-trivial strongly stationary sunspot equilibrium. This can be seen as a simple consequence of Proposition 1. For if y^* is the Golden Rule price of the deterministic model, namely the positive solution of $u_1'(a - y^*)y^* = u_2'(b + y^*)y^*$, then $q_s^* = y^*, \forall s \in \boldsymbol{S}$, is a stationary equilibrium (which is trivial) and by Proposition 1 it is unique.

then (7) can be written as

$$\lambda_s = \sum_{s' \in \boldsymbol{S}} \rho_{ss'} \Gamma_{ss'}(\psi_{ss'}(\lambda_s), \lambda_s)\psi_{ss'}(\lambda_s), \qquad \forall\, \lambda_s \in [0,1],\ \forall\, s \in \boldsymbol{S}, \tag{8}$$

A rational expectations equilibrium generated by an expectation function ϕ: $[0, q^*] \times \boldsymbol{S} \to [0, q^*]$ can thus be equivalently described by its associated scale function ψ as:

$$\begin{aligned} q(\sigma_t) &= \lambda(\sigma_t) q^*_{s_t}, \\ \lambda(\sigma_t, s') &= \psi_{s_t s'}(\lambda(\sigma_t)), \\ \forall\, \sigma_t &= (s_0, \ldots, s_t) \in \Sigma_t,\ \forall\, t \geq 0 \end{aligned} \tag{9}$$

where ψ satisfies (8).

Proposition 3. *Let $(q(\sigma_t))_{t \geq 0}$ be a rational expectations equilibrium associated with an expectation function ϕ. (i) If ϕ is continuous and such that the associated scale function ψ satisfies*

$$\lambda_s \geq \sum_{s' \in \boldsymbol{S}} \rho_{ss'} \psi_{ss'}(\lambda_s), \quad \forall \lambda_s \in [0,1],\ \forall\, s \in \boldsymbol{S} \tag{10}$$

then for almost all $\sigma \in \Sigma$ the equilibrium path $(q_t(\sigma))_{t \geq 0}$ converges to a strongly stationary equilibrium. (ii) If $0 \leq q_0 < q^*_{s_0}$, equilibrium paths converge to the no-trade equilibrium with positive probability, i.e. there exists $\Sigma' \subset \Sigma$ with $P(\Sigma') > 0$ such that $q_t(\sigma) \to 0$ for all $\sigma \in \Sigma'$.

Proof. Let q_0 be the initial condition with $q_0 = \lambda_0 q^*_{s_0}$, $0 \leq \lambda_0 \leq 1$ and let $\lambda(\sigma_t)$ be the stochastic process defined by (9), which can also be written in stochatic process notation

$$\lambda_{t+1}(\sigma) = \psi_{s_t s_{t+1}}(\lambda_t(\sigma))$$

Note that the two strongly stationary equilibria of the economy, the no-trade and the positive stationary equilibrium, are characterized respectively by $\lambda_t = 0$ and $\lambda_t = 1$ for all $t \geq 0$. Thus to prove convergence of an equilibrium path to a SE^* we need to prove that the sequence $(\lambda_t(\sigma))_{t \geq 0}$ converges either to 0 or 1. Condition (10) implies that for all $t \geq 0$, $\lambda_t \geq E(\lambda_{t+1} | \mathcal{F}_t)$, i.e. that the process $(\lambda_t)_{t \geq 0}$ is a supermartingale. Since it is bounded below by 0, for almost all $\sigma \in \Sigma$ the sequence $(\lambda_t(\sigma))_{t \geq 0}$ converges. Let σ be a path on which the sequence converges and let $\bar{\lambda} \in [0,1]$ denote the limit.

Let us show that for any s, s', $\psi_{ss'}(\bar{\lambda}) = \bar{\lambda}$. Note that

$$|\psi_{ss'}(\bar{\lambda}) - \bar{\lambda}| \leq |\psi_{ss'}(\bar{\lambda}) - \psi_{ss'}(\lambda_t(\sigma))| + |\psi_{ss'}(\lambda_t(\sigma)) - \bar{\lambda}|$$

Since $\rho_{ss'} > 0$, with probability 1 the succession of states s and s' occurs an infinite number of times on a trajectory. Thus w.l.o.g we can assume that the trajectory σ is such that for any $T > 0$ there exists $\tau \geq T$ such that $s_\tau = s,\ s_{\tau+1} = s'$. For any such τ

$$|\psi_{ss'}(\bar{\lambda}) - \bar{\lambda}| \leq |\psi_{ss'}(\bar{\lambda}) - \psi_{ss'}(\lambda_\tau(\sigma))| + |\lambda_{\tau+1}(\sigma) - \bar{\lambda}| \tag{11}$$

Since ψ is continuous and $\lambda_t(\sigma) \to \bar{\lambda}$, for any $\varepsilon > 0$, there exist $T > 0$ such that if $\tau \geq T$ each term of the right side of (11) is less than ε. Since this is true for any $\varepsilon > 0$ it follows that $\psi_{ss'}(\bar{\lambda}) = \bar{\lambda}$.

Since ψ satisfies (8)

$$\bar{\lambda} = \sum_{s' \in \boldsymbol{S}} \rho_{ss'} \Gamma_{ss'}(\bar{\lambda}, \bar{\lambda})\bar{\lambda} \Longleftrightarrow \bar{\lambda}\left(1 - \sum_{s' \in \boldsymbol{S}} \rho_{ss'} \Gamma_{ss'}(\bar{\lambda}, \bar{\lambda})\right) = 0, \forall s \in \boldsymbol{S} \tag{12}$$

If $0 < \bar{\lambda} < 1$, then $\Gamma_{ss'}(\bar{\lambda}, \bar{\lambda}) > \Gamma_{ss'}(1,1) = \frac{u_2'(\omega^2_{ss'} + q^*_{s'})q^*_{s'}}{u_1'(\omega^1_s - q^*_s)q^*_s}$ and, since $\sum_{s' \in \boldsymbol{S}} \rho_{ss'}\ \Gamma_{ss'}\ (1,1) = 1$, $1 - \sum_{s' \in \boldsymbol{S}} \rho_{ss'} \Gamma_{ss'}(\bar{\lambda}, \bar{\lambda}) < 0$. Thus the only solutions to (12) are $\bar{\lambda} = 0$ or $\bar{\lambda} = 1$, which proves that the equilibrium path converges either to the no-trade equilibrium or to the positive SE^*.

(ii) Let $\bar{\lambda}(\sigma)$ denote the random variable (defined almost everywhere) which is the limit of $(\lambda_t(\sigma))_{t \geq 0}$. If $\lambda_0 = 0$, then $\lambda_t(\sigma) = 0$ for all t and all $\sigma \in \Sigma$, so that (ii) clearly holds. If $0 < \lambda_0 < 1$, by property (10), $\lambda_t \geq E(\lambda_{t+1}|\mathcal{F}_t)$: it follows that the sequence $E(\lambda_t)$ is non-increasing and $E(\lambda_t) \leq \lambda_0$. Since $\lambda_t(\sigma)$ converges almost surely to $\bar{\lambda}(\sigma)$ and is dominated by the integrable constant function 1, by the dominated convergence theorem, $E(\lambda_t) \to E(\bar{\lambda})$. Thus $E(\bar{\lambda}) \leq \lambda_0$, which implies that $\bar{\lambda}(\sigma)$ can not be equal to 1 on a set of probability 1, and thus must be equal to zero on a set of positive probability. □

In order to use the martingale theorem to prove convergence of the equilibrium paths, we had to impose condition (10) on the expectation function.[6] The strength of Proposition 3 depends on whether or not condition (10) seriously restricts the admissible expectation functions. First note that this condition is not vacuous. Condition (8), which is satisfied by any admissible function ψ, can be written as

$$\begin{aligned} \lambda_s &= E_s\Big(\Gamma_s(\psi_s(\lambda_s), \lambda_s)\psi_s(\lambda_s)\Big) \\ &= E_s\Big(\Gamma_s(\psi_s(\lambda_s), \lambda_s)\Big)E_s\Big(\psi_s(\lambda_s)\Big) \\ &\quad + \mathrm{cov}_s\Big(\Gamma_s(\psi_s(\lambda_s), \lambda_s), \psi_s(\lambda_s)\Big) \end{aligned} \tag{13}$$

where $\psi_s(\lambda_s)$ denotes the S-vector $(\psi_{ss'}(\lambda_s))_{s' \in \boldsymbol{S}}$, $\Gamma_s(\psi_s(\lambda_s), \lambda_s)$ denotes the vector $(\Gamma_{ss'}(\psi_{ss'}(\lambda_s), \lambda_s))_{s' \in \boldsymbol{S}}$ and E_s is the expectation with respect to the probability conditional on state s. Since λ_s and $\psi_{ss'}(\lambda_s)$ are less or equal to 1 and $\Gamma_{ss'}$ is decreasing in both components, $E_s\ (\Gamma_s(\psi_s(\lambda_s), \lambda_s)) \geq E_s\ (\Gamma_s(1,1)) = 1$ with a strict inequality if either λ_s or some component of $\psi_{ss'}$ is strictly less than 1. Thus if the covariance term is either non-negative or negative and small, the inequality

[6] For any expectation function, the price process (q_t) is a supermartingale with respect to the measure $\tilde{P}$ on Σ induced by the Markov chain $\tilde{\rho}_{ss'} = \rho_{ss'}\Gamma_{ss'}(1,1)$. Thus for a set of measure 1 with respect to $\tilde{P}$, the equilibrium paths $q_t(\sigma)$ converge. Unfortunately in the infinite horizon case a change of conditional probability does not result in an "equivalent martingale measure" since the measure $\tilde{P}$ is not absolutely continuous with respect to P: convergence on a set of measure 1 for $\tilde{P}$ essentially proves nothing for the typical trajectory $\sigma \in \Sigma$ under the measure P. We are indebted to Jean Francois Mertens for pointing out this mistake in an earlier version of this paper.

(10) will hold. In particular if $\psi_{ss'}(\lambda_s)$ is constant as in the expectation function constructed in the proof of Proposition 2 then (10) is satisfied. (10) essentially places restrictions on how different the scale factors for each future state can be. When the scale factors are not constant, the greater the term $E_s\Big(\varGamma_s(\psi_s(\lambda_s),\lambda_s)\Big)$, the more negative the covariance term in (13) can be without violating (10). Intuitively since for given x, y, the function $\varGamma_{ss'}(x, y)$ increases when $\omega^2_{ss'}$ decreases (or ω^1_s increases), condition (10) will tend to be satisfied if the endowments of the old are sufficiently small relative to that of the young. Although we do not have an argument which applies to the general case, we can show that for log and power utilities, if the endowments of the old are sufficiently small relative to those of the young, then condition (10) is satisfied by any expectation function ϕ.

Example. Let $S = 2$, $\boldsymbol{S} = \{a, b\}$, and assume that the shocks are i.i.d with conditional probabilities ρ_a and ρ_b. The preferences of the representative agent born at any node are defined by the utility function

$$U(x) = \log(x^1) + \rho_a \log(x^2_a) + \rho_b \log(x^2_b)$$

We assume that the endowments $(\omega^1_a, \omega^1_b, \omega^2_a, \omega^2_b)$ satisfy Assumption 4 which, in this case, reduces to $\rho_a \dfrac{\omega^1_a}{\omega^2_a} + \rho_b \dfrac{\omega^1_b}{\omega^2_b} > 1$: if the endowments of the young are on average greater that those of the old, then agents want to transfer forward. A strongly stationary equilibrium is a solution of the system of equations

$$\frac{q_a}{\omega^1_a - q_a} = \frac{q_b}{\omega^1_b - q_b} = \frac{\rho_a q_a}{\omega^2_a + q_a} + \frac{\rho_b q_b}{\omega^2_b + q_b}$$

which is equivalent to the system

$$\frac{q_a}{\omega^1_a} = \frac{q_b}{\omega^1_b}, \quad q_a \left(\frac{1}{\omega^1_a - q_a} - \frac{\rho_a}{\omega^2_a + q_a} - \frac{\rho_b}{\frac{\omega^2_b \omega^1_a}{\omega^1_b} + q_a} \right) = 0 \tag{14}$$

which has the solution $q = 0$ and a positive solution q^* whose analytical expression is too complicated to be interesting. We will however use the fact that when $\omega^2 \to 0$ this solution tends to $(\omega^1_a/2, \omega^1_b/2)$.

An expectation function $\phi = (\phi_a, \phi_b)$ is such that, for $q_a \in [0, q^*_a]$, ϕ_a selects a point in the set

$$\varPhi_a(q_a) \cap [0, q^*] = \left\{ (q'_a, q'_b) \in [0, q^*] \,\middle|\, \frac{\rho_a q'_a}{\omega^2_a + q'_a} + \frac{\rho_b q'_b}{\omega^2_b + q'_b} = \frac{q_a}{\omega^1_a - q_a} \right\}$$

and, for $q_b \in [0, q^*_b]$, ϕ_b selects a point in the set

$$\varPhi_b(q_b) \cap [0, q^*] = \left\{ (q'_a, q'_b) \in [0, q^*] \,\middle|\, \frac{\rho_a q'_a}{\omega^2_a + q'_a} + \frac{\rho_b q'_b}{\omega^2_b + q'_b} = \frac{q_b}{\omega^1_b - q_b} \right\}$$

From the analysis above we know that the sets $\varPhi_a(q_a) \cap [0, q^*]$ and $\varPhi_b(q_b) \cap [0, q^*]$ are non empty if $0 \le q_a \le q^*_a$ and $0 \le q_b \le q^*_b$.

When the price vectors in the cube $[0, q^*]$ are expressed in terms of the scale factors, $q = \lambda \circ q^*$ with $\lambda \in [0,1]^2$, a function ϕ is equivalent to a scale function ψ where, for each $\lambda_a \in [0,1]$, ψ_a selects a point in

$$\Psi_a(\lambda_a) \cap [0,1]^2 = \left\{ (\lambda'_a, \lambda'_b) \in [0,1]^2 \,\middle|\, \frac{\rho_a \lambda'_a}{e_a^2 + \lambda'_a} + \frac{\rho_b \lambda'_b}{e_b^2 + \lambda'_b} = \frac{\lambda_a}{e_a^1 - \lambda_a} \right\}$$

and for each $\lambda_b \in [0,1]$ ψ_b selects a point in

$$\Psi_b(\lambda_b) \cap [0,1]^2 = \left\{ (\lambda'_a, \lambda'_b) \in [0,1]^2 \,\middle|\, \frac{\rho_a \lambda'_a}{e_a^2 + \lambda'_a} + \frac{\rho_b \lambda'_b}{e_b^2 + \lambda'_b} = \frac{\lambda_b}{e_b^1 - \lambda_b} \right\}$$

where e_s^i denote the normalized endowments: $e_s^i = \omega_s^i / q_s^*$, $i = 1, 2$, $s = a, b$. Note that by (14) $e_a^1 = e_b^1$, so that $\Psi_a(\lambda) = \Psi_b(\lambda)$ for any $\lambda \in [0,1]$. Furthermore since the function $x \to \frac{x}{\alpha + x}$ is concave for all positive values of α, the level curves $\Psi_s(\lambda)$ $(s = a, b)$ have the shape of standard indifference curves. A selection ψ_s is represented by a curve in the box $[0,1] \times [0,1]$, $\psi_s(\lambda)$ being at the intersection of the curve with the level curve $\Psi_s(\lambda)$. Figure 1 represents one of these possible selections. While the level curves $\Psi_a(\lambda)$ and $\Psi_b(\lambda)$ are the same, the selections ψ_a and ψ_b can differ.

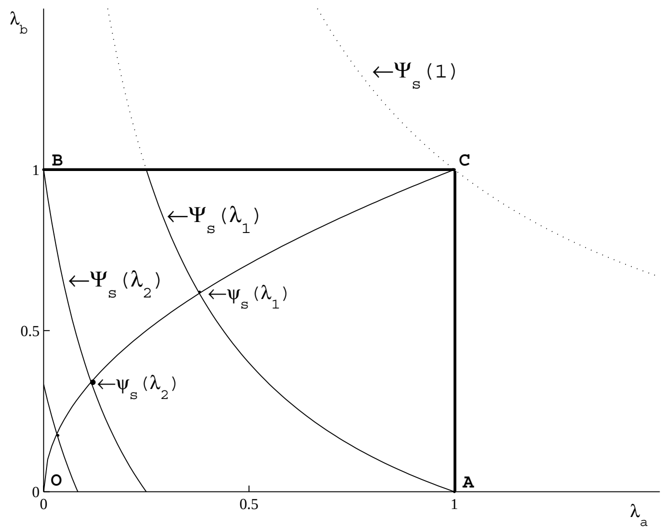

Figure 1

When the selection ψ_s is given by the intersection of the level curves with the diagonal of the box OABC then the covariance term in (13) is zero and (10) is satisfied. The further the selection ψ_s is from the diagonal, the greater the variance of ψ_s and the greater the likelihood that inequality (10) is violated. The maximum variance is obtained when the (scale) expectation function selects the point A or B of Figure 1. The point A with coordinates $(1, 0)$ is on the level curve corresponding to $\lambda_1 = \frac{\rho_a e_a^1}{1+\rho_a+e_a^2}$, while the point B with coordinates $(1,0)$ is on the level curve corresponding to $\lambda_2 = \frac{\rho_b e_b^1}{1+\rho_b+e_b^2}$. If the inequality (10) is to be satisfied when $\lambda_s = \lambda_1$ and the expectation function selects A, or when $\lambda_s = \lambda_2$ and the expectation function selects B, then the following inequalities must be satisfied

$$e_s^1 \geq 1 + \rho_s + e_s^2, \quad s = a, b \tag{15}$$

When these inequalities are satisfied any expectation function ψ satisfies inequality (10).

Lemma 1. *If conditions (15) are satisfied, then (10) holds for any expectation function* ψ.

Proof. See Appendix.

Since $e_s^i = \omega_s^i/q_s^*$, inequalities (15) involve the SE^* prices which are difficult to express in closed form. However when the endowments of the old tend to zero, $q^* \to (\omega_a^1/2, \omega_b^1/2)$, so that $e_s^1 \to 2 > 1 + \rho_s$ and in the limit (15) is satisfied with strict inequalities. Thus there exist ε such that if $\omega_s^2 \leq \varepsilon$, $s = a, b$, then (15) hold. In this case Proposition 3 implies that for any admissible expectation function almost all equilibrium paths converge to an SE^*. Note that the equilibrium path may not converge with probability 1 to the no-trade equilibrium. If the expectation function selects scale factors on the segment AC or BC and if the initial condition satisfies $\lambda_0 \geq \lambda_1$ or $\lambda_0 \geq \lambda_2$ then, with positive probability, some paths converge to the positive SE^* $(1,1)$.

For power utilities, conditions analogous to (15) can be derived, expressing the condition that inequality (10) is satisfied if the expectation function selects point A and/or B of Figure 1 and Lemma 1 holds with this appropriate version of (15). However it is no longer possible to compute explicitly the limit of the price vector q^* when $\omega^2 \to 0$. For all numerical examples that we considered we found that the appropriate version of (15) holds in the limit with strict inequalities. Thus it seems that, at least when ω^2 is small, convergence of the equilibrium path to an SE^* is independent of the expectation function chosen–although the probability of converging to the no-trade or to the positive SE^* may depend on the choice of the expectation function.

3 The model with equity

The problem of indeterminacy disappears if the asset used for intergenerational transfer gives positive instead of zero dividends, i.e., if the asset is "equity"[7] instead of being "money". To see this, consider the same economy as in Section 2, with agents' characteristics satisfying Assumptions 1–3, the only change being that the infinitely-lived asset gives a dividend of $D_s \geq 0$ units of good at every date when the state of nature is s, for all $s \in \boldsymbol{S}$, with at least one strict inequality. At date 0 the asset, whose supply is normalized to 1, belongs to the representative agent of the

[7] It can be the equity of any productive asset like "land" or a Lucas "tree".

old generation and is then exchanged at each date between the old and the young agent. The budget constraint of the representative agent born at node σ_t is now

$$x^1(\sigma_t) = \omega^1(\sigma_t) - q(\sigma_t)z, \; z \in \mathbf{R} \tag{16}$$

$$x^2(\sigma_t, s') = \omega^2(\sigma_t, s') + (D_{s'} + q(\sigma_t, s'))z, \; s' \in \boldsymbol{S} \tag{17}$$

where we retain the notation $q(\sigma_t)$ for the price of the asset. We assume free disposal of the (property right to the) asset so that the price $q(\sigma_t)$ must be non-negative. An equilibrium price process for the economy with equity is a process $(q(\sigma_t))_{\sigma_t \in \Sigma}$ such that $q(\sigma_t) \geq 0$ for all $\sigma_t \in \Sigma$ and

$$u_1'(\omega^1_{s_t} - q(\sigma_t))q(\sigma_t) = \sum_{s' \in \boldsymbol{S}} \rho_{s_t s'} u_2'(\omega^2_{s_t s'} + D_{s'} + q(\sigma_t, s')) \qquad (\mathcal{E}_Q)$$
$$(D_{s'} + q(\sigma_t, s')), \forall \sigma_t \in \Sigma$$

A strongly stationary equilibrium price vector is a non-negative vector $(q^*_s)_{s \in \boldsymbol{S}}$ such that

$$u_1'(\omega^1_s - q^*_s)q^*_s = \sum_{s' \in \boldsymbol{S}} \rho_{ss'} u_2'(\omega^2_{ss'} + D_{s'} + q^*_{s'})(D_{s'} + q^*_{s'}), \; \forall s \in \boldsymbol{S} \quad (\mathcal{E}^*_Q)$$

Proposition 4. *Under Assumptions 1-3 the economy with equity has a unique equilibrium which is a positive strongly stationary equilibrium.*

Proof. Let $(q(\sigma_t))_{\sigma_t \in \Sigma}$ be an equilibrium price process. Let σ_t be a node where the current shock is s. To be affordable by the young the price of the asset at a successor node (σ_t, s') of σ_t must be strictly less than $\omega^1_{s'}$. As before let $q^{(0)} = \omega^1$ be the vector of the initial endowments of the young. Then by monotonicity of the functions $y \to u_1'(\omega^1_s - y)y$ and $y \to u_2'(\omega^2_{ss'} + y)y$, $q(\sigma_t)$ must be less than $q^{(1)}_s$, where $q^{(1)}_s$ is defined by

$$u_1'(\omega^1_s - q^{(1)}_s)q^{(1)}_s = \sum_{s' \in \boldsymbol{S}} \rho_{ss'} u_2'(\omega^2_{ss'} + D_{s'} + q^{(0)}_{s'})(D_{s'} + q^{(0)}_{s'}), \; \forall s \in \boldsymbol{S}$$

Note that $q^{(1)} \ll q^{(0)}$. If we define the sequence $(q^{(n)})_{n \geq 0}$ by induction

$$u_1'(\omega^1_s - q^{(n+1)}_s)q^{(n+1)}_s = \sum_{s' \in \boldsymbol{S}} \rho_{ss'} u_2'(\omega^2_{ss'} + D_{s'} + q^{(n)}_{s'})(D_{s'} + q^{(n)}_{s'}), \; \forall s \in \boldsymbol{S}$$

then the equilibrium price process must be such that $q(\sigma_t) \leq q^{(n)}_{s_t}$ for all n and all $\sigma_t \in \Sigma$. Since $q^{(1)} \ll q^{(0)}$ the sequence is decreasing, and since it is bounded below by zero it converges to a vector q^* satisfying $(\mathcal{E}^*_Q)$. Since a vector with zero components cannot satisfy $(\mathcal{E}^*_Q)$, it follows that $q^* \gg 0$ and q^* is a positive SE^*. Since an equilibrium price process must satisfy $q(\sigma_t) \leq q^{(n)}_{s_t}$ for all n and all $\sigma_t \in \Sigma$, it must be such that $q(\sigma_t) \leq q^*_{s_t}$ for all $\sigma_t \in \Sigma$.

Since an equilibrium price process $(q(\sigma_t))_{\sigma_t \in \Sigma}$ must satisfy $(\mathcal{E}_Q)$ at each node and the right side of $(\mathcal{E}_Q)$ is strictly positive, the price must be positive at each node. Let $\tilde{q}^{(0)} = 0$ (the zero vector of $\mathbf{R}^S$). Since the prices at the successor nodes (σ_t, s') of σ_t must be strictly positive, by monotonicity, $q(\sigma_t)$ must be greater than $\tilde{q}_s^{(1)}$, where $\tilde{q}_s^{(1)}$ is defined by

$$u_1'(\omega_s^1 - \tilde{q}_s^{(1)})\tilde{q}_s^{(1)} = \sum_{s' \in \boldsymbol{S}} \rho_{ss'} u_2'(\omega_{ss'}^2 + D_{s'} + \tilde{q}_{s'}^{(0)})(D_{s'} + \tilde{q}_{s'}^{(0)}),\ \forall s \in \boldsymbol{S}$$

Note that $\tilde{q}^{(1)} \gg \tilde{q}^{(0)}$. If we define the sequence $(\tilde{q}^{(n)})_{n \geq 0}$ by induction

$$u_1'(\omega_s^1 - \tilde{q}_s^{(n+1)})\tilde{q}_s^{(n+1)} = \sum_{s' \in \boldsymbol{S}} \rho_{ss'} u_2'(\omega_{ss'}^2 + D_{s'} + \tilde{q}_{s'}^{(n)})(D_{s'} + \tilde{q}_{s'}^{(n)}),\ \forall s \in \boldsymbol{S}$$

then the equilibrium price process must be such that $q(\sigma_t) \geq \tilde{q}_{s_t}^{(n)}$ for all n and all $\sigma_t \in \Sigma$. Since $\tilde{q}^{(1)} \gg \tilde{q}^{(0)}$ the sequence is increasing, and since it is bounded above by $q^{(0)}$ it converges to a SE^* price vector $\tilde{q}^* \gg 0$. Any equilibrium price process must be such that $q(\sigma_t) \geq \tilde{q}_{s_t}^*$ for all $\sigma_t \in \Sigma$. Combining the two previous steps we find that all the equilibria must satisfy

$$\tilde{q}_{s_t}^* \leq q(\sigma_t) \leq q_{s_t}^*, \quad \forall\, \sigma_t \in \Sigma \tag{18}$$

For a vector $q \in \mathbf{R}^S$ such that $0 \leq q \ll \omega^1$ define the matrix

$$\hat{\Pi}(q) = [\hat{\pi}_{ss'}(q)]_{s,s' \in \boldsymbol{S}} = \left[\frac{\rho_{ss'} u_2'(\omega_{ss'}^2 + D_{s'} + q_{s'})}{u_1'(\omega_s^1 - q_s)}\right]_{s,s' \in \boldsymbol{S}}$$

If q is a SE^*, it satisfies

$$(I - \hat{\Pi}(q))q = \hat{\Pi}(q)D$$

Since $\hat{\Pi}(q)$ is a positive matrix and $D > 0$, $\hat{\Pi}(q)D \gg 0$. Thus for the vector $q \gg 0$, $(I - \hat{\Pi}(q))q \gg 0$. It follows that the matrix $I - \hat{\Pi}(q)$ is diagonal dominant, invertible, and $(I - \hat{\Pi}(q))^{-1} = I + \hat{\Pi}(q) + \ldots + \hat{\Pi}^n(q) + \ldots$ (see McKenzie, 1960; Takayama, 1974). Thus for any SE^* vector q

$$q = (\hat{\Pi}(q) + \hat{\Pi}^2(q) + \ldots + \hat{\Pi}^n(q) + \ldots)D \tag{19}$$

Consider the stationary equilibria q^* and $\tilde{q}^*$. Since $q^* \geq \tilde{q}^*$, and since the functions $q \to \hat{\pi}_{ss'}(q)$ are decreasing, $\hat{\Pi}(q^*) \leq \hat{\Pi}(\tilde{q}^*)$. By (19), $q^* \leq \tilde{q}^*$. Combining with (18) gives $q^* = \tilde{q}^*$, so that the equilibrium is unique. □

Note that, for an economy with equity, the existence of a positive SE^* does not require Assumption 4, or rather its modified version $\lambda_f(\hat{\Pi}(0)) > 1$. Even if the agents have greater endowments in old age than in their youth, there is always a positive price – perhaps small – at which young agents will want to buy an asset yielding positive dividends.

From the positive point of view, the model with positive dividends is better behaved than the model with money, since it does not exhibit the same indeterminacy.[8] It would be interesting to know if there are assumptions under which the uniqueness result extends to the more complicated models with heterogenous agents and several securities – typically bonds and equity – which are used in financial economics. This is left for future research.

Appendix

Proof of Lemma 1. Since $e_a^1 = e_b^1$ and for all $\lambda \in [0,1]$, $\Psi_a(\lambda) = \Psi_b(\lambda)$ we omit the subscript a or b from e^1, Ψ and λ. Because of the shape of the level curves, if the 2 points at which a level curve $\Psi(\lambda)$ intersects the boundary of the box $OABC$ are below the line with equation $\rho_a\lambda'_a + \rho_b\lambda'_b = \lambda$, i.e. if these points have coordinates such that $\rho_a\lambda'_a + \rho_b\lambda'_b \leq \lambda$, then the same inequality will hold for all the points on the level curve $\Psi(\lambda)$ which are inside the box.

The level curve $\Psi(\lambda)$ intersects OB at the point with coordinates $\lambda'_a(\lambda) = 0$, $\lambda'_b(\lambda) = \frac{\lambda e_b^2}{\rho_b e^1 - \lambda(1+\rho_b)}$, if λ is such that $\frac{\lambda e_b^2}{\rho_b e^1 - \lambda(1+\rho_b)} \leq 1$ or equivalently

$$\lambda \leq \lambda_2 = \frac{\rho_b e^1}{e_b^2 + 1 + \rho_b} \tag{20}$$

Note that when (15) is satisfied, $\lambda_2 \geq \rho_b$. When $\lambda \geq \lambda_2$ then $\Psi(\lambda)$ does not interset OB and it intersects BC at the point $(\lambda'_a(\lambda), 1)$ where $\lambda'_a(\lambda)$ is the solution of the equation

$$\frac{\rho_a\lambda'_a}{e_a^2 + \lambda'_a} = \frac{\lambda}{e^1 - \lambda} - \frac{\rho_b}{e_b^2 + 1} \tag{21}$$

Let us show that in both cases the inequality $\rho_a\lambda'_a(\lambda) + \rho_b\lambda'_b(\lambda) \leq \lambda$ holds. If $\lambda \leq \lambda_2$, then $\rho_a\lambda'_a(\lambda) + \rho_b\lambda'_b(\lambda) = \rho_b\lambda'_b(\lambda)$ and

$$\rho_b\lambda'_b(\lambda) = \frac{\lambda\rho_b e_b^2}{\rho_b e^1 - \lambda(1+\rho_b)} \leq \frac{\lambda\rho_b e_b^2}{\rho_b e^1 - \lambda_2(1+\rho_b)} = \frac{\lambda\rho_b e_b^2}{\lambda_2 e_b^2} = \frac{\lambda\rho_b}{\lambda_2} \leq \lambda$$

since $\rho_b/\lambda_2 \leq 1$. If $\lambda \geq \lambda_2$, $\rho_a\lambda'_a(\lambda) + \rho_b\lambda'_b(\lambda) = \rho_a\lambda'_a(\lambda) + \rho_b$. Differentiating (21) gives $\frac{d}{d\lambda}(\lambda'_a(\lambda)) = \frac{e^1(e_a^2 + \lambda'_a(\lambda))^2}{e_a^2(e^1-\lambda)^2}$, which, since $\lambda'_a(\lambda)$ is an increasing function of λ, is increasing. Thus the function $\lambda \to \rho_a\lambda'_a(\lambda) + \rho_b - \lambda$ is convex and if it is non-positive for $\lambda = \lambda_2$ and for $\lambda = 1$ it is non-positive in the interval $[\lambda_2, 1]$. $\lambda'_a(\lambda_2) = 0$ and $\lambda_2 \geq \rho_b$ implies that it is non-positive for $\lambda = \lambda_2$, while $\lambda'_a(1) = 1$ implies that it is non-positive for $\lambda = 1$. The analysis for the intersection

[8] However the positive SE^* of the model with equity has less good normative properties than that of the model with money. Since $\lambda_f(\hat{\Pi}(q^*)) < 1$, it can be deduced from Aiyagari and Peled (1991) that the equilibrium is Pareto optimal, or dynamically efficient, but is not optimal in the set of all feasible stationary allocations. This lack of the "Golden Rule property" can be deduced from Peled (1984) or seen directly by considering a transfer $dx = (dx_1, \ldots, dx_S) \gg 0$ from the old to the young, where dx is an eigenvector associated with the eigenvalue $\lambda_f(\hat{\Pi}(q^*))$.

of $\Psi(\lambda)$ with either OA or AC is similar and leads to the result that the inequality (10) holds for any function ψ which selects points on the boundary or in the interior of $OABC$. □

References

Aiyagari, S. R., Peled, D.: Dominant root characterization of Pareto optimality and the existence of optimal equilibria in stochastic overlapping generations models. Journal of Economic Theory **54**, 69–83 (1991)

Cass, D., Green, R. C., Spear, S. E.: Stationary equilibria with incomplete markets and overlapping generations. International Economic Review **33**, 495–512 (1992)

Chattopadhyay, S., Gottardi, P.: Stochastic OLG models, market structure and optimality. Journal of Economic Theory **89**, 21–67 (1999)

Chiappori, P. A., Guesnerie, R.: Sunspot equilibria in sequential markets models. In: Hildenbrand, W., Sonnenschein, H. (eds.) Handbook of mathematical economics, Vol. IV. Amsterdam: North Holland 1991

Demange, G., Laroque, G.: Social security and demographic shocks. Econometrica **67**, 527–542 (1999)

Duffie, D., Geanakoplos, J., Mas-Colell, A., McLennan, A.: Stationary Markov equilibria. Econometrica **62**, 745–781 (1994)

Gale, D.: Pure exchange equilibrium of dynamic economic models. Journal of Economic Theory **6**, 12–36 (1973)

Gantmacher, F. R.: The theory of matrices. New York: Chelsea 1959

Geanakoplos, J. D., Polemarchakis, H. M.: Overlapping generations. In: Hildenbrand, W., Sonnenschein, H. (eds.) Handbook of mathematical economics, Vol. IV. Amsterdam: North Holland 1991

Gottardi, P.: Stationary monetary equilibria in overlapping generations models with incomplete markets. Journal of Economic Theory **71**, 75–89 (1996)

Guesnerie, R., Woodford, M.: Endogenous fluctuations. In: Laffont, J. J. (ed.) Advances in economic theory. Sixth World Congress, Vol. 2. Cambridge: Cambridge University Press 1992

Keynes, J. M.: A treatise on money, Vol. I. The pure theory of money. London: Macmillan 1936

Kurz, M.: Endogenous economic fluctuations: studies in the theory of rational beliefs. Berlin Heidelberg New York: Springer 1997

McKenzie, L. W.: Matrices with dominant diagonal and economic theory. In: Arrow, K., Karlin S., Suppes, P. (eds.) Mathematical methods in the social sciences. Stanford: Stanford University Press 1959

Peled, D.: Stationary Pareto optimality of stochastic asset equilibria with overlapping generations. Journal of Economic Theory **34**, 396–403 (1984)

Takayama, A.: Mathematical economics. Hinsdale. IL: The Dryden Press 1974