Chapter 30

INCOMPLETE MARKETS

MICHAEL MAGILL and WAYNE SHAFER*

University of Southern California

Contents

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1. Introduction

The principal objective of general equilibrium theory is to study the allocation of resources achievable via a system of markets. If all activity in an economy could be viewed as taking place in a single period then it would perhaps be reasonable to assume that markets are *complete*; that is, there is a market and an associated price for each good. This is the environment of the classical theory of resource allocation which finds its most elegant synthesis in the Arrow–Debreu theory [Debreu (1959)]. As soon as we are concerned with a world in which *time* and *uncertainty* enter in an essential way it is no longer reasonable to assume the existence of such a complete set of markets: we must enter the world of *incomplete markets*. The object of this chapter is to lay out briefly the principal contributions that have been made to this branch of general equilibrium theory since the survey article of Radner (1982).

The basic objective of the theory of incomplete markets is to extend the general equilibrium analysis of markets from the classical Arrow–Debreu framework (GE) to a more general model with real and financial markets in which the structure of the markets is incomplete (GEI). The idea is to retain the simplicity, coherence and generality which are the hallmarks of the Arrow–Debreu construction while moving the nature of actual markets, contracts and constraints on agent participation into closer conformity with the actual structure of markets observed in the real world. Thus in addition to the traditional real spot markets for goods, there is a rich array of financial markets such as bond and equity markets, not to speak of options, futures and insurance contracts, as well as contracts between firms, between employees and firms and so on. To model all these markets and contracts in a way that enhances our understanding of the roles they play in the overall problem of arriving at an actual allocation of resources over time is a challenging task on which significant progress has recently been made. Far more of course remains to be understood. But we are not alone in the profession in our conviction that a microeconomic foundation for macroeconomics may ultimately come from a more concrete version of general equilibrium theory in which there is trading on real and financial markets, where nominal contracts and money enter in an essential way, but where the ability to trade into the future is limited by the *incompleteness* of markets and by the unwillingness or inability of agents to make more than limited commitments into the future [see Keynes (1936), Arrow (1974), Tobin (1980)].

Much of economic theory can be viewed as a study of the *causes* and *consequences* of *market failure*, with special emphasis on the consequences of market failure for subgroups of agents or for society as a whole. In such an investigation the Arrow–Debreu theory provides the idealised framework in which markets function at their best. The phenomenon of incomplete markets taken in its broadest sense includes the classical concept of *missing markets* arising from externalities and public goods. However for the purposes of this survey we interpret the *theory of incomplete markets* in the narrower sense of being that branch of economic theory which studies the causes and consequences of incomplete financial markets in a general equilibrium framework of risk and uncertainty over time. The general equilibrium model that forms the basis for the analysis satisfies all the *idealised* assumptions of the standard Arrow–Debreu model except that it has *incomplete markets*. While the model is thus unrealistic in that it retains the remaining idealised assumptions, it provides a setting in which the effects of this particular market failure can be isolated and studied.

Classical general equilibrium theory (GE) as synthesised by Arrow–Debreu has the property of being theoretically the most elegant and yet empirically one of the least satisfactory parts of the economic theory. It is elegant, because within the context of a precisely formulated set of hypotheses it leads to a clear and simple explanation of how an idealised system of markets allocates resources and achieves what amounts to a best possible solution to the problem of resource allocation. GE crystallises a classical tradition in economic theory that has its origin in Adam Smith's theory of the *invisible hand*, by which a competitive system with market prices coordinates the otherwise independent activities of consumers and producers acting purely in their self-interest. A central conclusion is the idea of *laisser-faire*: the government should not interfere with the system of markets that allocates resources in the private sector of the economy.

GE however stands on shaky empirical foundations: one of its key hypotheses is far from being satisfied. We live in a world in which *time* and *uncertainty* enter in an essential way and in which the system of markets is *incomplete*. What is needed is an extension of classical GE which explicitly allows for the fact that markets are incomplete and it is to this issue that the analysis that follows addresses itself. We begin by recalling the market structure of GE, the system of contingent markets. We then introduce the more general market structure of GEI consisting of a system of spot markets for real goods coupled with a system of financial markets.

Most of the theory is very recent, having its origins in the classical papers of Arrow (1953), Diamond (1967), Radner (1972), Drèze (1974) and Hart (1975). An extensive array of new results has been obtained in the last five years which seems to call for a re-examination of the status of the theory. *What are the central issues which emerge*? In this survey we focus principally on the *consequences* as opposed to the causes of incomplete markets: from this perspective, three basic messages stand out.

(1) *The non-neutrality of financial instruments and the role of money*
(2) *The conflicting objectives of firms*
(3) *The potential inadequacy of a decentralised system of markets*

These three topics motivate the basic layout of the paper. Thus in Sections 2 and 3 which analyse the GEI model of an exchange economy we find that when markets are incomplete, changing the financial instruments, or when nominal assets are present, changing the money supply leads to a change in the equilibrium allocation; in short financial instruments and money are non-neutral. Section 2 also contains a systematic analysis of the concepts and mathematical techniques needed for a proper understanding of the behavior of GEI equilibria. While real assets are inflation proof, nominal assets are not. The economic consequence is the striking property exhibited by the GEI model with nominal assets: indeterminacy if the model is left unchanged (Section 3.1) and non-neutrality of money if a role is introduced for money as a medium of exchange (Section 3.2).

Section 4 presents an analysis of the GEI model of a production economy: it is here that the theory still encounters great difficulties. When markets are incomplete each firm faces a public goods problem with respect to its constituency of shareholders (and employees) for which there is no evident solution. We try to bring together the different theories under a common framework, but cannot claim to have advanced the theory much beyond the contribution of Grossman and Hart (1979).

When markets are incomplete it should hardly be surprising that equilibrium allocations are inefficient. What is interesting is to understand the cause of the inefficiency: this is the subject of Section 5. From a policy point of view (i.e. should the government intervene or not) what is significant is the *magnitude* of the distortions which the inefficiency theorems assert are generically present at an equilibrium. While the analysis of Section 5 indicates in principle how estimates of these magnitudes could be made, to our knowledge no such estimates have yet been made.

We have attempted to present a reasonably coherent view of the current status of the theory of incomplete markets. In emphasising conceptual continuity we have had to sacrifice a number of important ideas which are dealt with in only a cursory way in Section 6.

A clarifying comment is perhaps in order regarding the relation between the concept of equilibrium which forms the basis for the analysis which follows and that which is used in the related area of *temporary equilibrium theory*. In a model in which time and uncertainty enter in an essential way, a concept of *market equilibrium* involves two subordinate concepts: one regarding *expectation formation* and one regarding *market clearing*. Agents must form expectations about future prices in order to determine their market demand decisions:

These demand decisions are then used via market clearing to determine prices. In a *temporary equilibrium* agents form expectations (ex ante) about future spot prices which are not necessarily fulfilled (ex post): in addition, at a given date, only the current spot markets are required to clear, no condition being imposed on future spot markets. This framework provides a natural and powerful tool for analysing the consequences of incorrect and hence changing price expectations: it has been the subject of an extensive literature which is surveyed in Grandmont (1982, 1988). However when financial markets enter in an essential way (that is when arbitrage and information are important), a richer theory can be developed if the much stronger assumption regarding expectation formation is made that agents correctly anticipate future prices and all future markets are also cleared. This leads to the concept which Radner (1972, 1982) has called an *equilibrium of plans, prices and price expectations* which forms the basis for the analysis that follows. It should be noted that this concept permits agents to hold different probability assessments regarding future events. In the special case where all agents hold common probability assessments this concept reduces to what is referred to in macroeconomics as a *rational expectations equilibrium*.

2. Real assets

2.1. Two period exchange economy

In this section we introduce the basic exchange economy and the concepts of a GE and a GEI equilibrium. The model which underlies the first part of our analysis is the simplest two period exchange economy under uncertainty. The economy consists of a finite number of agents ($i = 1, \ldots, I$) and a finite number of goods ($l = 1, \ldots, L$). To capture both time and uncertainty in the simplest way we consider a model with two time periods ($t = 0, 1$) in which one of S states of nature ($s = 1, \ldots, S$) occurs at date 1. For convenience we call date $t = 0$, state $s = 0$ so that in total there are $S + 1$ states. The main results that follow can be extended to a stochastic process over many time periods (Section 2.4).

Since there are L commodities available in each state ($s = 0, \ldots, S$) the *commodity space* is $\mathbb{R}^n$ with $n = L(S + 1)$. Each consumer i ($i = 1, \ldots, I$) has an *initial endowment* of the L goods in each state, $\omega^i = (\omega_0^i, \omega_1^i, \ldots, \omega_S^i)$. Since consumer i does not know which state of nature will occur at date 1, the endowment at date 1, $\omega_1^i = (\omega_1^i, \ldots, \omega_S^i)$, is a random variable. For concreteness we can think of agent i's endowment $\omega^i \in \mathbb{R}^n$ as giving the output at dates 0 and 1 of a *farm* owned by agent i. The preference ordering of agent i is represented by a *utility function*,

$$u^i : \mathbb{R}^n_+ \to \mathbb{R}\,, \quad i = 1, \ldots, I\,,$$

defined over *consumption bundles* $x^i = (x^i_0, x^i_1, \ldots, x^i_S)$ lying in the two-period consumption set $X^i = \mathbb{R}^n_+$. A useful example of a utility function is given by the von Neumann–Morgenstern expected utility function

$$u^i(x^i_0, x^i_1, \ldots, x^i_S) = \sum_{s=1}^{S} \rho_s U^i(x^i_0, x^i_s)$$

where $\rho_s > 0$ denotes the probability of state s and $\Sigma_{s=1}^{S} \rho_s = 1$. But the results that follow in no way depend on such a special form.

Since most of the mathematical proofs that follow are based on the use of *differential topology* [see Guillemin and Pollack (1974)] we invoke the classical *smooth preferences* introduced by Debreu (1972). The *characteristics* of agent i are thus summarised by a utility function and endowment vector (u^i, ω^i) satisfying:

Assumption 1 (*agent characteristics*). (1) $u^i : \mathbb{R}^n_+ \to \mathbb{R}$ is continuous on $\mathbb{R}^n_+$ and $\mathscr{C}^\infty$ on $\mathbb{R}^n_{++}$; (2) if $U^i(\xi) = \{x \in \mathbb{R}^n_+ \mid u^i(x) \geq u^i(\xi)\}$ then $U^i(\xi) \subset \mathbb{R}^n_{++}$, $\forall \xi \in \mathbb{R}^n_{++}$; (3) for each $x \in \mathbb{R}^n_{++}$, $Du^i(x) \in \mathbb{R}^n_{++}$ and $h^T D^2 u^i(x) h < 0$ for all $h \neq 0$ such that $Du^i(x)h = 0$; (4) $\omega^i \in \mathbb{R}^n_{++}$.

Let $(u, \omega) = (u^1, \ldots, u^I, \omega^1, \ldots, \omega^I)$. The collection of I agents with their characteristics (u, ω) constitutes the smooth *exchange economy* $\mathscr{E}(u, \omega)$ which forms the basis for our initial analysis.

An *allocation of resources* for the economy $\mathscr{E}(u, \omega)$ is a vector of consumption bundles $x = (x^1, \ldots, x^I) \in \mathbb{R}^{nI}_+$. Equilibrium theory can be viewed as the qualitative study of the allocations that arise when we adjoin different market structures to the basic exchange economy $\mathscr{E}(u, \omega)$. We will study two such market structures: first a system of *contingent markets* and second a system of *spot and financial markets*. The former leads to the standard *general equilibrium model* (*GE*) of Arrow–Debreu, the latter to the *general equilibrium model with incomplete markets* (*GEI*). For the exchange economy $\mathscr{E}(u, \omega)$ the difference between these two models reflects itself in the different budget sets that agents face in these two market environments.

Contingent markets (*GE*)

A *contingent commodity* for good l ($l = 1, \ldots, L$) in state s ($s = 0, \ldots, S$) is a contract which promises to deliver one unit of good l in state s and nothing otherwise. The price of this contract P_{sl} (measured in the unit of account) is payable at date 0. If there is available at date 0 a complete set of such contingent contracts (one for each good in each state) then each agent i can sell his endowment $\omega^i = (\omega_0^i, \omega_1^i, \ldots, \omega_S^i)$ at the prices $P = (P_0, P_1, \ldots, P_S)$ where $P_s = (P_{s1}, \ldots, P_{sL})$, to obtain the income $P\omega^i = \Sigma_{s=0}^S P_s \omega_s^i$ and can purchase any consumption vector $x^i = (x_0^i, x_1^i, \ldots, x_S^i)$ satisfying $Px^i = \Sigma_{s=0}^S P_s x_s^i \leqslant P\omega^i$. (Note that we will always write *prices* as *row* vectors and *quantities* as *column* vectors so that Px^i is the standard scalar product of P and x^i.) Agent i's contingent market (GE) budget set is thus defined by

$$B(P, \omega^i) = \{x^i \in \mathbb{R}_+^n \mid P(x^i - \omega^i) = 0\} .$$

Definition 1. A *contingent market (CM) equilibrium* for the economy $\mathscr{E}(u, \omega)$ is a pair of actions and prices $(\bar{x}, \bar{P}) = (\bar{x}^1, \ldots, \bar{x}^I, \bar{P})$ such that

(i) $\bar{x}^i$, $i = 1, \ldots, I$ satisfy

$$\bar{x}^i = \arg\max\{u^i(x^i) \mid x^i \in B(\bar{P}, \omega^i)\}$$

(ii) $\Sigma_{i=1}^I (\bar{x}^i - \omega^i) = 0$.

We also refer to such an equilibrium as a *GE equilibrium*.

Spot-financial markets (GEI)

A system of contingent markets is a market structure that is principally of theoretical interest: it can be viewed as an *ideal* system of markets. It is far removed however from the *sequential* structure of markets that is typical of actual decentralised market economies. To model such a sequential structure we introduce a collection of *real spot markets* for each of the L goods at date 0 and in each state s at date 1, together with a system of *financial markets*. The spot markets lead to a system of $S + 1$ budget constraints; the financial markets provide instruments that enable each agent, at least to some extent, to redistribute income across the states, thereby reducing the constraints imposed by the basic spot market equations. More precisely, let $p = (p_0, p_1, \ldots, p_S) \in \mathbb{R}_{++}^n$ denote the vector of *spot* prices, where $p_s = (p_{s1}, \ldots, p_{sL})$ and p_{sl} denotes the price (measured in units of account) payable in state s for one unit of good l. The essential distinction between a spot market in state s and a contingent market for state s is that in the former the payment is made at date 1 in state s (if $s \geqslant 1$), while in the latter it is always made at date 0. It is this property that leads to the system of $S + 1$ budget constraints under a system of spot markets and to a single budget constraint with a system of contingent markets.

The financial assets we consider will be one of three basic types or a combination of these three: *real* assets (such as the equity of firms or futures contracts on real goods), *nominal* assets (such as bonds or financial futures), and *secondary* or *derivative* assets (such as call and put options). In each case we assume that there is given a system of J *financial assets* where asset j can be purchased for the price q_j (units of account) at date 0 and delivers a random return $V^j = (V_1^j, \ldots, V_S^j)^T$ across the states at date 1 where the transpose T indicates that V^j is written as a column vector and where V_s^j is measured in the unit of account. The J column vectors V^j can be combined to form the date 1 matrix of returns

$$V = [V^1 \quad \cdots \quad V^J] = \begin{bmatrix} V_1^1 & \cdots & V_1^J \\ \vdots & & \vdots \\ V_S^1 & \cdots & V_S^J \end{bmatrix}. \tag{1}$$

V generates the *subspace of income transfers* $\langle V \rangle$, namely the subspace of $\mathbb{R}^S$ spanned by the J columns of V

$$\langle V \rangle = \{\tau \in \mathbb{R}^S \mid \tau = Vz,\ z \in \mathbb{R}^J\}. \tag{2}$$

Definition 2. If the subspace of income transfers satisfies $\langle V \rangle = \mathbb{R}^S$ then the asset structure is called *complete*. If $\langle V \rangle \neq \mathbb{R}^S$ then it is *incomplete*.

Let $z^i = (z_1^i, \ldots, z_J^i) \in \mathbb{R}^J$ denote the number of units of each of the J assets purchased by agent i (where $z_j^i < 0$ means *short-selling* asset j), then the $S + 1$ budget constraints can be written as

$$\begin{aligned} p_0(x_0^i - \omega_0^i) &= -qz^i \\ p_s(x_s^i - \omega_s^i) &= V_s z^i, \quad s = 1, \ldots, S \end{aligned} \tag{3}$$

where $q = (q_1, \ldots, q_J)$ and $V_s = (V_s^1, \ldots, V_s^J)$ is row s of the matrix V. If we define the full matrix of returns (i.e. date 0 and date 1)

$$W(q, V) = \begin{bmatrix} -q \\ V \end{bmatrix} = \begin{bmatrix} -q_1 & \cdots & -q_J \\ V_1^1 & \cdots & V_1^J \\ \vdots & & \vdots \\ V_S^1 & \cdots & V_S^J \end{bmatrix} \tag{4}$$

and for $p \in \mathbb{R}^{L(S+1)}$, $x^i \in \mathbb{R}^{L(S+1)}$ define the *box product*

$$p \,\square\, x^i = (p_0 x_0^i, p_1 x_1^i, \ldots, p_S x_S^i)$$

then agent i's GEI *budget set* is given by

$$\mathscr{B}(p, q; \omega^i) = \{x^i \in \mathbb{R}^n_+ \mid p \,\square\, (x^i - \omega^i) = W(q, V)z^i, z^i \in \mathbb{R}^J\} . \tag{5}$$

Definition 3. A *spot-financial market* (*FM*) *equilibrium* for the economy $\mathscr{E}(u, \omega)$ is a pair of actions and prices $((\bar{x}, \bar{z}), (\bar{p}, \bar{q})) = ((\bar{x}^1, \ldots, \bar{x}^I, \bar{z}^1, \ldots, \bar{z}^I), (\bar{p}, \bar{q}))$ such that

(i) $(\bar{x}^i, \bar{z}^i)$, $i = 1, \ldots, I$ satisfy

$$\bar{x}^i = \arg\max\{u^i(x^i) \mid x^i \in \mathscr{B}(\bar{p}, \bar{q}; \omega^i)\} \quad \text{and} \quad \bar{p} \,\square\, (\bar{x}^i - \omega^i) = W\bar{z}^i ,$$

(ii) $\Sigma_{i=1}^I (\bar{x}^i - \omega^i) = 0$,
(iii) $\Sigma_{i=1}^I \bar{z}^i = 0$.

We also refer to such an equilibrium as a *GEI equilibrium*.

Real assets

The first class of financial assets that we want to analyse is the class of real assets. A *real asset j* is a contract which promises to deliver a vector of the L goods (written as a column vector)

$$A_s^j = (A_{s1}^j, \ldots, A_{sL}^j)^T \in \mathbb{R}^L , \quad s = 1, \ldots, S$$

in each state $s = 1, \ldots, S$ at date 1. A real asset is thus characterised by a date 1 commodity vector $A^j = (A_1^j, \ldots, A_S^j)^T \in \mathbb{R}^{LS}$ (written as a column vector). The revenue it yields in state s is proportional to the spot price p_s

$$V_s^j = p_s \cdot A_s^j , \quad s = 1, \ldots, S$$

If there are J real assets then the date 1 matrix of returns (1) is given by

$$V = V(p_1) = \begin{bmatrix} p_1 A_1^1 & \cdots & p_1 A_1^J \\ \vdots & & \vdots \\ p_S A_S^1 & \cdots & p_S A_S^J \end{bmatrix}$$

where $p_1 = (p_1, \ldots, p_S) \in \mathbb{R}^{LS}$ is the date 1 vector of spot prices. If we let p_s denote the row vector $(p_{s1}, \ldots, p_{sL})$ then we can also write V as

$$V(p_1) = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & p_S \end{bmatrix} \begin{bmatrix} A_1^1 & \cdots & A_1^J \\ A_2^1 & \cdots & A_2^J \\ \vdots & & \vdots \\ A_S^1 & \cdots & A_S^J \end{bmatrix} .$$

Real assets are inflation-proof in the sense that doubling the spot prices in state s doubles their income. Thus with real assets if $(\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_S, \bar{q})$ is an equilibrium price vector then $(\alpha_0\bar{p}_0, \alpha_1\bar{p}_1, \ldots, \alpha_S\bar{p}_S, \alpha_0\bar{q})$ with $\alpha_s > 0$, $s = 0, \ldots, S$ is also an equilibrium price vector. In short, *in an economy with only real assets, price levels are unimportant.*

If we let the J column vectors $A^j \in \mathbb{R}^{LS}$, $j = 1, \ldots, J$, be the columns of an $LS \times J$ matrix

$$A = [A^1 \cdots A^J] = \begin{bmatrix} A^1_{11} & \cdots & A^J_{11} \\ \vdots & & \vdots \\ A^1_{SL} & \cdots & A^J_{SL} \end{bmatrix}$$

then the *real asset structure* is summarised by the matrix $A \in (\mathbb{R}^{LS})^J$. We let $\mathscr{E}(u, \omega; A)$ denote the exchange economy with real asset structure A.

Example 1 (*contingent commodities*). Introduce $J = SL$ assets, one for each good in each state. Asset $j = (s, l)$, $s = 1, \ldots, S$, $l = 1, \ldots, L$ promises to deliver one unit of good l in state s and nothing otherwise. Thus $A^{s,l}_{s,l} = 1$ and $A^{s,l}_{\sigma,h} = 0$ if $(\sigma, h) \neq (s, l)$. Here $A = I_{SL}$ (the $SL \times SL$ *identity* matrix) and

$$V = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & p_S \end{bmatrix}.$$

Thus $\langle V \rangle = \mathbb{R}^S$. Since $z^i \in \mathbb{R}^{SL}$, commodities are purchased forward directly and there is no need to exchange on spot markets at date 1. It is clear that it suffices to consider the subset of assets which delivers *only the first good in each state*: this leads to the next example.

Example 2 (*numeraire assets*). Suppose each asset j delivers contingent amounts of only one of the goods, say the first. In this case $A^j_s = (A^j_{s1}, 0, \ldots, 0)^T$ and V can be written as

$$V(p_1) = \begin{bmatrix} p_{11} & 0 & \cdots & 0 \\ 0 & p_{21} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_{S1} \end{bmatrix} \begin{bmatrix} A^1_{11} & \cdots & A^J_{11} \\ A^1_{21} & \cdots & A^J_{21} \\ \vdots & & \vdots \\ A^1_{S1} & \cdots & A^J_{S1} \end{bmatrix}.$$

Note that in general, that is for most $S \times J$ matrices (A^j_s), changing the prices p_{s1} $(s = 1, \ldots, S)$ changes the subspace $\langle V \rangle$ spanned by the columns of V but leaves the dimension of the subspace $\langle V \rangle$ unchanged (i.e. $\dim\langle V \rangle = J$ for all $p_{s1} > 0$, $s = 1, \ldots, S$). However since with real assets price levels do not matter it is often convenient to normalise the spot prices so that $p_{s1} = 1$ $(s = 0, \dots, S)$. In this case the matrix V becomes

$$V = \begin{bmatrix} A_{11}^1 & \cdots & A_{11}^J \\ \vdots & & \vdots \\ A_{S1}^1 & \cdots & A_{S1}^J \end{bmatrix}$$

so that the subspace $\langle V \rangle$ is independent of p_1. For most real asset structures not only does the subspace $\langle V(p_1) \rangle$ vary as p_1 changes, but also the *dimension* of $\langle V(p_1) \rangle$ can change as p_1 changes and this creates some quite new phenomena. In this sense the next two examples are more representative of the general class of real asset structures.

Example 3 (*futures contracts*). Suppose there are $J \leq L$ futures contracts on the goods. If the jth asset is a futures contract for good j then $A_{sj}^j = 1$, $A_{sl}^j = 0$, $l \neq j$, $s = 1, \dots, S$, $j = 1, \dots, J$. In this case

$$V = \begin{bmatrix} p_{11} & \cdots & p_{1J} \\ \vdots & & \vdots \\ p_{S1} & \cdots & p_{SJ} \end{bmatrix}.$$

Note that if the spot prices p_s are all collinear ($p_s = \alpha_s \rho$ for $\alpha_s > 0$, $\rho \in \mathbb{R}_{++}^L$, $s = 1, \dots, S$) then $\langle V \rangle$ is a one-dimensional subspace; with no price variability across the states no spanning is achieved with futures contracts.

Example 4 (*equity contracts*). Consider the simplest production economy in which agents hold initial ownership shares of firms. Let there be J firms and suppose the production decision $y^j \in Y^j$ (firm j's production set) has already been made where $Y^j \subset \mathbb{R}^n$; then the *equity of firm j* is a real asset with $A_s^j = y_s^j$, $s = 1, \dots, S$. Let $\theta^i = (\theta_1^i, \dots, \theta_J^i)$ denote the portfolio of shares purchased by agent i and let $\zeta^i = (\zeta_1^i, \dots, \zeta_J^i)$ denote his initial ownership shares, with $\zeta_j^i \geq 0$ and $\Sigma_{i=1}^I \zeta_j^i = 1$, $j = 1, \dots, J$. We assume that if agent i buys the share θ_j^i of firm j then he also finances the share $\theta^i(-p_0 y_0^i)$ of the input cost at date 0. A *stock market equilibrium* is then defined in the obvious way. In a stock market equilibrium the assets (equities) are in *positive net supply*: the change of variable $z^i = \theta^i - \zeta^i$, $\underline{\omega}^i = \omega^i + y\zeta^i$ where $y = [y^1 \quad \cdots \quad y^J]$ converts the stock-market equilibrium into an FM equilibrium in which assets are in *zero net supply*. In this case the returns matrix W in (4) is given by

$$W = \begin{bmatrix} -q_1 + p_0 y_0^1 & \cdots & -q_J + p_0 y_0^J \\ p_1 y_1^1 & \cdots & p_1 y_1^J \\ \vdots & & \vdots \\ p_S y_S^1 & \cdots & p_S y_S^J \end{bmatrix}.$$

Clearly Examples 3 and 4 can be combined to create an asset structure consisting of a system of futures contracts and equity.

No-arbitrage equilibrium

The idea of *arbitrage* and the absence of arbitrage opportunities is a basic concept of finance. *Applied in an abstract way in the present model, it leads to an alternative (and equivalent) concept of equilibrium that is analytically simpler to work with than an FM equilibrium.* Let us show how this new concept of equilibrium is derived. We say that $q \in \mathbb{R}^J$ is a *no-arbitrage asset price* if there does not exist a portfolio $z \in \mathbb{R}^J$ such that $W(q, V)z \geqslant 0$ (where for $y \in \mathbb{R}^{S+1}$, $y \geqslant 0$ means $y_s \geqq 0$, $s = 0, \ldots, S$ and $y_s > 0$ for at least one s). Agent i's utility maximising problem in Definition 3(i) has a solution if and only if q is a no-arbitrage asset price. Recall the following version of the Minkowski–Farkas lemma [see Gale (1960, p. 49)].

Lemma 1. *If W is an $(S+1) \times J$ matrix then either there exists $z \in \mathbb{R}^J$ such that $Wz \geqslant 0$ or there exists $\beta \in \mathbb{R}^{S+1}_{++}$ such that $\beta W = 0$.*

Thus the absence of arbitrage opportunities in the trading of the financial assets implies the existence of a present value vector (positive state prices) $\beta = (\beta_0, \beta_1, \ldots, \beta_S)$ such that $\beta W = 0$ which is equivalent to $\beta_0 q_j = \sum_{s=1}^{S} \beta_s V_s^j$, $j = 1, \ldots, J$ so that *the price of each asset equals the present value of its future income stream*. From the budget equations (3), the date 0 equation becomes

$$\beta_0 p_0(x_0^i - \omega_0^i) = -\beta_0 q z^i = -\sum_{s=1}^{S} \beta_s V_s z^i = -\sum_{s=1}^{S} \beta_s p_s(x_s^i - \omega_s^i)\,. \tag{6}$$

If we define the new vector of *date* 0 *present value prices*

$$P = \beta \,\square\, p\,. \tag{7}$$

Then the date 0 budget equation (6) reduces to the GE budget constraint

$$P(x^i - \omega^i) = 0\,.$$

In the case of *real* assets, since the date 1 equations are homogeneous functions of the spot prices, the date 1 equations can be written as

$$P_1 \,\square\, (x_1^i - \omega_1^i) \in \langle V(P_1) \rangle$$

where $P_1 = (P_1, \ldots, P_S)$ is the vector of present value prices for date 1. Thus under the new vector of prices (7) each agent can be viewed as maximising utility over the budget set

$$\mathbb{B}(P; \omega^i) = \left\{ x^i \in \mathbb{R}^n_+ \;\middle|\; \begin{array}{c} P(x^i - \omega^i) = 0 \\ P_1 \,\square\, (x_1^i - \omega_1^i) \in \langle V(P_1) \rangle \end{array} \right\}. \tag{8}$$

It is clear that the budget set (8) is the same for all those β and β' such that

$$\left(\frac{\beta_1'}{\beta_0'}\right)V = \left(\frac{\beta_1}{\beta_0}\right)V = q\,.$$

It thus suffices to choose one no-arbitrage β. In particular since the first order conditions for maximising utility subject to the constraints (3) lead to a vector of marginal utilities of income (Lagrange multipliers) $\lambda^i = (\lambda_0^i, \lambda_1^i) = (\lambda_0^i, \lambda_1^i, \ldots, \lambda_S^i)$ for agent i which satisfies

$$\left(\frac{\lambda_1^i}{\lambda_0^i}\right)V = q\,,$$

we may choose $\beta = \lambda^1$. It is easy to check that with this choice of β agent 1's budget set reduces to a GE budget set (i.e. the date 1 constraints are automatically satisfied).

For reasons that will become clear shortly we need to consider equilibria in which the subspace of income transfers $\langle V \rangle$ is of fixed dimension ρ, where $0 \leq \rho \leq S$. Let $G^\rho(\mathbb{R}^S)$ denote *the set consisting of all linear subspaces of* $\mathbb{R}^S$ *of dimension* ρ. Let $\mathscr{L} \in G^\rho(\mathbb{R}^S)$ denote a ρ-dimensional subspace of $\mathbb{R}^S$. Replacing the actual subspace of income transfers $\langle V(P_1) \rangle$ by a surrogate subspace $\mathscr{L}$, the budget set (8) becomes

$$\mathbb{B}(P, \mathscr{L}; \omega^i) = \left\{ x^i \in \mathbb{R}^n_+ \;\middle|\; \begin{array}{c} P(x^i - \omega^i) = 0 \\ P_1 \,\square\, (x_1^i - \omega_1^i) \in \mathscr{L} \end{array} \right\}. \tag{9}$$

We are thus led to the following alternative concept of equilibrium.

Definition 4. A *normalised no-arbitrage* (*NA*) *equilibrium of rank* ρ with $0 \leq \rho \leq S$ for the economy $\mathscr{E}(u, \omega; A)$ is a pair $(\bar{x}, \bar{P}, \bar{\mathscr{L}}) \in \mathbb{R}^{nI}_+ \times R^n_{++} \times G^\rho(\mathbb{R}^S)$ such that

(i) $\bar{x}^1 = \arg\max\{u^1(x^1) \mid x^1 \in B(\bar{P}, \omega^1)\}$
$\quad\;\bar{x}^i = \arg\max\{u^i(x^i) \mid x^i \in \mathbb{B}(\bar{P}, \bar{\mathscr{L}}; \omega^i)\}\,, \quad i = 2, \ldots, I$

(ii) $\sum_{i=1}^{I} (\bar{x}^i - \omega^i) = 0$

(iii) $\langle V(\bar{P}_1) \rangle = \bar{\mathscr{L}}$

Remark. Normalising the no-arbitrage equilibrium by choosing the no-arbitrage present value vector $\beta = \lambda^1$ has two important consequences. First it gives a GE demand function for agent 1 satisfying the standard boundary condition. Second it eliminates a condition of dependence for the aggregate demands at date 1 (S date 1 *Walras Law* equations) that would otherwise arise from the fact that each agent satisfies $P_1 \square (x_1^i - \omega_1^i) \in \mathscr{L}$. This allows transversality arguments to be applied directly. The following lemma shows that the concepts of an FM equilibrium of rank ρ that is, with rank $V(\bar{P}_1) = \rho$, and an NA equilibrium of rank ρ are equivalent. NA equilibria are analytically easier to handle.

Lemma 2. (i) *If* $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}))$ *is an FM equilibrium of rank* ρ *then there exists a* ρ*-dimensional subspace* $\bar{\mathscr{L}} \in G^{\rho}(\mathbb{R}^S)$ *and a no-arbitrage* $\bar{\beta} \in \mathbb{R}^{S+1}_{++}$ *such that* $(\bar{x}, \bar{\beta} \square \bar{p}, \bar{\mathscr{L}})$ *is an NA equilibrium of rank* ρ.

(ii) *If* $(\bar{x}, \bar{P}, \bar{\mathscr{L}})$ *is an NA equilibrium of rank* ρ *then there exist portfolios* $\bar{z} = (\bar{z}^1, \ldots, \bar{z}^I)$ *and an asset price* $\bar{q}$ *such that* $((\bar{x}, \bar{z}), (\bar{P}, \bar{q}))$ *is an FM equilibrium of rank* ρ.

Dual subspaces

Define the *subspace of income transfers* in $\mathbb{R}^{S+1}$ generated by the columns of the matrix W

$$\langle W \rangle = \{\tau \in \mathbb{R}^{S+1} \mid \tau = Wz,\ z \in \mathbb{R}^J\}$$

and the *orthogonal* (*dual*) *subspace of present value vectors* (*state prices*)

$$\langle W \rangle^{\perp} = \{\beta \in \mathbb{R}^{S+1} \mid \beta W = 0\}\,.$$

Each agent i's *income transfer vector* $\tau^i = Wz^i$ arising from asset trading lies in $\langle W \rangle$ and his (normalised) *present value vector* $\pi^i = (1/\lambda_0^i)\lambda^i$, arising from the portfolio first-order conditions, lies in $\langle W \rangle^{\perp}$. A key idea that underlies the analysis of incomplete markets can now be given a precise geometric statement. Since $\mathbb{R}^{S+1} = \langle W \rangle \oplus \langle W \rangle^{\perp}$, *the greater* (*smaller*) *the dimension of the space of income transfers, the smaller* (*greater*) *the space of present value vectors*. In short the greater the opportunities for income transfer, the smaller the differences of opinion among agents about the present value of a stream of date 1 income. We say that W is a *no-arbitrage* matrix if $\langle W \rangle \cap (\mathbb{R}_+^{S+1} \backslash 0) = \varnothing$. Lemma 1 can then be stated as

$$\text{either } \langle W \rangle \cap (\mathbb{R}_+^{S+1} \backslash 0) \neq \varnothing \quad \text{or} \quad \langle W \rangle^{\perp} \cap \mathbb{R}_{++}^{S+1} \neq \varnothing .$$

Thus if W is a no-arbitrage matrix then $\dim\langle W \rangle^{\perp} \geq 1$. In the case of complete markets $\dim\langle W \rangle = S$ and $\dim\langle W \rangle^{\perp} = 1$ so that there is a unique normalised vector $\pi \in \mathbb{R}_{++}^{S+1}$ (with $\pi_0 = 1$) satisfying $\pi W = 0$. With complete markets, all agents' present value vectors coincide $\pi^1 = \cdots = \pi^I = \pi$; there is complete agreement about the present value of a stream of date 1 income. This property leads to the Pareto optimality of a GEI equilibrium when asset markets are complete. When the markets are incomplete, if $\dim\langle W \rangle = J < S$ then $\dim\langle W \rangle^{\perp} = S - J + 1 > 1$. We will show that *generically in an associated GEI equilibrium, agents' normalised present value vectors are distinct*. With incomplete asset markets there is disagreement about the present value of a stream of date 1 income. It is this difference in the π^i vectors which leads to the Pareto-inefficiency of a GEI equilibrium when asset markets are incomplete. *More generally it is the differences in the π^i vectors that drive the key results in the theory of incomplete markets.*

Existence of GEI equilibrium

From the classical GE existence theorem we know that for all characteristics (u, ω) satisfying Assumption 1 the exchange economy $\mathscr{E}(u, \omega)$ has a contingent market (CM) equilibrium [Debreu (1959)]. Does a GEI equilibrium exist for all such economies? Not necessarily, as Hart (1975) first showed. The key intuition behind Hart's non-existence example can be illustrated as follows.

Example 5 (*non-existence of a GEI equilibrium*). Suppose the only activity at date 0 is the trading of financial assets. We consider an economy with two agents, two commodities and two states of nature ($I = L = S = 2$). The utility functions, endowments and asset structure of the economy $\mathscr{E}(u, \omega; A)$ are as follows:

(a) *Utility functions*:

$$u^i(x_0^i, x_1^i, x_2^i) = \sum_{s=1}^{2} \rho_s U^i(x_s^i) , \quad \rho_s > 0,\ \rho_1 + \rho_2 = 1 ,$$

$$U^i(\xi) = \alpha_1^i \log \xi_1 + \alpha_2^i \log \xi_2 , \quad i = 1, 2 ,$$

$$\alpha^i = (\alpha_1^i, \alpha_2^i) , \qquad \alpha_l^i > 0 , \qquad \alpha_1^i + \alpha_2^i = 1 .$$

(b) *Endowments*: for $0<\epsilon<1$, $1-\epsilon+h>0$,

$$\omega^1=\begin{bmatrix}\omega^1_{01} & \omega^1_{11} & \omega^1_{21}\\ \omega^1_{02} & \omega^1_{12} & \omega^1_{22}\end{bmatrix}=\begin{bmatrix}0 & 1-\epsilon+h & \epsilon\\ 0 & 1-\epsilon & \epsilon\end{bmatrix}$$

$$\omega^2=\begin{bmatrix}\omega^2_{01} & \omega^2_{11} & \omega^2_{21}\\ \omega^2_{02} & \omega^2_{12} & \omega^2_{22}\end{bmatrix}=\begin{bmatrix}0 & \epsilon & 1-\epsilon\\ 0 & \epsilon & 1-\epsilon\end{bmatrix}$$

(c) *Real assets*: futures contracts for goods 1 and 2:

$$A_s=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\qquad s=1,2 \quad \text{so that } V=\begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix}.$$

Remark. For the economy with characteristics $(u, \omega; A)$ defined by (a)–(c), a GEI equilibrium exists if and only if either $h\neq 0$ or $\epsilon=\frac{1}{2}$ or $\alpha^1_1=\alpha^2_1$. Thus if $h=0$, $\epsilon\neq\frac{1}{2}$, $\alpha^1_1\neq\alpha^2_1$ then no GEI equilibrium exists.

In this economy there is *aggregate risk* if and only if $h\neq 0$ and *individual risk* if and only if $\epsilon\neq\frac{1}{2}$. The condition $\alpha^1_1\neq\alpha^2_1$ states that the two agents have *distinct preferences* for the two goods. The assertion is thus that if there is no aggregate risk ($h=0$), if both agents face individual risk ($\epsilon\neq\frac{1}{2}$) and if the agents have distinct preferences for the two goods ($\alpha^1_1\neq\alpha^2_1$), then a risk-sharing (GEI) equilibrium cannot be obtained through a system of futures markets. Let us indicate briefly two ways of showing that no GEI equilibrium exists. First three observations:

(1) If $\alpha^1_1\neq\alpha^2_1$ then in a pure spot market equilibrium the spot prices are linearly independent.

(2) If a GEI equilibrium satisfies rank $V=2$, then a CM equilibrium can be constructed with the same allocation and prices.

(3) If $h=0$ then in a CM equilibrium the prices in the two states are collinear.

If a GEI equilibrium price $(\bar{p}, \bar{q})$ exists then either rank $V=1$ or rank $V=2$. If rank $V=1$, then the equilibrium must be a pure spot market equilibrium, since nothing can be gained from asset trading. By (1) spot prices are linearly independent, implying rank $V=2$, a contradiction. If rank $V=2$ then by (2) a CM equilibrium can be constructed with identical prices, but by (3) the prices are collinear, implying rank $V=1$, a contradiction. Thus neither case can arise and no GEI equilibrium exists when $h=0$, $\epsilon\neq\frac{1}{2}$, $\alpha^1_1\neq\alpha^2_1$.

A second argument can be obtained by examining the properties of a GEI equilibrium when $h\neq 0$. When $h\neq 0$ then in a CM equilibrium the prices in the two states are linearly independent: these equilibrium prices can be calculated. When $h\to 0$ (i.e. as the aggregate risk goes to zero) the prices become more and more collinear, so that the agents have to trade progressively more to achieve a given transfer of income. In fact as $h \to 0$, $\|z^i(h)\| \to \infty$ so that in the limit no equilibrium exists.

There is a simple economic message that underlies this example. Futures markets are not the appropriate markets for sharing individual risk when there is no underlying aggregate risk. For in the absence of aggregate risk, spot prices are not sufficiently variable across the states to permit the proper functioning of a system of futures markets.

References

The basic two period exchange economy of this section together with the concepts of a CM and an FM equilibrium (in the case where the assets are the nominal assets called *Arrow securities*) was first introduced in the classic paper of Arrow (1953). While Diamond (1967) was the first to explicitly model *incomplete* markets, the first fully articulated general equilibrium model with incomplete markets is that of Radner (1972); he established existence of an equilibrium by placing *a priori* bounds on the agents' trades in asset markets. Hart (1975) subsequently developed a more convenient model by introducing the class of *real assets*: this led to his famous examples of nonexistence and ranking of equilibria. The concept of no-arbitrage and the associated existence of prices is as old as economics and finance. Perhaps the earliest mathematical formalisation appears in the *activity analysis* literature of the 1950s [see Koopmans (1951)]. If the columns of W denote activities then the choice of a portfolio is equivalent to the choice of an activity vector. The absence of arbitrage is equivalent to the requirement that it is not possible to produce any good in positive amount without using some other good as an input – a condition that Koopmans (1951) called the *impossibility of the land of Cockaigne* – this is shown to imply the existence of positive prices for the commodities. The idea of a no-arbitrage equilibrium appears in Fischer (1972) and is made into a basic tool of analysis in Cass (1985) and Magill and Shafer (1985).

2.2. Generically complete markets

In this section we shall develop some basic techniques for handling the GEI model and show how these techniques can be used to establish the conditions under which the GEI and GE equilibrium allocations coincide. These techniques will play a basic role in all the analysis that follows.

Consider the exchange economy $\mathscr{E}(u, \omega; A)$ with financial structure A. Let us fix the profile of utility functions $u = (u^1, \ldots, u^I)$ with each u^i satisfying

Assumption 1 and the asset structure $A \in \mathbb{R}^{LSJ}$. If we let the vector of endowments $\omega = (\omega^1, \ldots, \omega^I)$ lie in the open set

$$\Omega = \mathbb{R}^{nI}_{++}$$

(called the *endowment space*) then we obtain a parametrised family of economies $\{\mathscr{E}_A(\omega), \omega \in \Omega\}$. We say that a property holds *generically* if it is true on an open set of full measure in the parameter space Ω.

Definition 5. Let $E_A(\omega)$ denote the *set of financial market (FM) equilibrium allocations* (i.e. the vector of consumption bundles $x = (x^1, \ldots, x^I)$ for each FM equilibrium) for the economy $\mathscr{E}_A(\omega)$. Similarly let $E_C(\omega)$ denote the *set of contingent market (CM) equilibrium allocations* for the parameter value ω.

The most natural way to begin an analysis of the properties of the set (correspondence) $E_A(\omega)$ is to try to relate them to the properties of the set (correspondence) $E_C(\omega)$, which are well known. From the classical GE theory we have the following three properties

(P1) *Existence*: $E_C(\omega) \neq \varnothing$ for all $\omega \in \Omega$.

(P2) *Pareto optimality*: $x \in E_C(\omega) \Rightarrow x$ is Pareto optimal, for all $\omega \in \Omega$.

(P3) *Comparative statics*: generically $E_C(\omega)$ is a finite set and each equilibrium is locally a smooth function of the parameter ω.

The problem of studying the relation between sets $E_A(\omega)$ and $E_C(\omega)$ can be posed as the solution of the following:

Characterisation problem. (*a*) *What condition on the real asset structure* $A \in \mathbb{R}^{LSJ}$ *ensures that there exist generic sets* Ω', Ω'' *such that*
(1) $E_C(\omega) \subset E_A(\omega)$ *for all* $\omega \in \Omega'$,
(2) $E_A(\omega) \subset E_C(\omega)$ *for all* $\omega \in \Omega''$.

(*b*) *If there exists a generic set* Ω^* *such that* $E_C(\omega) = E_A(\omega)$ *for all* $\omega \in \Omega^*$, *what restriction does this imply on the real asset structure* A?

The theorems of this section give the solution to the characterisation problem obtained by Magill and Shafer (1985). We begin with the key condition on the asset structure A.

Definition 6. The real asset structure $A \in \mathbb{R}^{LSJ}$ is *regular* if for each state of nature $s = 1, \ldots, S$, a row $\tilde{a}_s$ can be selected from the $L \times J$ matrix $A_s =$

$[A_s^1 \quad \cdots \quad A_s^J]$ such that the collection $(\tilde{a}_s)_{s=1}^S$ is linearly independent. Note that this requires $J \geqslant S$.

Example 6. The asset structure with $J = L = S$ futures contracts

$$A_s = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_L \,, \quad s = 1, \ldots, S$$

is regular.

Theorem 1. *If the real asset structure $A \in \mathbb{R}^{LSJ}$ is regular then there exists a generic set $\Omega' \subset \Omega$ such that*

$$E_C(\omega) \subset E_A(\omega) \,, \quad \forall \omega \in \Omega' \,.$$

Proof. The basic idea is simple.

(1) We first establish the following property: *if the asset structure A is regular then the set of critical date 1 prices*

$$K_1 = \{p_1 \in \mathbb{R}^{LS} \mid \operatorname{rank}(V(p_1)) < S\}$$

is a closed set of measure zero in $\mathbb{R}^{LS}$. Define $K = \mathbb{R}^L \times K_1$.

(2) It follows from Lemma 2(ii) that if $(\bar{x}, \bar{P})$ is a CM equilibrium for which $\operatorname{rank}(V(\bar{P}_1)) = S$, where $\bar{P} = (\bar{P}_0, \bar{P}_1)$ then there exist portfolios and prices $\bar{p} = \bar{P}$, $\bar{q} = \Sigma_{s=1}^S V_s(\bar{p}_1)$ such that $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}))$ is an FM equilibrium.

(3) If we can show that there is a generic set Ω' such that for all economies $\omega \in \Omega'$, the CM equilibrium prices do not end up in the critical set K, then the proof will be complete. To show this we use the following property: Let $U \subset \mathbb{R}^m$ and let $\phi : U \to \mathbb{R}^n$, $m \geqslant n$ be a submersion (i.e. $D_x\phi : U \to \mathbb{R}^n$ is surjective for all $x \in U$). If $K \subset \mathbb{R}^n$ is a closed set of measure zero then $\phi^{-1}(K)$ is a closed set of measure zero in $\mathbb{R}^m$.

The natural tool for completing step (3) is the theory of *regular economies* introduced by Debreu (1970); the basic ideas are explained in the article of Dierker (1982). Let the function $F : \mathbb{R}^n_{++} \times \mathbb{R}^{nI}_{++} \to R^n$ defined by

$$F(P, \omega^1, \ldots, \omega^I) = \sum_{i=1}^I (f^i(P, P \cdot \omega^i) - \omega^i)$$

denote the GE *aggregate excess demand* function and let $\hat{F} = (F_1, \ldots, F_{n-1})$ denote the truncation of F defined on the normalised price domain $\mathscr{P} = \{P \in \mathbb{R}^n_{++} \mid P_n = 1\}$. An economy $\omega \in \Omega$ is *regular* if $\operatorname{rank}(D_P\hat{F}(P, \omega)) = n - 1$ for all equilibrium prices, i.e. P satisfying $\hat{F}(P, \omega) = 0$. It is shown that the set of regular economies Ω_R is an open set of full measure in Ω. Pick $\bar{\omega} \in \Omega_R$ then by the Implicit Function Theorem there exists a neighborhood $U_{\bar{\omega}}$ of $\bar{\omega}$ and smooth functions $\psi^j : U_{\bar{\omega}} \to \mathscr{P}$, $j = 1, \ldots, r$ defining the equilibrium prices, so that $\hat{F}(\psi^j(\omega), \omega) = 0$ for all $\omega \in U_{\bar{\omega}}$, $j = 1, \ldots, r$. Thus $D_\omega \psi^j = -(D_P \hat{F})^{-1} D_\omega \hat{F}$, where $(D_P \hat{F})^{-1}$ is well defined and of rank $n-1$ since ω is a regular economy.

Since $\text{rank}(D_\omega \hat{F}) = n - 1$ it follows that $\text{rank}(D_\omega \psi^j) = n - 1$ for all $\omega \in U_{\bar{\omega}}$, so that ψ^j is a submersion. Applying the property given above $(\psi^j)^{-1}(K)$, $j = 1, \ldots, r$ are closed sets of measure zero, so that $U'_{\bar{\omega}} = U_{\bar{\omega}} \setminus \bigcup_{j=1}^{r} (\psi^j)^{-1}(K)$ is an open set of full measure in $U_{\bar{\omega}}$. Repeating the argument over a countable sequence of regular values leads to a sequence of open sets $U'_1, U'_2, \ldots$ and $\Omega' = \bigcup_{k=1}^{\infty} U'_k$ is then the desired generic set for step (3).

Remark. The key intuition behind step (3) lies in the fact that the price functions ψ^j are locally onto: this implies that ψ^j can be moved in any direction in $\mathscr{P}$ by a small perturbation in ω, thereby ensuring that all the critical prices K can be avoided.

Generic existence

Using property P1 of GE theory and Theorem 1 gives the following existence theorem for GEI equilibria.

Theorem 2 (existence). *If the real asset structure $A \in \mathbb{R}^{LSJ}$ is regular then there exists a generic set $\Omega' \subset \Omega$ such that*

$$E_A(\omega) \neq \emptyset, \quad \forall \omega \in \Omega'.$$

Remark. Property (P2) of GE theory and Theorem 1 imply that whenever $\omega \in \Omega'$ there is at least one allocation $x \in E_A(\omega)$ which is Pareto optimal. Can there be inefficient equilibria under the regularity condition? Hart (1975) showed that this can occur. Let us modify the asset structure in Example 5 and show how this can happen.

Example 5 (*continued*). Replace the futures contracts by the following real assets

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to check that inserting the GE equilibrium prices leads to a V matrix of rank 2. Thus the GE equilibrium allocation can be achieved as an FM equilibrium. But it can be shown that there exist $(\alpha_1^1, \alpha_1^2, \epsilon)$ such that there is in addition a *pure spot market* equilibrium ($\bar{z} = 0$) and this equilibrium is inefficient (in fact Pareto inferior to the full rank equilibrium). Examples of this kind are exceptional as the next theorem shows.

Theorem 3. *If the real asset structure* $A \in \mathbb{R}^{LSJ}$ *is regular then there exists a generic set* $\Omega'' \subset \Omega$ *such that*

$$E_A(\omega) \subset E_C(\omega)\,, \quad \forall \omega \in \Omega''\,.$$

Remark. The proofs of Theorems 1 and 2 use known results of GE theory to obtain a result for a GEI equilibrium. *To prove Theorem 3 we need a new tool for handling the GEI model – a method for systematically handling equilibria of all possible ranks* ρ $(0 \leq \rho \leq S)$ *while avoiding the discontinuities created by changes in the rank of* V. This can be done by using the concept of an NA equilibrium of rank ρ (Definition 4). With this new concept "market clearing" involves not just the prices P, but also the new variable $\mathscr{L}$, which consists of a *ρ-dimensional subspace* of $\mathbb{R}^S$.

We can write an NA equilibrium as the solution of a collection of conditions on the new price variables $(P, \mathscr{L}) \in \mathbb{R}^n_{++} \times G^\rho(\mathbb{R}^S)$. Let $f^1(P, \omega^1)$ denote the standard GE demand function of agent 1 and define the NA *demand functions* of the remaining agents $i = 2, \ldots, I$

$$f^i(P, \mathscr{L}; \omega^i) = \arg\max\{u^i(x^i) \mid x^i \in \mathbb{B}(P, \mathscr{L}; \omega^i)\} \tag{10}$$

and the *aggregate excess NA demand function*

$$F(P, \mathscr{L}; \omega^1, \ldots, \omega^I) = f^1(P, \omega^1) - \omega^1 + \sum_{i=2}^{I} (f^i(P, \mathscr{L}; \omega^i) - \omega^i)\,. \tag{11}$$

Then $(\bar{P}, \bar{\mathscr{L}}) \in \mathbb{R}^n_{++} \times G^\rho(\mathbb{R}^S)$ is an NA equilibrium of rank ρ if and only if

$$F(\bar{P}, \bar{\mathscr{L}}; \omega) = 0\,, \qquad \langle V(\bar{P}; A) \rangle = \bar{\mathscr{L}} \tag{12}$$

where we have included the fact that V depends on the returns matrices A, just as F depends on the parameters ω, an observation that we shall not use immediately, but which is important in Section 2.3. *Equation* (12) *gives the fundamental conditions characterising equilibria with incomplete markets.*

Representation of subspaces

Up till now the set $G^\rho(\mathbb{R}^S)$ has been viewed *purely formally* as the collection of all ρ-dimensional linear subspaces of the Euclidean space $\mathbb{R}^S$. To prove Theorem 3 and to establish the existence of equilibrium with incomplete markets (Section 2.3) *we need an explicit way of representing all ρ-dimensional subspaces in the neighborhood of a given subspace* $\mathscr{L} \in G^\rho(\mathbb{R}^S)$.

Associated with any ρ-dimensional subspace $\mathscr{L} \in G^{\rho}(\mathbb{R}^S)$ there is a unique $(S - \rho)$-dimensional subspace $\mathscr{L}^{\perp}$, its *orthogonal complement*, consisting of all vectors at right angles to $\mathscr{L}$

$$\mathscr{L}^{\perp} = \{v \in \mathbb{R}^S \mid v \perp \mathscr{L}\} ,$$

i.e. their inner product with any vector from $\mathscr{L}$ is zero. Pick any collection of linearly independent vectors $B_j \in \mathbb{R}^S$, $j = 1, \ldots, S - \rho$ such that $\{B_1, \ldots, B_{S-\rho}\}$ is a *basis* for the orthogonal space $\mathscr{L}^{\perp}$. Let B be the $(S - \rho) \times S$ matrix whose $(S - \rho)$ rows are the vectors B_j; then $\mathscr{L}^{\perp} = \langle B^T \rangle$, where B^T denotes the transpose of B and

$$\mathscr{L} = \{v \in \mathbb{R}^S \mid Bv = 0\} . \tag{13}$$

Thus $\mathscr{L}$ is represented as the solution of a system of equations using the coefficients of the matrix $B \in \mathbb{R}^{(S-\rho)S}$. But there are many ways of choosing the basis B. In fact if B is a basis for $\mathscr{L}^{\perp}$ then so is CB for any non-singular $(S - \rho) \times (S - \rho)$ matrix C. We need to *factor out* this redundancy in the representation of $\mathscr{L}^{\perp}$. Note that since rank $B = S - \rho$ we can always perform a *permutation* of the columns of B (this amounts to permuting the states $s = 1, \ldots, S$) in such a way that the permuted matrix $B' = [B_1 \mid B_2]$ where B_1 is an $(S - \rho) \times (S - \rho)$ matrix of rank $S - \rho$ and B_2 is an $(S - \rho) \times \rho$ matrix. Let $C = B_1^{-1}$ then $CB = [B_1^{-1}B_1 \mid B_1^{-1}B_2] = [I \mid E]$ where I is the $(S - \rho) \times (S - \rho)$ identity matrix and E is an $(S - \rho) \times \rho$ matrix. We now have a normalised way of representing $\mathscr{L}$ (see Figure 30.1):

$$\mathscr{L} = \{v \in \mathbb{R}^S \mid [I \mid E]v = 0\} \tag{14}$$

which involves $(S - \rho) \cdot \rho$ parameters (the matrix E) rather than the $(S - \rho) \cdot S$

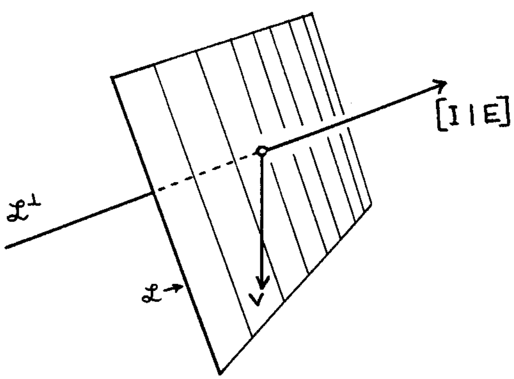

Figure 30.1. Representation of subspace.

parameters (the matrix B) in the representation (13). *It is now true that there exists a neighborhood of $\mathscr{L}$ in $G^{\rho}(\mathbb{R}^S)$ such that for any $\bar{\mathscr{L}}$ close to $\mathscr{L}$ there exists a unique matrix $\bar{E}$ in $\mathbb{R}^{(S-\rho)\rho}$ such that $\bar{\mathscr{L}}$ is represented via* (14) *with $\bar{E}$. Conversely with any E in $\mathbb{R}^{(S-\rho)\rho}$ we can associate a unique $\mathscr{L} \in G^{\rho}(\mathbb{R}^S)$.*

Proof of Theorem 3. The idea of the proof is simple. We show that there is a generic set Ω'' such that for all $\omega \in \Omega''$, every GEI equilibrium satisfies $\dim\langle V(p_1)\rangle = S$. This is equivalent to proving that equilibria with $\dim\langle V(p_1)\rangle = \rho$ for $0 \leq \rho < S$ cannot arise. By Lemma 2 we know that analysing GEI equilibria of rank ρ is equivalent to analysing NA equilibria of rank ρ. *We show that NA equilibria of rank ρ can be defined (locally) as solutions of a system of equations in which the number of equations exceeds the number of variables and in which the number of linearly independent equations exceed the number of variables*. Once this is established, the existence of the desired generic set Ω'' follows from a standard *transversality argument*.

(1) It can be shown that there exists a finite collection of manifolds M_k, $k = 1, \ldots, r$ with $\dim M_k = SL - 1$ such that the $K_1 = \{P_1 \in \mathbb{R}^{SL} \mid \text{rank } V(P_1) < S\}$ satisfies $K_1 \subset \bigcup_{k=1}^{r} M_k$. Let V be partitioned as $V = [\begin{smallmatrix} V_\alpha \\ V_\beta \end{smallmatrix}]$ where V_β is $\rho \times J$ and let $M'_k = \{P_1 \in M_k \mid \text{rank } V_\beta(P_1) = \rho\}$.

(2) Using (14) we can write aggregate demand as a function of E so that the local equations for an NA equilibrium of rank ρ become

$$\hat{F}(P, E, \omega) = 0\,, \qquad G(P, E) = 0 \tag{15}$$

where $\hat{F} = (F_1, \ldots, F_{n-1})$ and $G(P, E) = [I \mid E]V(P_1)$. Thus $(\hat{F}, G): \mathscr{P}_0 \times M'_k \times \mathbb{R}^{(S-\rho)\rho} \times \mathbb{R}^{nI}_{++} \to \mathbb{R}^{L-1} \times \mathbb{R}^{SL} \times \mathbb{R}^{J(S-\rho)}$ where $\mathscr{P}_0 = \{P_0 \in \mathbb{R}^L_{++} \mid P_{01} = 1\}$. Since $\text{rank}(D_\omega \hat{F}) = L - 1 + SL$ and $\text{rank}(D_E G) = \rho(S - \rho)$, the number of independent equations exceeds the number of variables by 1.

Using property (P2) of GE equilibria or directly using the fact that rank $V = S$ implies $\pi^1 = \cdots = \pi^I$, gives the following.

Theorem 4 (Pareto optimality). *If the real asset structure $A \in R^{LSJ}$ is regular then there exists a generic set $\Omega'' \subset \Omega$ such that $x \in E_A(\omega)$ implies x is Pareto optimal, for all $\omega \in \Omega''$.*

Combining Theorems 1–4 and defining $\Omega^* = \Omega' \cap \Omega''$ shows that the regularity condition ensures that generically GE and GEI market equilibrium allocations coincide.

Theorem 5 (equivalence under regularity). *If the asset structure $A \in \mathbb{R}^{LSJ}$ is regular then there exists a generic set $\Omega^* \subset \Omega$ such that*

$$E_A(\omega) = E_C(\omega), \quad \forall \omega \in \Omega^* .$$

Let $\mathscr{A}_R \subset \mathbb{R}^{LSJ}$ denote the *set of regular asset structures*. It is clear that $\mathscr{A}_R$ is open: but beyond this it seems to be a complicated set. It is natural to ask what happens to equilibrium allocations as we let A vary in the set $\mathscr{A}_R$.

Theorem 6 (invariance of financial structure). *Let $A \in \mathscr{A}_R$ then there exists a generic set $\Omega_A \subset \Omega$ such that for $\omega \in \Omega_A$*

$$E_{A'}(\omega) = E_A(\omega) \quad \textit{for almost all } A' \in \mathscr{A}_R .$$

Furthermore $E_A(\omega) = E_{A+\mathrm{d}A}(\omega)$ for all local changes $\mathrm{d}A \in \mathbb{R}^{LSJ}$.

Remark. Theorem 6 reveals a remarkable invariance property: *under the regularity condition equilibrium allocations of the GEI model are invariant with respect to changes in the return structure of the financial assets*. We shall see that when markets are incomplete Theorem 6 fails dramatically, for then the basic dichotomy that it reflects between the *real* and *financial* sectors of the economy is no longer valid.

Theorem 5 combined with property (P3) for GE equilibria [namely Debreu's (1970) theorem] leads to the following result.

Theorem 7 (comparative statics). *If the asset structure $A \in \mathbb{R}^{LSJ}$ is regular then generically $E_A(\omega) \neq \varnothing$ is a finite set and each equilibrium is locally a $\mathscr{C}^1$ function of the parameter ω.*

We have shown that regularity is a *sufficient* condition for ensuring that generically GE and GEI equilibrium allocations coincide: the next result shows that regularity is also a *necessary* condition for this property to hold. We have thereby obtained a complete solution to the characterisation problem posed at the beginning of this section.

Theorem 8 (necessity of regularity). *If there exists a generic set Ω^* such that $E_C(\omega) = E_A(\omega)$ for all $\omega \in \Omega^*$ then the asset structure A is regular.*

References

The techniques and results of this section were obtained by Magill and Shafer (1985). A special case of Theorems 1 and 2 where A represents futures contracts was obtained by Magill and Shafer (1984) and independently by McManus (1984) and Repullo (1986). Magill and Shafer (1985) also extended these results to the case of a stochastic exchange economy (Section 2.4).

2.3. Incomplete markets

In this section we will study the properties of the GEI model when the markets are incomplete. The key technique for establishing the generic existence of a GEI equilibrium is the global analysis of a slight weakening of the concept of a no-arbitrage (NA) equilibrium which we call a pseudo-equilibrium. The theorems of this section reveal the very different qualitative properties of the set $E_A(\omega)$ when *asset markets are incomplete* $(J < S)$.

In the previous section *genericity* was with respect to the space of endowments $\Omega = \mathbb{R}^{nI}_{++}$. In this section the parameter space is augmented by adding the *space of (real) asset structures* $\mathscr{A} = \mathbb{R}^{LSJ}$. Thus genericity is with respect to the parameters

$$(\omega, A) \in \Omega \times \mathscr{A} .$$

To emphasise this choice of parameters we let $E(\omega, A) = E_A(\omega)$ denote the set of GEI equilibrium allocations for the economy $\mathscr{E}(\omega, A) = \mathscr{E}_A(\omega)$. In view of this extended concept of genericity the theorems that follow are weaker than those presented in the previous section in the case where the markets are complete. The first result asserts the generic existence of a GEI equilibrium and will be proved later in this section.

Theorem 9 (existence). *Let $\mathscr{E}(u, \omega; A)$ be a GEI exchange economy satisfying Assumption 1. If $J < S$ then there exists a generic set $\Delta' \subset \Omega \times \mathscr{A}$ such that*

$$E(\omega, A) \neq \varnothing , \quad \forall (\omega, A) \in \Delta' .$$

The next theorem asserts that when markets are incomplete GEI equilibrium allocations are generically Pareto inefficient. A more thorough analysis of the precise sense in which GEI equilibria are inefficient is postponed to Section 5. The second property asserted by the theorem is that all agents have distinct (normalised) present value vectors. As we shall see in Section 4 this has particularly important consequences when we introduce firms that need to make decisions (at date 0) on production plans at date 1. Agents will hold quite different opinions on the present value of any such productions plans.

Theorem 10 (Pareto inefficiency). *If $J < S$ then there exists a generic set $\Delta'' \subset \Omega \times \mathscr{A}$ such that $x \in E(\omega, A)$ implies x is Pareto inefficient, for all $(\omega, A) \in \Delta''$. Furthermore the present value coefficients of the agents*

$$\pi^i \in \mathbb{R}^{S+1} , \quad i = 1, \ldots, I$$

are distinct for each $x \in E(\omega, A)$, $\forall (\omega, A) \in \Delta''$.

Proof (*idea*). It is the fact that $\dim\langle V\rangle = J \Leftrightarrow \dim\langle V\rangle^{\perp} = S - J > 0$ that allows agents (normalised) π^i vectors to be distinct in equilibrium. This is proved by adjoining the equations $\pi_1^1 - \pi_1^i = 0$ to the equations of equilibrium and showing that the resulting system of equations involves more independent equations than unknowns.

Let $\Delta^* = \Delta' \cap \Delta''$, then we have the following analogue of Theorem 5 which compares the GE and GEI equilibrium allocations.

Theorem 11. *If $J < S$ then there is a generic set $\Delta^* \subset \Omega \times \mathscr{A}$ such that*

$$E(\omega, A) \cap E_C(\omega) = \varnothing\,, \quad \forall\,(\omega, A) \in \Delta^*\,.$$

The invariance theorem of the previous section asserted that when markets are complete, changing the asset structure does not alter the equilibrium allocations: in short, *with complete markets financial changes have no real effects*. This property of invariance with respect to financial structure is no longer true when markets are incomplete. In this case, changing the structure of financial assets in general alters the equilibrium allocations: in short, *when markets are incomplete financial changes have real effects*. Unlike Theorem 6, the following result is confined to a statement about the effects of *local* changes in the asset structure.

Theorem 12 (real effects of financial assets). *If $J < S$ then there exists a generic set $\Delta^* \subset \Omega \times \mathscr{A}$ such that for all $(\omega, A) \in \Delta^*$*

$$E(\omega, A) \cap E(\omega, A + \mathrm{d}A) = \varnothing$$

for almost all local changes $\mathrm{d}A \in \mathbb{R}^{JLS}$.

Proof of Theorem 12 (*idea*). Consider an NA equilibrium of rank J with price vector $\bar{P}$. For generic $\mathrm{d}A$, $\langle V(\bar{P}, A)\rangle \neq \langle V(\bar{P}, A + \mathrm{d}A)\rangle$ since $J < S$. Since generically $\bar{P}_1 \square (\bar{x}_1^i - \omega_1^i) \not\in \langle V(\bar{P}, A + \mathrm{d}A)\rangle$ for some i, $\bar{P}$ cannot remain an equilibrium price vector. But any new equilibrium price $P' \neq \bar{P}$ must change the demand of agent 1 and hence the equilibrium allocation.

Example 7. Consider the following simple example: $I = 2$, $L = 1$, $J = 1$, agents have identical log-linear utility functions $\log x_0 + \log x_1 + \log x_2$, and endowments $\omega^1 = (1, 2, \epsilon)$, $\omega^2 = (1, \epsilon, 2)$. The single asset delivers one unit of the good in state 1 and $1 + \delta$ units in state 2. If $0 < \epsilon < 2$, it is not difficult to verify that if $\delta = 0$, the unique equilibrium is the no trade equilibrium, and that if $\delta \neq 0$ is small, trade takes place in equilibrium. *The $\delta \neq 0$ equilibrium is Pareto superior to the $\delta = 0$ equilibrium.*

Theorem 12 and the above example make it clear that the nature of assets has both *private* and *social* consequences, and that a general theory of trade with financial assets needs to include a specification of the process by which assets are designed and introduced into the economy. It is a restrictive assumption to suppose that the financial structure of an economy is given independent of the characteristics of the agents that constitute the economy. An interesting question is whether private incentives to offer assets are compatible with social welfare criteria, when it is not possible to complete the markets.

Proof of Theorem 9. Recall the strategy of the proof with potentially complete markets. (1) Show that a GE equilibrium exists. (2) Show that generically in a GE equilibrium $\dim\langle V(P, A)\rangle = S$. The strategy with incomplete markets is the same. (1)* Show that a *pseudo-equilibrium* exists. (2)* Show that generically in a pseudo-equilibrium $\dim\langle V(p, A)\rangle = J$. The concept in (1)* which generalises a GE equilibrium is defined as follows.

Definition 7. A *pseudo-equilibrium* (*ψ-equilibrium*) for the economy $\mathscr{E}(\omega, A)$ is a pair $(\bar{x}, \bar{P}, \bar{\mathscr{L}}) \in \mathbb{R}^{nI}_{+} \times \mathbb{R}^{n}_{++} \times G^{J}(\mathbb{R}^{S})$ which satisfies conditions (i) and (ii) of an NA equilibrium of rank J (Definition 4), condition (iii) being replaced by

$$\langle V(\bar{P}, A)\rangle \subset \bar{\mathscr{L}} . \tag{16}$$

Thus a pseudo-equilibrium is a *constrained* GE equilibrium: each agent satisfies the standard GE budget constraint (under $\bar{P}$) and in addition for agents $i = 2, \ldots, I$ the date 1 excess expenditures (evaluated with the date 0 prices $\bar{P}_1$) must lie in the *subspace* $\bar{\mathscr{L}}$. Just as in a GE equilibrium it can happen that $\langle V(\bar{P}, A)\rangle \subsetneq \mathbb{R}^{S}$, so in a ψ-equilibrium it can happen that $\langle V(P, A)\rangle \subsetneq \bar{\mathscr{L}}$. The transversality arguments in (2) and (2)* show that generically neither of these strict inclusions can occur.

From the homogeneity of the budget equations (8) in a ψ-equilibrium, it is clear that the prices P can be normalised to lie in the *positive unit* $(n-1)$-*sphere*

$$\mathscr{S}^{n-1}_{++} = \left\{ P \in \mathbb{R}^{n}_{++} \;\middle|\; \sum_{j=1}^{n} P_j^2 = 1 \right\}.$$

Since the GE budget constraint $P(x^i - \omega^i) = 0$ holds for each agent, Walras law holds and we truncate the aggregate excess demand function (11): $F \rightarrow \hat{F} = (F_1, \ldots, F_{n-1})$. *Thus* $(\bar{P}, \bar{\mathscr{L}}) \in \mathscr{S}^{n-1}_{++} \times G^{J}(\mathbb{R}^{S})$ *is a ψ-equilibrium price-subspace pair if and only if*

$$\begin{aligned} &\text{(i)}\;\; \hat{F}(\bar{P}, \bar{\mathscr{L}}, \omega) = 0 \\ &\text{(ii)}\;\; \langle V(\bar{P}, A)\rangle \subset \bar{\mathscr{L}} \end{aligned} \tag{17}$$

The first step is to show that there exists a pair $(\bar{P}, \bar{\mathscr{L}})$ which is a solution to (17). *This is the key step in establishing the existence of a GEI equilibrium.*

Theorem 13. *Let $\mathscr{E}(u, \omega, A)$ be a GEI exchange economy satisfying Assumption* 1, *then a ψ-equilibrium exists for all $(\omega, A) \in \Omega \times \mathscr{A}$.*

The second step is to show that generically the ψ-equilibria are smooth functions of the parameters and that ψ-equilibria can always be perturbed so that generically rank $V(\bar{P}, A) = J$. These two properties may be summarised as follows.

Lemma 3. *There exists a generic set $\Delta \subset \Omega \times \mathscr{A}$ such that for each $(\omega, A) \in \Delta$ there are at most a finite number of ψ-equilibria, each of which is locally a $\mathscr{C}^1$ function of the parameters (ω, A).*

Lemma 4. *There is a generic set $\Delta' \subset \Delta$ such that for each $(\omega, A) \in \Delta'$, $\langle V(\bar{P}, A)\rangle = \bar{\mathscr{L}}$ for each ψ-equilibrium.*

Remark. When markets are potentially complete (17)(ii) is automatically satisfied since $\mathscr{L} = \mathbb{R}^S$. Thus the conditions (17) reduce to the standard aggregate excess demand equation characterising a GE equilibrium

$$\hat{F}(\bar{P}, \omega) = 0 \,. \tag{18}$$

The problem of proving the existence of a solution to (17) thus reduces to the problem of proving that (18) has a solution. The classical GE argument uses Brouwer's Theorem to prove that (18) has a solution.

Grassmanian manifold

The main difficulty in proving Theorem 13 is the presence of the complicated set $G^J(\mathbb{R}^S)$. The reader familiar with the concept of a *manifold* will note that in the section *Representation of subspaces*, we performed the key steps in constructing an atlas for a smooth manifold structure on $G^J(\mathbb{R}^S)$ when we showed how all subspaces $\bar{\mathscr{L}}$ in the neighborhood of any subspace $\mathscr{L} \in G^J(\mathbb{R}^S)$ can be put into one-to-one correspondence with $(S-J) \times J$ matrices $\bar{E} \in \mathbb{R}^{(S-J)J}$. Consistent with its natural topology, the set $G^J(\mathbb{R}^S)$ can be given the structure of a smooth compact manifold of dimension $J(S-J)$, called the *Grassmanian manifold* of J-dimensional subspaces of $\mathbb{R}^S$. The Grassmanian is a canonical manifold which plays a key role in many parts of modern mathematics.

The presence of the Grassmanian makes it inappropriate to attempt to apply conventional fixed point theorems (Brouwer, Kakutani) to prove Theorem 13. The convexity assumption that underlies these theorems is simply not relevant. Grassmanian manifolds are in general not even acyclic, so that even the Eilenberg–Montgomery fixed point theorem would not be applicable.

We outline two approaches to proving Theorem 13. The first is due to Duffie and Shafer (1985) and gives Lemma 3 as a by-product. The second due to Husseini, Lasry and Magill (1986) and Hirsch, Magill and Mas-Colell (1987) shows that Theorem 13 is a special case of a much more general theorem. This theorem (which can be stated in a number of equivalent forms) leads to a striking generalisation of the classical Borsuk–Ulam theorem and contains Brouwers theorem as a special case – we refer the reader to the above mentioned papers for details. Before presenting these two approaches to the existence problem it will be useful to introduce two additional concepts that play an important role in the differential topology approach to general equilibrium theory. The first is the concept of the *equilibrium manifold*, the second is the concept of *degree*.

Debreu's regular economies

In studying the problem of *uniqueness* of equilibrium in the GE model, Debreu (1970) was led to introduce a new approach to the qualitative analysis of equilibrium which has proved to have far-reaching consequences. Previously the analysis of equilibrium for an exchange economy $\mathscr{E}(u, \omega)$ had focussed on existence and optimality for *fixed* characteristics (u, ω). Debreu conceived the idea of leaving the profile of preferences $u = (u^1, \ldots, u^I)$ fixed, but allowing the endowments $\omega = (\omega^1, \ldots, \omega^I)$ to be viewed as *parameters*. He was thus led to introduce the approach of differential topology. Using Sard's Theorem and the Implicit Function Theorem he showed that generically in ω, there is at most a finite number of equilibrium prices, each of which is locally a smooth function of the parameter ω. This established the property of local uniqueness, *but even more importantly it laid the correct foundation for carrying out comparative static analysis in general equilibrium theory.*

An abstract formulation of this approach was developed by Balasko (1976, 1988). The key idea is the introduction of the *equilibrium manifold*

$$\mathbb{E} = \{(P, \omega) \in \mathscr{S}^{n-1}_{++} \times \Omega \mid \hat{F}(P, \omega) = 0\} \tag{19}$$

induced by the excess demand equation (18) and the *projection*

$$\pi : \mathbb{E} \to \Omega \tag{20}$$

defined by $\pi(P, \omega) = \omega$. The equilibrium prices are then given by $\pi^{-1}(\omega)$.

Since $\mathbb{E}$ and Ω are smooth manifolds, differential topology is applicable and since $\mathbb{E}$ and Ω have the same dimension, the powerful tool of *degree* can be applied. *This approach to equilibrium theory provides a unified framework for analysing the comparative statics properties of equilibria and their existence in a setting of great generality*. It is the contribution of Duffie and Shafer (1985) to have shown almost ten years later that this abstract formulation provides a natural setting for establishing the generic existence of equilibrium with incomplete markets.

Just as in the GE model (18) leads to the equilibrium manifold (19), so in the GEI model (17) leads to the *ψ-equilibrium manifold*

$$\mathbb{E} = \left\{ (P, \mathcal{L}, \omega, A) \in \mathcal{S}^{n-1}_{++} \times G^J(\mathbb{R}^S) \times \Omega \times \mathcal{A} \;\middle|\; \begin{array}{l} \hat{F}(P, \mathcal{L}, \omega) = 0 \\ \langle V(P, A) \rangle \subset \mathcal{L} \end{array} \right\} \tag{21}$$

and the projection

$$\pi : \mathbb{E} \to \Omega \times \mathcal{A} \tag{22}$$

defined by $\pi(P, \mathcal{L}, \omega, A) = (\omega, A)$. *Proving Theorem* 13 *is equivalent to proving* $\pi^{-1}(\omega, A) \neq \emptyset$ *for all* $(\omega, A) \in \Omega \times \mathcal{A}$. The idea is to apply mod 2 degree theory to the map $\pi : \mathbb{E} \to \Omega \times \mathcal{A}$.

Mod 2 degree of map

Recall that if $f : M \to N$ is a $\mathscr{C}^1$ proper map between $\mathscr{C}^1$ manifolds M and N with $\dim M = \dim N$ and N connected, then we can associate with f an important *topological invariant* called the *mod 2 degree of f* (written $\#_2 f$) such that the number of points mod 2 in the pre-image set $f^{-1}(y)$ (written $\#_2 f^{-1}(y)$) is the same for all $y \in \mathbb{R}_f$ (the set of *regular values* of f). Furthermore if $\#_2 f \neq 0$ then $f^{-1}(y) \neq \emptyset$ for all $y \in N$. The standard way of applying degree theory is to make an astute choice of $\bar{y} \in \mathbb{R}_f$ for which it is straightforward to show $\#_2 f^{-1}(\bar{y}) = 1$.

Let $f = \pi$, $M = \mathbb{E}$, $N = \Omega \times \mathcal{A}$. It is clear that $\Omega \times \mathcal{A}$ is a smooth connected manifold with $\dim(\Omega \times \mathcal{A}) = nI + JLS$. Thus in order to prove Theorem 13 (and Lemma 3 by applying Sard's theorem) it suffices to show the following:

(i) $\mathbb{E}$ *is a* $\mathscr{C}^1$ *submanifold of* $\mathcal{S}^{n-1}_{++} \times G^J(\mathbb{R}^S) \times \Omega \times \mathcal{A}$ *with* $\dim \mathbb{E} = nI + LSJ$;

(ii) π *is proper*;

(iii) *there exists* $(\omega, A) \in \mathbb{R}_\pi$ *such that* $\#_2 \pi^{-1}(\omega, A) = 1$.

Proof. (i) Let $(\bar{p}, \bar{\mathcal{L}}, \bar{\omega}, \bar{A}) \in \mathbb{E}$, and let $H(p, E, \omega, A) = 0$ denote the system of equations (15) which represents $\mathbb{E}$ in a neighborhood of $(\bar{p}, \bar{\mathcal{L}}, \bar{\omega}, \bar{A})$.

Direct calculation shows $\text{rank}(D_{\omega,A}H(\bar{p}, E, \bar{\omega}, \bar{A})) = n - 1 + J(S - J)$, so that 0 is a regular value of H. Thus $H^{-1}(0) \subset \mathbb{E}$ is a manifold with $\dim H^{-1}(0) = \dim(\Omega \times \mathscr{A})$. $\mathbb{E}$ is the union of all such $H^{-1}(0)$, so $\mathbb{E}$ is a manifold with $\dim \mathbb{E} = \dim \Omega \times \mathscr{A}$.

(ii) We need to show $\pi^{-1}(K)$ is compact for any compact set $K \subset \Omega \times \mathscr{A}$. Since $G^J(\mathbb{R}^S)$ is compact, $\pi^{-1}(K)$ can fail to be compact only if $\pi^{-1}(K) \cap \partial\mathscr{S}^{n-1}_{++} \times G^J(\mathbb{R}^S) \times K \neq \emptyset$. But this is impossible by the *boundary behavior* of $\hat{F}$ inherited from the boundary behavior of agent 1's demand f^1.

(iii) Pick a *Pareto optimal allocation* $\bar{\omega} \in \Omega$ and let $\bar{P} \in \mathscr{S}^{n-1}_{++}$ denote the unique associated price system. Pick $\bar{A} \in \mathscr{A}$ so that $V(\bar{P}, \bar{A})$ is in general position. Let $\bar{\mathscr{L}} = \langle V(\bar{P}, \bar{A}) \rangle$ then $(\bar{P}, \bar{\mathscr{L}}, \bar{\omega}, \bar{A}) \in \mathbb{E}$ and $(\bar{P}, \bar{\mathscr{L}})$ is the unique equilibrium price pair for $(\bar{\omega}, \bar{A})$. Showing that $(\bar{\omega}, \bar{A})$ is a regular value of π reduces to showing that $\text{rank}(D_{P,E}H(\bar{P}, \bar{E}, \bar{\omega}, A)) = n - 1 + J(S - J)$ where $\bar{E}$ represents $\bar{\mathscr{L}}$.

Oriented degree

Mod 2 degree theory, rather than *oriented* degree theory was used in the above argument because it is not known, in general, if $\mathbb{E}$ is an *orientable* manifold. If $\mathbb{E}$ is orientable the same proof which shows that $\#_2\pi = 1$ shows that the *oriented degree* is 1 for a suitable choice of orientation. The advantage of being able to use the oriented degree of π is that it would permit the construction of an *index theorem* analogous to Dierker's index theorem for a GE exchange economy and would permit a study of conditions under which equilibrium is globally unique. $\mathbb{E}$ will certainly be orientable if $V(P, A)$ always has full rank, and an index theorem could be written out for this case (we do not know of anyone who has done this). If A is such that $V(P, A)$ can change rank with P, then two problems arise in attempting to verify if $\mathbb{E}$ is orientable. The construction of $\mathbb{E}$ in Duffie–Shafer simply shows that $\mathbb{E}$ can be locally represented as a solution of a transverse system of equations, from which it is difficult to obtain information about orientability. Secondly, $G^J(\mathbb{R}^S)$ itself is orientable if and only if S is odd, although it is difficult to believe that being able to write down an index formula should depend on the parity of S, which is not of immediate economic significance.

Vector bundle approach

There is an abstract approach to the GEI existence problem which may prove to have applications in other branches of equilibrium theory and to which we would now like to draw the readers attention. The idea is to lift the problem into an abstract setting where finding a solution of (17) can be viewed as showing that a *system of equations* has a solution. The mathematical object which allows one to do precisely that is called a *vector bundle* and is a powerful generalisation of the concept of a manifold. A (smooth) vector bundle is a space which is locally homeomorphic to the cartesian product of a vector space and a manifold. To each point in the manifold is associated a vector space which "twists" in a certain way as we move over the manifold: but *locally* we can always untwist the vector space so that the vector bundle looks like the above mentioned product. By introducing this concept (as we show below) we can reduce the existence problem to a simple *topological property* of an appropriately defined vector bundle.

In the above analysis Walras Law led us to replace F by its truncation $\hat{F}$. Alternatively Walras Law ($PF(P, \mathscr{L}) = 0$ for all $(P, \mathscr{L}) \in \mathscr{S}^{n-1}_{++} \times G^J(\mathbb{R}^S)$) implies that F defines a *vector field* on $\mathscr{S}^{n-1}_{++}$ for each $\mathscr{L} \in G^J(\mathbb{R}^S)$. By a familiar argument, the boundary behavior of F (namely $(P_m, \mathscr{L}_m) \in \mathscr{S}^{n-1}_{++} \times G^J(\mathbb{R}^S)$, $(P_m, \mathscr{L}_m) \to (P, \mathscr{L})$ with $P \in \partial \mathscr{S}^{n-1}_{+}$, implies $\|F(P_m, \mathscr{L}_m)\| \to \infty$) implies that F can be modified to a function $\tilde{F}$ with the following properties:

(i) $\tilde{F}$ is defined on $\mathscr{S}^{n-1}_{+} \times G^J(\mathbb{R}^J)$
(ii) $\tilde{F}$ is *inward pointing* on the boundary $\partial \mathscr{S}^{n-1}_{+}$ for each $\mathscr{L} \in G^J(\mathbb{R}^S)$
(iii) $\tilde{F}$ has the same zeros as F.

The existence of a pseudo-equilibrium then follows from Theorem 14 by setting

$$(\Phi, \Psi) = (\tilde{F}, V)\,.$$

Theorem 14. *If* $\Phi : \mathscr{S}^{n-1}_{+} \times G^J(\mathbb{R}^S) \to \mathbb{R}^n$ *is a continuous vector field on* $\mathscr{S}^{n-1}_{+}$ *which for each fixed* $\mathscr{L} \in G^J(\mathbb{R}^S)$ *is inward pointing and if the* $S \times J$ *matrix valued function* $\Psi : \mathscr{S}^{n-1}_{+} \times G^J(\mathbb{R}^S) \to \mathbb{R}^{SJ}$ *is continuous then there exists* $(\bar{P}, \bar{\mathscr{L}}) \in \mathscr{S}^{n-1}_{+} \times G^J(\mathbb{R}^S)$ *such that*

$$\Phi(\bar{P}, \bar{\mathscr{L}}) = 0\,, \quad \langle \Psi(\bar{P}, \bar{\mathscr{L}}) \rangle \subset \bar{\mathscr{L}}\,. \tag{23}$$

Proof. The idea is to construct a *vector bundle* over the manifold $\mathscr{S}^{n-1}_{+} \times G^J(\mathbb{R}^S)$ and a *section* such that $(\bar{P}, \bar{\mathscr{L}})$ is a solution of (23) if and only if this section intersects the *zero section*. The idea is then to show that the *topological structure* of this vector bundle is such that every continuous section must intersect the zero section. Hence the solution (23).

For a discussion of the properties of vector bundles we refer the reader to Bröcker and Jänich (1983) and Hirsch (1976). The following definitions may help to make some of what follows intelligible. An *m-dimensional vector bundle* $\xi = (E, M, \pi)$ over a manifold M is a triple where E (the total space) and M (the base space) are manifolds, $\pi : E \to M$ is a continuous surjective map and where $\pi^{-1}(x) = E_x$ (the fibre at x) is an m-dimensional vector space for all $x \in M$, which satisfies:

(a) for each $x \in M$ there exists an open set U containing x and a homeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{R}^m$,

(b) the restriction $h_x : E_x \to x \times \mathbb{R}^m$ is an isomorphism of vector spaces.

When the vector space E_x is the tangent space to M at x, then the vector bundle ξ is called the *tangent bundle* of M (we write $\xi = \tau_M$). A *section* of the vector bundle ξ is a map $\sigma : M \to E$ satisfying $\sigma(x) \in E_x$ for all $x \in M$. The *zero section* $\sigma_0 : M \to E$ is defined by $\sigma_0(x) = 0 \in E_x$ for all $x \in M$. A *vector field* f on a manifold M defines a section of the tangent bundle τ_M by $\sigma(x) = (x, f(x))$ for all $x \in M$.

Let $\tau_{\mathscr{S}_+^{n-1}}$ denote the *tangent bundle* of $\mathscr{S}_+^{n-1}$ and let $\gamma^{J,S} = (\Gamma^{J,S}, G^J(\mathbb{R}^S), \tilde{\pi})$ denote the *vector bundle* over the Grassmanian with total space

$$\Gamma^{J,S} = \left\{ (\mathscr{L}, w) \in G^J(\mathbb{R}^S) \times \mathbb{R}^{SJ} \;\middle|\; \begin{array}{l} w = (w^1, \ldots, w^J), \\ w^j \in \mathscr{L}^\perp, \; j = 1, \ldots, J \end{array} \right\}.$$

Let $\xi = \tau_{\mathscr{S}_+^{n-1}} \times \gamma^{J,S}$ denote the *cartesian product bundle* and define the section σ of ξ by

$$\sigma(P, \mathscr{L}) = (P, \mathscr{L}, \Phi(P, \mathscr{L}), \Pi_{\mathscr{L}^\perp} \Psi^1(P, \mathscr{L}), \ldots, \Pi_{\mathscr{L}^\perp} \Psi^J(P, \mathscr{L}))$$

where $\Pi_{\mathscr{L}^\perp}$ denotes the projection onto $\mathscr{L}^\perp$. Clearly $\sigma(\bar{P}, \bar{\mathscr{L}}) = \sigma_0(\bar{P}, \bar{\mathscr{L}})$ (where σ_0 denotes the *zero section*) if and only if $(\bar{P}, \bar{\mathscr{L}})$ solves (23).

Mod 2 Euler number of vector bundle

A vector bundle η whose fiber dimension equals the dimension of the base has a numerical invariant associated with it called the *mod 2 Euler number* [written $e_2(\eta)$] such that the number of points mod 2 at which any section σ transverse to σ_0 intersects σ_0 (written $\#_2(\sigma, \sigma_0)$) is the same for all transverse sections ($\sigma \pitchfork \sigma_0$) and $\#_2(\sigma, \sigma_0) = e_2(\eta)$. Furthermore if $e_2(\eta) = 1$ then $\sigma \cap \sigma_0 \neq \varnothing$ for all continuous sections σ. The standard way of applying the mod 2 Euler number is to make an astute choice of section $\bar{\sigma}$ for which it is straightforward to show $\#_2(\bar{\sigma}, \sigma_0) = 1$. In the case of a *manifold with boundary* (for example $\mathscr{S}_+^{n-1}$) the equality $\#_2(\sigma, \sigma_0) = e_2(\eta)$ remains true provided the vector bundle is the *tangent bundle* of the manifold and provided the sections are restricted to vector fields which are *inward pointing on the boundary*. The two geometric properties which explain why Theorem 14 works are then following.

Lemma 5. (i) $e_2(\tau_{\mathscr{S}_+^{n-1}}) = 1$, (ii) $e_2(\gamma^{J,S}) = 1$.

Proof. (i) For any $\bar{P} \in \mathscr{S}_{++}^{n-1}$ the vector field $\phi(P) = (\bar{P}/\bar{P} \cdot P) - P$ is inward pointing and defines a section $\bar{\sigma}$ of the tangent bundle $\tau_{\mathscr{S}_+^{n-1}}$ which satisfies $\bar{\sigma}(P) = \sigma_0(P)$ if and only if $P = \bar{P}$. Since $D_{\bar{P}}\phi$ has rank $n-1$, $\bar{\sigma} \pitchfork \sigma_0$. Since we have exhibited a section $\bar{\sigma}$ with a unique transverse intersection with the zero section σ_0, it follows that $e_2(\tau_{\mathcal{S}_+^{n-1}}) = 1$.

(ii) Pick any $\bar{\mathcal{L}} \in G^J(\mathbb{R}^S)$ and let $u^1, \ldots, u^J$ denote J orthonormal vectors in $\mathbb{R}^S$ such that $\bar{\mathcal{L}} = \langle u^1, \ldots, u^J \rangle$. Consider the section $\bar{\sigma}$ of $\gamma^{J,S}$ defined by $\bar{\sigma}(\mathcal{L}) = (\mathcal{L}, \Pi_{\mathcal{L}^\perp} u^1, \ldots, \Pi_{\mathcal{L}^\perp} u^J)$. Clearly $\bar{\sigma}(\mathcal{L}) = \sigma_0(\mathcal{L})$ if and only if $\mathcal{L} = \bar{\mathcal{L}}$. It can be shown by calculation that $\bar{\sigma} \pitchfork \sigma_0$ so that $e_2(\gamma^{J,S}) = 1$.

From the multiplicative property of the mod 2 Euler number with respect to a cartesian product of vector bundles, $e_2(\tau_{\mathcal{S}_+^{n-1}} \times \gamma^{J,S}) = e_2(\tau_{\mathcal{S}_+^{n-1}}) \cdot e_2(\gamma^{J,S})$. The proof of Theorem 14 follows by applying Lemma 5.

Geometric interpretation. Consider the case where $n = 2$, $J = 1$, $S = 2$. $\gamma^{1,2}$ is homeomorphic to the unit circle, $\tau_{S_+^1}$ is the tangent bundle to the positive part of the unit circle, $\Gamma^{1,2}$ is the open Möbius band (see Figure 30.2). It is the *boundary behavior* of excess demand Φ which ensures $e_2(\tau_{\mathcal{S}_+^{n-1}}) = 1$ and it is

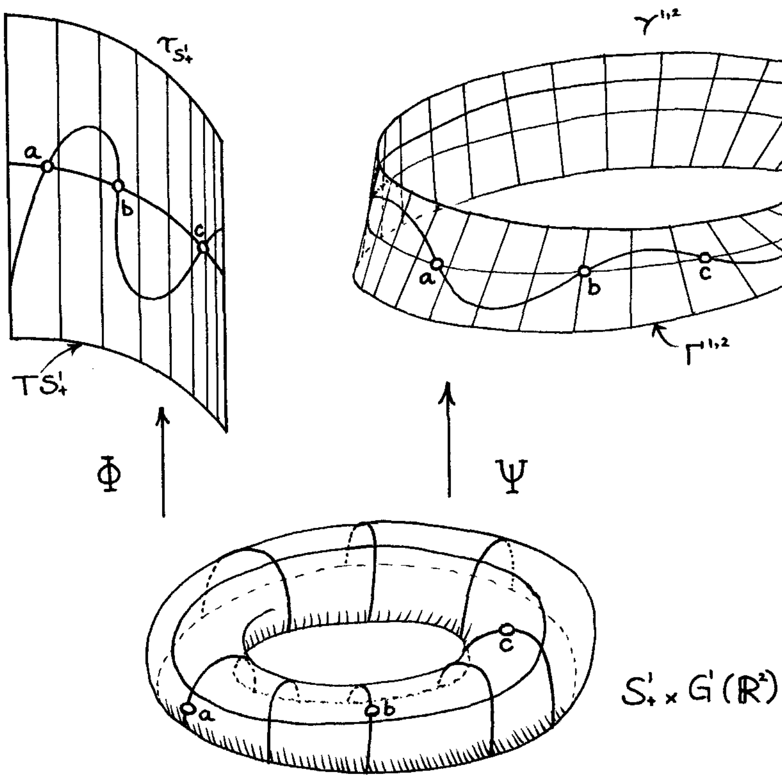

Figure 30.2. a, b, c pseudo-equilibria.

the *twisting* of the fiber as we move along the zero section (the basic *topological property* of the vector bundle $\gamma^{1,2}$) which ensures $e_2(\gamma^{1,2}) = 1$.

Remark. Lemma 5(i) is the *inward-pointing vector field theorem* which gives *existence for the GE model* (the Arrow–Debreu theorem) and is equivalent to Brouwer's theorem. Lemma 5(ii) is the new property induced by the GEI model: it can be viewed as a *subspace fixed-point theorem. The cartesian product bundle*

$$\xi = \tau_{\mathscr{S}_+^{n-1}} \times \gamma^{J,S}$$

thus provides a geometric decomposition of the problem of existence of equilibrium when markets are incomplete: the first component is the vector bundle for equilibrium with *complete markets*, or more generally for the *real market* component of the *ψ-equilibrium*, the second component is the subspace vector bundle introduced by the *incomplete financial markets* namely the *subspace compatibility condition* of a ψ-equilibrium.

References

The first existence results with incomplete markets without constraints on agents' asset trades were obtained for the GEI model with nominal assets (see *References* in Section 3). In the special case of numeraire assets an existence theorem was given by Geanakopolos and Polemarchakis (1986) [see also Chae (1988)]. In this case an equilibrium exists for all parameter values, since (with prices normalised) the rank of the returns matrix V never changes. For the general case considered in this section, in addition to the papers of Duffie and Shafer (1985), Husseini, Lasry and Magill (1986) and Hirsch, Magill and Mas-Colell (1987) mentioned above, Geanakopolos and Shafer (1987) have presented a general existence theorem which includes Theorem 13 as a special case.

2.4. *Stochastic exchange economy*

The model of the previous sections can be enriched by viewing the equilibrium in the economy as a stochastic process over many time periods, $t = 0, 1, \ldots, T$. The underlying exchange economy can be extended to a stochastic economy by modelling the uncertainty via an event-tree. There is a finite set of states of nature $\mathbf{S} = \{1, \ldots, S\}$ and a collection of partitions $F = (F_t)_{t=0}^T$ of $\mathbf{S}$ where F_{t+1} is a refinement of F_t and $F_0 = \mathbf{S}$, $F_T = \{\{s\}\}_{s \in \mathbf{S}}$. F defines an *information structure* in that at each date $t = 0, \ldots, T$ exactly one of the "events" $\sigma \in F_t$ has occurred and this is known to each agent in the economy.

If $\sigma \in F_t$ has occurred the possible events $\sigma' \in F_{t+1}$ that can occur at $t+1$ are those satisfying $\sigma' \subset \sigma$. The filtration F then defines an event-tree as follows. Let $D = \bigcup_{t=0}^{T} F_t$ (disjoint union) be the set of *nodes*. For each node $\xi \in D$ there is exactly one t and one $\sigma \in F_t$ such that $\xi = (t, \sigma)$. The unique node $\xi_0 = (0, \sigma)$ is called the *initial* node. For each $\xi \in D \backslash \xi_0$, $\xi = (t, \sigma)$ there is for $t-1$ a unique $\sigma' \in F_{t-1}$ such that $\sigma' \supset \sigma$; the node $\xi^- = (t-1, \sigma')$ is called the *predecessor* of ξ. Let $D^- \equiv \bigcup_{t=0}^{T-1} F_t$ (disjoint union) denote the set of all non-terminal nodes. For each $\xi \in D^-$ with $\xi = (t, \sigma)$, let $\xi^+ = \{\xi' \equiv (t+1, \sigma') \mid \sigma' \subset \sigma\}$ denote the set of *immediate successors* of ξ. The number of elements in the set ξ^+ is called the *branching number* of the node ξ and is written $b(\xi)$. Finally we say that ξ *succeeds* ξ' (*weakly*) if $\xi = (t, \sigma)$, $\xi' = (\tau, \sigma')$ satisfy $t > \tau$ ($t \geq \tau$), $\sigma \subset \sigma'$ and we write $\xi > \xi'$ ($\xi \geq \xi'$).

With this notation the commodity space $C(D, \mathbb{R}^L)$ consists of all functions $f: D \to \mathbb{R}^L$, namely the collection of all $\mathbb{R}^L$-valued stochastic processes, which for brevity we write as C. Each consumer i ($i = 1, \ldots, I$) has a stochastic endowment process $\omega^i \in C_{++}$ (the strictly positive orthant of C) and a utility function $u^i: C_+ \to \mathbb{R}$ satisfying Assumption 1 on the commodity space C_+. Given the information structure F if we let $(u, \omega) = (u^1, \ldots, u^I, \omega^1, \ldots, \omega^I)$ then $\mathscr{E}(u, \omega; F)$ denotes the associated stochastic exchange economy.

GE and GEI equilibrium

As in Section 2.1 we can define two market structures for the economy $\mathscr{E}(u, \omega; F)$, that of GE and that of GEI. If we define a *contingent price process* $P \in C_{++}$ then the contingent market (GE) budget set of agent i is defined by

$$B(P, \omega^i) = \{x^i \in C_+ \mid P(x^i - \omega^i) = 0\} .$$

A *contingent market* (*CM*) *equilibrium* is then given as before by Definition 1. We also refer to such an equilibrium as a GE equilibrium.

To keep the description of a GEI equilibrium simple we assume that there are J assets all initially issued at date 0. With slight complication of notation the case where assets are introduced at subsequent nodes $\xi \neq \xi_0$ can also be treated. Real asset j is characterised by a map $A^j: D \to \mathbb{R}^L$ with $A^j(\xi_0) = 0$. One unit of asset j held at ξ_0 promises to deliver the commodity vector $A^j(\xi)$ at node ξ, for $\xi \in D$. Assets are retraded at all later dates, so that one unit of asset j purchased at node ξ promises the delivery of $A^j(\xi')$ for all $\xi' > \xi$. We let $A = (A^1, \ldots, A^J)$ denote the asset structure and we let $\mathscr{A}$ denote the set of all asset structures. If $A(\xi) = [A^1(\xi) \quad \cdots \quad A^J(\xi)]$, $\xi \in D$ and $p \in C_+$ is a stochastic *spot price process* then

$$V^j(\xi) = p(\xi) A^j(\xi) , \quad \xi \in D \tag{24}$$

is the *dividend* (in units of account) paid by asset j ($j = 1, \ldots, J$) at node ξ. A *security price process* is a map $q : D \to \mathbb{R}^J$ with $q(\xi) \equiv 0$ for $\xi \not\in D^-$ (the terminal value condition); $q(\xi)$ is the vector of after-dividend prices of the J assets at node ξ. The *trading strategy* of agent i is a map $z^i : D \to \mathbb{R}^J$ with $z^i(\xi) = 0$ for $\xi \not\in D^-$; $z^i(\xi)$ is the portfolio of the J assets purchased by agent i at node ξ after the previous portfolio has been liquidated. With this notation agent i's decision problem in the GEI model is:

$$(\mathcal{P}) \begin{cases} \max\limits_{x^i, z^i} u^i(x^i) \text{ subject to} \\ p(\xi_0)(x^i(\xi_0) - \omega^i(\xi_0)) = -q(\xi_0) z^i(\xi_0)\,, \\ p(\xi)(x^i(\xi) - \omega^i(\xi)) = [p(\xi)A(\xi) + q(\xi)] z^i(\xi^-) - q(\xi) z^i(\xi)\,, \\ \quad \forall\, \xi \in D \backslash \xi_0\,. \end{cases}$$

Definition 8. A *financial market (FM) equilibrium* for the stochastic exchange economy $\mathscr{E}(u, \omega; F)$ is a pair $((\bar{x}, \bar{z}), (\bar{p}, \bar{q})) = ((\bar{x}^1, \ldots, \bar{x}^I, \bar{z}^1, \ldots, \bar{z}^I), (\bar{p}, \bar{q}))$ such that

(i) $(\bar{x}^i, \bar{z}^i)$ solves $(\mathcal{P})$, $i = 1, \ldots, I$,
(ii) $\Sigma_{i=1}^I (\bar{x}^i - \omega^i) = 0$,
(iii) $\Sigma_{i=1}^I \bar{z}^i = 0$.

We also refer to such an equilibrium as a GEI equilibrium.

No-arbitrage equilibrium

As in the two period case, the asset price process $\bar{q}$ in an FM equilibrium satisfies a no-arbitrage condition and this property allows the equilibrium to be transformed into an analytically more tractable form. Let us show how this new concept of equilibrium is derived. Given the asset structure A and a spot price process p, we say that the security price process q admits *no arbitrage* possibilities (NA) if there is no trading strategy generating a non-negative return at all nodes and a positive return for at least one node. By the same argument as in the two period case, q satisfies NA given (A, p) if and only if there exists a *stochastic state price (present value) process*

$$\beta : D \to \mathbb{R}_{++}$$

such that

$$\beta(\xi) q(\xi) = \sum_{\xi' \in \xi^+} \beta(\xi')[p(\xi')A(\xi') + q(\xi')]\,, \quad \forall\, \xi \in D^- \tag{25}$$

so that the present value (i.e. the value at date 0) *of the asset prices at node* ξ *is the present value of their dividend and capital values over the set of immediate successors* ξ^+. Solving this system of equations recursively over the nodes and using the terminal condition $q(\xi)=0$, $\forall\, \xi \notin D^-$ leads to the equivalent statement

$$q(\xi)=\frac{1}{\beta(\xi)}\sum_{\xi'>\xi}\beta(\xi')p(\xi')A(\xi'), \quad \forall\, \xi \in D^-, \tag{25$'$}$$

namely that the current value of each asset at node ξ *is the present value of its future dividend stream over all succeeding nodes* $\xi' > \xi$.

It is clear from (24) that real assets yield a return at each node ξ which is proportional to the current spot price $p(\xi)$. Thus the budget constraints in $(\mathcal{P})$ remain unchanged if the system of prices $(p(\xi), q(\xi))$, $\xi \in D$ is replaced by the system of prices $(\alpha(\xi)p(\xi), \alpha(\xi)q(\xi))$, $\xi \in D$ for any positive scalar process $\alpha : D \to \mathbb{R}_{++}$. *In a stochastic economy with only real assets price levels are unimportant.*

As in Section 2.1 the key idea is to introduce what amount to generalized Arrow–Debreu (GE) prices so that the GEI equilibrium is transformed into a constrained GE equilibrium. We define a stochastic *date* 0 *present value price process* $P : D \to \mathbb{R}^L$ by

$$P=\beta \,\square\, p=(\beta(\xi)p(\xi))_{\xi\in D} \tag{26}$$

where the box-product now extends over all nodes in the event tree. For $P : D \to \mathbb{R}^L$ and $x : D \to \mathbb{R}^L$ it is convenient to define for each $\xi \in D^-$ the *box-product over the successors of* ξ

$$P \underset{\xi}{\square} x=\Big(\sum_{\xi''\geq\xi'} P(\xi'')x(\xi'')\Big)_{\xi'\in\xi^+} \in \mathbb{R}^{b(\xi)}\,.$$

$P \,\square_\xi\, x$ is thus the vector of present values of the consumption stream x, started at each of the immediate successor nodes ξ' of ξ. For each $\xi \in D^-$ we may define the $b(\xi) \times J$ matrix of asset returns

$$V_\xi(P, A)=\Big(\sum_{\xi''\geq\xi'} P(\xi'')A(\xi'')\Big)_{\xi'\in\xi^+}$$

where the jth column is the $b(\xi)$-vector of present values of dividends from the jth asset, starting at each of the immediate successors of ξ, namely $\xi' \in \xi^+$. If we let $\langle V_\xi \rangle$ denote the subspace of $\mathbb{R}^{b(\xi)}$ spanned by the J columns of the matrix V_ξ and if we substitute (25)$'$ and (26) into the budget constraints in $(\mathcal{P})$ then we are led to the equivalent decision problem for agent i

$$(\mathscr{P})^* \begin{cases} \max\limits_{x^i} u^i(x^i) \text{ subject to} \\ P(x^i - \omega^i) = 0 \,, \\ P \mathop{\Box}\limits_{\xi} (x^i - \omega^i) \in \langle V_\xi(P, A) \rangle \,, \quad \xi \in D^- \,. \end{cases}$$

As in the two period case, the budget set implied by these constraints is the same for all no-arbitrage state price processes β. Let $\lambda^i : D \to \mathbb{R}_{++}$ denote the multiplier process induced by the constraints in $(\mathscr{P})$. If we choose $\beta = \lambda^1$ then agent 1's budget set reduces to the GE budget set $B(P, \omega^1)$ defined above.

We need to be able to consider equilibria in which for each non-terminal node $\xi \in D^-$, the subspace of income transfers $\langle V_\xi \rangle$ is of fixed dimension $\rho(\xi)$ with $0 \leq \rho(\xi) \leq \min(J, b(\xi))$. Define the product of the Grassmanian manifolds over the non-terminal nodes

$$G^{\rho,b} = \prod_{\xi \in D^-} G^{\rho(\xi)}(\mathbb{R}^{b(\xi)}) \,, \quad (\rho, b) = (\rho(\xi), b(\xi))_{\xi \in D^-} \tag{27}$$

then for any $\mathscr{L} \in G^{\rho,b}$, $\mathscr{L} = (\mathscr{L}_\xi)_{\xi \in D^-}$. We define the NA budget set of agent i for each $(P, \mathscr{L}, \omega^i) \in C_{++} \times G^{\rho,b} \times C_{++}$ by

$$\mathbb{B}(P, \mathscr{L}, \omega^i) = \left\{ x^i \in C_+ \;\middle|\; \begin{array}{c} P(x^i - \omega^i) = 0 \,, \\ P \mathop{\Box}\limits_{\xi} (x^i - \omega^i) \in \mathscr{L}_\xi, \ \xi \in D^- \end{array} \right\}, \tag{28}$$

which reduces to (9) when $T = 1$. Then a *normalized no-arbitrage (NA) equilibrium of rank* $\rho = (\rho(\xi))_{\xi \in D^-}$ with $0 \leq \rho(\xi) \leq \min(J, b(\xi))$, $\forall \xi \in D^-$, is a pair $(\bar{x}, \bar{P}, \bar{\mathscr{L}}) \in C_+^I \times C_{++} \times G^{\rho,b}$ satisfying conditions (i) and (ii) of Definition 4 with (iii) replaced by

$$\langle V_\xi(\bar{P}, A) \rangle = \bar{\mathscr{L}}_\xi \,, \quad \forall \, \xi \in D^- \,. \tag{29}$$

Lemma 2 is again true: thus an FM equilibrium of rank ρ is equivalent to an NA equilibrium of rank ρ. To prove the generic existence of a GEI equilibrium one proceeds as in the two period case, defining a *pseudo-equilibrium* (ψ-equilibrium) as a normalized no-arbitrage equilibrium of maximal rank (i.e. $\rho(\xi) = \min(J, b(\xi))$, $\forall \xi \in D^-$) in which (29) is replaced by the weaker condition

$$\langle V_\xi(\bar{P}, A) \rangle \subset \bar{\mathscr{L}}_\xi \,, \quad \forall \, \xi \in D^- \,. \tag{30}$$

The kernel of the proof of the existence of a GEI equilibrium lies in showing that a ψ-equilibrium exists for all parameter values (ω, A). Once this is established a transversality argument shows that there is a generic subset of the parameters such that for all economies in this subset every ψ-equilibrium is an NA equilibrium of maximum rank.

Generically complete markets

In a two period economy there are potentially complete markets if $J \geqslant S$, namely when the number of assets is sufficient to cover the possible contingencies (see Section 2.2). In the stochastic economy there are potentially complete markets if $J \geqslant b(\xi)$ (or more generally when the number of tradeable assets varies over the nodes if $J(\xi) \geqslant b(\xi)$) for all $\xi \in D^-$. Thus if we take J as fixed, what matters is the amount of information revealed at each node ξ measured by $b(\xi)$. If assets can be traded sufficiently often and if at each date-event ξ information is revealed sufficiently slowly then the condition can in principle be satisfied: this of course is the idea of *frequent trading* in a few assets which is the key idea underlying the *Black–Scholes theory* [for a discussion of this in the context of finance see Chapter 31].

If $J \geqslant b(\xi)$ for all non-terminal nodes ξ then the budget set $\mathbb{B}(P, \mathscr{L}, \omega^i)$ in an NA equilibrium of maximum rank reduces to the GE budget set $B(P, \omega^i)$ so that a ψ-equilibrium is a GE equilibrium. In this case the existence of a ψ-equilibrium follows from the standard GE existence theorem. To establish the existence of a GEI equilibrium it thus only remains to find conditions on the asset structure A which ensure that for most price processes P,

$$\operatorname{rank} V_\xi(P, A) = b(\xi)\,, \quad \forall\, \xi \in D^-\,, \tag{31}$$

so that (29) holds. Just as in the two period case there is a notion of regularity which does this.

Definition 9. The asset structure A in a stochastic economy is *regular* if for each node $\xi \in D^-$ and each immediate successor $\xi' \in \xi^+$ one can choose a J-vector $\tilde{a}(\xi')$ from the rows of the collection of matrices $(A(\xi''))_{\xi'' \geqslant \xi'}$ such that the collection of induced vectors over the immediate successors $(\tilde{a}(\xi'))_{\xi' \in \xi^+}$ is linearly independent.

Remark. An asset structure A is regular if and only if there exists a price process $P: D \to \mathbb{R}^L$ such that (31) holds. Thus regularity can only be satisfied if

$$J \geqslant b(\xi)\,, \quad \forall\, \xi \in D^- \tag{32}$$

when the number of assets is fixed and more generally if $J(\xi) \geqslant b(\xi)$, $\forall \xi \in D^-$ when the number of assets varies. When this condition holds it can be shown that regularity is a generic property of asset structures. In fact it is a generic property of asset structures A for which the assets pay dividends only at the terminal date T.

As in Definition 5 let $E_A(\omega)$ denote the set of financial market equilibrium allocations for the stochastic economy $\mathscr{E}_A(\omega)$ and let $E_C(\omega)$ denote the set of contingent market equilibrium allocations for the parameter value ω. The *characterisation problem* of Section 2.2 has also been completely solved for a stochastic economy (recall Theorems 5 and 8).

Theorem 15. *There is a generic subset $\Omega^* \subset \Omega$ such that*

$$E_A(\omega) = E_C(\omega)\,, \quad \forall\, \omega \in \Omega^*$$

if and only if the asset structure A is regular.

Remark. The difficult part in proving Theorem 15 lies in showing $E_A(\omega) \subset E_C(\omega)$, $\forall \omega \in \Omega^*$ (the analogue of Theorem 3). It is here that the concept of an NA equilibrium of rank ρ with $\rho(\xi) < \min(J, b(\xi))$ for some $\xi \in D^-$ is used. *The key idea* (*as with Theorem* 3) *is that for such equilibria the number of equations exceeds the number of unknowns and such systems of equations generically have no solution.*

Incomplete markets

When (31) is not satisfied we say that the asset markets in the stochastic economy are *incomplete*. In this case there is at least one non-terminal node ξ at which

$$\operatorname{rank} V_\xi(P, A) < b(\xi)$$

and at such a node agents have limited ability to redistribute their income over the immediately succeeding nodes. Thus if A is not regular, which is the case if $J < b(\xi)$ for some $\xi \in D^-$, then the asset markets are incomplete. In this case the GE existence theorem is not applicable to prove the existence of a ψ-equilibrium. The two approaches outlined in Section 2.3 can be extended to a stochastic economy.

If $n = (\#D)L$ denotes the number of spot markets over the event-tree, we let $\mathscr{S}^{n-1}_{++} = \{P \in C_{++} \mid \Sigma_{l,\xi} P_l^2(\xi) = 1\}$ denote the associated positive unit sphere in C and define $G^{\rho,b}$ as the product of Grassmanian manifolds (27) with $\rho(\xi) = \min(J, b(\xi))$, $\forall \xi \in D^-$. The first approach is based on a consideration of the ψ-equilibrium manifold

$$\mathbb{E} = \{(P, \mathscr{L}, \omega, A) \in \mathscr{S}^{n-1}_{++} \times G^{p,b} \times \Omega \times \mathscr{A} \mid (P, \mathscr{L}) \text{ is a } \psi\text{-equilibrium for } (\omega, A)\}$$

and the associated projection map $\pi : \mathbb{E} \to \Omega \times \mathscr{A}$. The argument follows the same steps as in the two period case. In the second approach the existence of a ψ-equilibrium is an immediate consequence of the following generalization of Theorem 14.

Theorem 16. *Let* $(a, b) = (a(\xi), b(\xi))_{\xi \in D^-}$, $a(\xi) \leq b(\xi)$, $\forall \xi \in D^-$ *and let* $G^{a,b} = \Pi_{\xi \in D^-} G^{a(\xi)}(\mathbb{R}^{b(\xi)})$. *If* $\Phi : \mathscr{S}^{n-1}_{+} \times G^{a,b} \to \mathbb{R}^n$ *is a continuous vector field on* $\mathscr{S}^{n-1}_{+}$ *which for each fixed* $\mathscr{L} \in G^{a,b}$ *is inward pointing and if the* $b(\xi) \times a(\xi)$ *matrix valued functions*

$$\Psi_\xi : \mathscr{S}^{n-1}_{+} \times G^{a,b} \to \mathbb{R}^{b(\xi)a(\xi)}, \quad \forall\, \xi \in D^-$$

are continuous, then there exists $(\bar{P}, \bar{\mathscr{L}}) \in \mathscr{S}^{n-1}_{+} \times G^{a,b}$ *such that*

$$\Phi(\bar{P}, \bar{\mathscr{L}}) = 0\,, \quad \langle \Psi_\xi(\bar{P}, \bar{\mathscr{L}}) \rangle \subset \bar{\mathscr{L}}_\xi,\ \forall\, \xi \in D^-\,.$$

Remark. Consider the collection of vector bundles $\gamma^{a(\xi),b(\xi)}$, $\xi \in D^-$ over the Grassmanians $G^{a(\xi)}(\mathbb{R}^{b(\xi)})$, $\xi \in D^-$. The proof is based on the multiplicative property of the mod 2 Euler number of the cartesian product and the use of Lemma 5 which gives

$$e_2\Big(\prod_{\xi \in D^-} \gamma^{a(\xi),b(\xi)}\Big) = \prod_{\xi \in D^-} e_2(\gamma^{a(\xi),b(\xi)}) = 1\,.$$

The second step consists of using a perturbation (transversality) argument to show that there is a generic subset of the parameter space $\Omega \times \mathscr{A}$ for which (29) holds at every ψ-equilibrium. For a fixed information structure F, let $E(\omega, A)$ denote the set of FM equilibrium allocations of the stochastic economy with parameters (ω, A).

Theorem 17. *If* $J < b(\xi)$ *for some* $\xi \in D^-$ *then there exists a generic set* $\Delta \subset \Omega \times \mathscr{A}$ *such that* $E(\omega, A)$ *consists of a positive finite number of equilibria for each* $(\omega, A) \in \Delta$.

Remark. The perturbation argument requires that at any non-terminal node ξ for which $J < b(\xi)$, there be $J(b(\xi) - J)$ free parameters in $(A(\xi'))_{\xi' > \xi}$ in order to perturb the matrix $V_\xi(P, A)$. Thus, in particular, it is not possible to replace $\mathscr{A}$ by the subset $\mathscr{A}'$ consisting of assets which pay dividends only at the terminal date T. This is in contrast to Theorem 15 which permits such asset structures. It would thus be of considerable interest if the following conjecture could be proved.

Conjecture. *For all J and all asset structures A with J assets, there is a generic set* $\Omega_A \subset \Omega$ *such that* $E(\omega, A) \neq \emptyset$ *for all* $\omega \in \Omega_A$.

Remark. This section has presented a brief summary of the GEI model with *real* assets for the case of a multiperiod exchange economy. There is a wealth of interesting properties of the underlying GEI model (such as the martingale and volatility properties of asset prices [LeRoy (1989)] that we have not attempted to analyse. The reader will recognise that in the one good case ($L = 1$) the resulting GEI model is essentially the basic model of the *theory of finance*. In his analysis of the relation between the Black–Scholes theory and the GEI model Kreps (1982) made clear that even in such a one good model, when there are three or more periods one can at best expect to obtain a generic existence theorem. The techniques and concepts of this section are thus likely to provide an appropriate analytical framework for a broader class of GEI models than might at first be expected.

References

The basic event-tree model of an exchange economy together with the concept of a CM equilibrium was given by Debreu (1959). The idea that frequent trading in a few securities can dramatically increase spanning was first systematically exploited by Black and Scholes (1973). Kreps (1982) presented a general equilibrium model and showed that if the condition $J \geqslant b(\xi)$ for all ξ holds then any CM equilibrium for $\mathscr{E}(u, \omega, F)$ can be implemented as an FM equilibrium for almost all A with J assets. The equivalence result (Theorem 15) was given by Magill and Shafer (1985). The proof of existence with incomplete markets was given by Duffie and Shafer (1986a, 1986b).

3. Nominal assets

The object of this section is to study the nature of GEI equilibria when some or all of the assets are nominal. For simplicity we consider only the case of a two-period economy. Asset j is called a *nominal* asset if it promises to deliver an exogenously given stream $N^j = (N_1^j, \ldots, N_S^j)^T$ of units of account (dollars) across the states at date 1. The riskless bond, for which $N^j = (1, \ldots, 1)^T$ is the simplest example of such an asset. It should not be surprising that the equilibria of a model with nominal assets behave very differently from the equilibria of a model with real assets. Basic economic intuition suggests the reason. Real assets are contracts promising dividends which are proportional to the prices in each state: doubling prices in any state doubles the dividend income that these assets generate. In short real assets are inflation proof. This is not the case with nominal assets: if the spot prices (in some state) are doubled since the dividend income remains unchanged, the purchasing power of the nominal asset's return is halved. What are the consequences of this for the resulting GEI equilibria?

Walras' test

A good way of obtaining a rough (and as we shall see, basically correct) answer is to go back to an old idea of Walras: *let's count the number of unknowns and equations, being careful to factor out any redundancy.* Let

$$\tilde{x}^i(p, q, \omega^i),\ \tilde{z}^i(p, q, \omega^i),\quad i = 1, \ldots, I \tag{33}$$

denote the I agents demand functions for $L(S+1)$ goods and the J assets. A vector of GEI equilibrium prices (p, q) is a solution of the system of equations

$$\begin{aligned} F(p, q, \omega) &= \sum_{i=1}^{I} (\tilde{x}^i(p, q, \omega^i) - \omega^i) = 0\,, \\ G(p, q, \omega) &= \sum_{i=1}^{I} \tilde{z}^i(p, q, \omega^i) = 0\,. \end{aligned} \tag{34}$$

Are the $L(S+1)+J$ equations in (34) independent? Certainly not. Let $F = (F_0, F_1, \ldots, F_S)$, then the fact that each agent fully spends his income in each state implies that we have $S+1$ Walras' Laws

$$p_0F_0 + qG = 0\,, \qquad p_sF_s - V_sG = 0\,, \quad s = 1, \ldots, S\,. \tag{35}$$

Thus there are at most $L(S+1)+J-(S+1)$ independent equations. This is true regardless of the type of assets we are considering, whether real or nominal.

What is the dimension of the set of prices (p, q)? Let us lay aside the fact that we need to restrict attention to no-arbitrage asset prices: this will not alter the argument that follows. Consider first the case where all the financial assets are real assets. Pick any vector of inflation factors $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_S) \in \mathbb{R}^{S+1}_{++}$ then we have seen that since each agent's budget set is independent of the price levels

$$\begin{aligned} F(\alpha \,\square\, p, \alpha_0 q, \omega) &= F(p, q, \omega)\,, \\ G(\alpha \,\square\, p, \alpha_0 q, \omega) &= G(p, q, \omega)\,. \end{aligned} \tag{36}$$

These $S+1$ homogeneity conditions correspond to the fact that there are $S+1$ directions in which price changes have no real effects. If we factor out these $S+1$ dimensions of redundant prices then the equilibrium equations (34) become a system of equations which typically has $(L-1)(S+1)+J$ independent equations in the same number of "relative" prices (p, q). Hence the conclusion: *with real assets there is generically a finite number of GEI equilibria* (see Lemmas 3 and 4 in Section 2.3). Thus the concept of a GEI equilibrium with real assets is well defined (determinate).

Suppose now that all the financial assets are nominal. Then provided the matrix of nominal asset returns satisfies a non-degeneracy condition, *there are at least two directions (easily checked from the budget equations) and in fact only two directions (proved in Section 3.1) in which price changes have no real effects*, namely those defined by the scalars $\alpha_0, \alpha_1 \in \mathbb{R}_{++}$ with

$$(p_0, q) \to (\alpha_0 p_0, \alpha_0 q), \ (p_1, q) \to \left(\alpha_1 p_1, \frac{q}{\alpha_1}\right),$$

the vector p_1 denoting the vector of spot prices at date 1. The equilibrium equations (34) thus typically have $L(S+1)+J-(S+1)$ independent equations in $L(S+1)+J-2$ unknown prices. *Walras' test applied to the GEI model with nominal assets implies that there are $S-1$ less equations than unknowns.* Hence the conclusion: with nominal assets the set of GEI equilibrium prices is generically an $(S-1)$-dimensional set. Since we have factored out the price changes which leave the budget sets unchanged it would seem that these $S-1$ dimensions of prices should correspond to $S-1$ dimensions of distinct real equilibrium allocations. If this is the case, then surely we are led to conclude that the concept of a GEI equilibrium with nominal assets as it stands is not well-defined? We shall see that this is indeed the case (Section 3.1).

The Walrasian test applied to the GEI model reveals an essential distinction between real and nominal assets. In the model with real assets since price levels are unimportant there is no need to explicitly introduce a role for money: indeed in such a model money is unimportant. *However in a model with nominal assets to obtain a well-defined concept of equilibrium we need to explicitly introduce a role for money as a medium of exchange.* Thus nominal assets in the GEI model lead us to the concept of a monetary equilibrium in which money influences the equilibrium allocation in an essential way. In this way the indeterminacy of the nominal asset equilibrium is translated into the property that money has real effects in the monetary equilibrium (Section 3.2).

3.1. Indeterminacy of GEI equilibrium with nominal assets

The object of this section is to make precise the sense in which there is indeterminacy in the GEI model with nominal assets and to reveal why the indeterminacy arises. We will see that the indeterminacy of equilibrium can be traced to the conjunction of the following three properties of the model.

(1) Nominal assets are contracts which promise returns denominated in the unit of account (say dollars).

(2) Variations in the purchasing power of the unit of account across the states at date 1 give rise to different equilibria.

(3) There is no mechanism endogenous to the model which determines the purchasing power of the unit of account across the states at date 1.

(1) is obvious and (3) is clear given (2). Understanding the indeterminacy of equilibrium thus amounts to understanding (2).

When all the J assets are nominal the date 1 returns matrix (1) can be written as

$$V = N = \begin{bmatrix} N_1^1 & \cdots & N_1^J \\ \vdots & & \vdots \\ N_S^1 & \cdots & N_S^J \end{bmatrix} .$$

We assume that there are no redundant assets so that rank $N = J$: our principal interest lies in the case where the asset markets are incomplete so that $J < S$. A GEI equilibrium in which all assets are nominal is called a *nominal asset equilibrium*.

Let $((x, z), (p, q); N)$ denote such an equilibrium when the nominal asset structure is given by N. The key to understanding (2) lies in noting that a nominal asset equilibrium can be viewed as a GEI equilibrium in which all J assets are real numeraire assets (Example 2). This is in fact immediate: for nominal asset j pays N_s^j units of account in state s and this is equivalent to a real numeraire asset which pays N_s^j/p_{s1} units of good 1. Thus if we define the diagonal matrix (representing the *purchasing power* of a unit of account across the states at date 1)

$$[\nu_1] = \begin{bmatrix} \nu_1 & & 0 \\ & \ddots & \\ 0 & & \nu_S \end{bmatrix} \quad \text{where } \nu_s = \frac{1}{p_{s1}}, \; s = 1, \ldots, S , \tag{37}$$

then $((x, z), (p, q); [\nu_1]N)$ is a real numeraire asset equilibrium with good 1 returns matrix

$$\begin{bmatrix} A_{11}^1 & \cdots & A_{11}^J \\ \vdots & & \vdots \\ A_{S1}^1 & \cdots & A_{S1}^J \end{bmatrix} = \begin{bmatrix} \nu_1 & & 0 \\ & \ddots & \\ 0 & & \nu_S \end{bmatrix} \begin{bmatrix} N_1^1 & \cdots & N_1^J \\ \vdots & & \vdots \\ N_S^1 & \cdots & N_S^J \end{bmatrix} . \tag{38}$$

Conversely if we pick any positive diagonal matrix $[\nu_1]$ (i.e. $\nu_s > 0$, $s =$

$1, \ldots, S$) and if $((x, z), (p, q); [\nu_1]N)$ is a real numeraire asset equilibrium with good 1 returns matrix defined by (38) which in addition satisfies (37) (we can always assume this since with real assets we are free to adjust the equilibrium price levels) then $((x, z), (p, q); N)$ is a nominal asset equilibrium. *Thus* $((x, z), (p, q); N)$ *is a nominal asset equilibrium if and only if there exists a positive diagonal matrix* $[\nu_1]$ *such that* $((x, z), (p, q); [\nu_1]N)$ *is a real numeraire asset equilibrium.*

Let $E'(\omega, N)$ denote the set of nominal asset GEI equilibrium allocations for the returns matrix N and let $E(\omega, [\nu_1]N)$ denote the set of numeraire asset equilibrium allocations for the matrix (38). The choice of a positive diagonal matrix

$$[\nu_1] \in \mathcal{N} = \mathcal{S}^{S-1}_{++}$$

lying in the positive $(S-1)$-dimensional unit sphere corresponds to the choice of a profile of purchasing power for the unit of account across the states at date 1. As we shall show below for most choices of the parameters (ω, ν_1) we obtain a finite number of equilibrium allocations. *Thus for a given profile of purchasing power* ν_1 *the GEI model becomes well defined.* Since

$$E'(\omega, N) = \bigcup_{\nu_1 \in \mathcal{N}} E(\omega, [\nu_1]N),$$

analysing the GEI nominal asset equilibrium allocations reduces to studying the family of real numeraire asset equilibrium allocations $E(\omega, [\nu_1]N)$ as ν_1 varies in $\mathcal{N}$. We shall view this as a problem of *comparative statics of equilibria* for which the equilibrium manifold approach of Section 2.3 provides the canonical framework.

To this end we transform the equilibrium into an NA equilibrium by introducing (date 0) present value prices

$$P = \beta \,\square\, p \ \text{ with } \ \beta = \lambda^1$$

and define the diagonal matrix of present value prices of good 1 across the states at date 1

$$[P_1] = \begin{bmatrix} P_{11} & & 0 \\ & \ddots & \\ 0 & & P_{S1} \end{bmatrix}.$$

It is easy to check that since rank $[P_1][\nu_1]N = J$ for all $\nu_1 \in \mathcal{N}$ and all strictly positive matrices $[P_1]$, if we substitute equation (17)(ii) (which now holds with equality) into equation (17)(i) by defining $\hat{\mathbb{F}} : \mathcal{S}^{n-1}_{++} \times \Omega \times \mathcal{N} \to \mathbb{R}^{n-1}$ with

$$\hat{\mathbb{F}}(P, \omega, \nu_1) = \hat{F}(P, \langle [P_1][\nu_1]N\rangle; \omega\rangle ,$$

then the equilibrium equations (17) reduce to

$$\hat{\mathbb{F}}(P, \omega, \nu_1) = 0 . \tag{39}$$

The following result can be viewed as a consequence of Theorem 13; it can also be established directly using the standard techniques of GE.

Theorem 18. *Under Assumption* 1, $E'(\omega, N) \neq \emptyset$ *for all* $(\omega, N) \in \Omega \times \mathbb{R}^{SJ}$.

We now begin a study of the "size" of $E'(\omega, N)$. A familiar argument shows that equations (39) can be "controlled" by appropriately varying the endowments ω, so that $\hat{\mathbb{F}} \pitchfork 0$. Thus the equilibrium manifold (21) reduces to

$$\mathbb{E} = \{(P, \omega, \nu_1) \in \mathcal{S}^{n-1}_{++} \times \Omega \times \mathcal{N} \mid \hat{\mathbb{F}}(P, \omega, \nu_1) = 0\}$$

which is a manifold of dimension $nI + S - 1$. The projection $\pi : \mathbb{E} \to \Omega \times \mathcal{N}$ is proper. Thus by Sard's theorem the set Δ of regular values of π is a generic subset of $\Omega \times \mathcal{N}$. In a neighborhood $\Delta_{(\bar{\omega}, \bar{\nu}_1)}$ of each regular value $(\bar{\omega}, \bar{\nu}_1) \in \Delta$, each equilibrium price vector P can be written as a smooth function $P(\omega, \nu_1)$ of the parameters. Let $\tilde{x}^1(P, \omega) = f^1(P, \omega^1)$ denote agent 1's GE demand function and for $i = 2, \ldots, I$, let $\tilde{x}^i(P, \omega, \nu_1) = f^i(P, \langle [P_1][\nu_1]N\rangle; \omega^i)$ denote agent i's demand function [where f^i is defined by (10)], then the equilibrium allocation $x = (x^1, \ldots, x^I)$ is a smooth function $\tilde{x}(P(\omega, \nu_1), \omega, \nu_1)$ of (ω, ν_1).

Let $\tilde{z}^i(P, \omega, \nu_1)$ denote the portfolio which finances agent i's net expenditure at date 1, i.e.

$$P_1 \,\square\, (\tilde{x}^i_1(P, \omega, \nu_1) - \omega^i_1) = [P_1][\nu_1]N\tilde{z}^i(P, \omega, \nu_1) , \quad i = 2, \ldots, I .$$

We want to show that if there are sufficiently many agents relative to the number of assets ($I > J$) then there is a generic subset $\Delta^* \subset \Delta$ such that in an equilibrium the J vectors

$$\{\tilde{z}^i(P(\omega, \nu_1), \omega, \nu_1)_{i=2}^{J+1}\}$$

are linearly independent. To this end for $\alpha \in \mathcal{S}^{J-1}$ [the $(J-1)$-dimensional unit sphere] consider the function $g : \mathcal{S}^{n-1}_{++} \times \Delta_{(\bar{\omega}, \bar{\nu}_1)} \times \mathcal{S}^{J-1} \to \mathbb{R}^J$ defined by

$$g(P, \omega, \nu_1, \alpha) = \sum_{i=2}^{J+1} \alpha_i \tilde{z}^i(P, \omega, \nu_1)$$

and $h = (\hat{\mathbb{F}}, g) : \mathscr{S}^{n-1}_{++} \times \Delta_{(\bar{\omega},\bar{\nu}_1)} \times \mathscr{S}^{J-1} \rightarrow \mathbb{R}^{n-1} \times \mathbb{R}^J$. The asset demands of the agents can be "controlled" without affecting the demands for goods, by appropriately redistributing endowments of the agents: thus $h \pitchfork 0$.

If we consider the manifold

$$\tilde{\mathbb{E}} = \{(P, \omega, \nu_1, \alpha) \in \mathscr{S}^{n-1}_{++} \times \Delta_{(\bar{\omega},\bar{\nu}_1)} \times \mathscr{S}^{J-1} \mid h(P, \omega, \nu_1, \alpha) = 0\} ,$$

then we find that the projection $\tilde{\pi} : \tilde{\mathbb{E}} \rightarrow \Delta_{(\bar{\omega},\bar{\nu}_1)}$ is proper so that by Sard's theorem the set of regular values $\Delta^*_{(\bar{\omega},\bar{\nu}_1)}$ is generic. Since $\dim(\mathbb{R}^{n-1} \times \mathbb{R}^J) > \dim(\mathscr{S}^{n-1}_{++} \times \mathscr{S}^{J-1})$, i.e. the number of equations exceeds the number of unknowns, $\tilde{\pi}^{-1}(\omega, \nu_1) = \varnothing$, $\forall (\omega, \nu_1) \in \Delta^*_{(\bar{\omega},\bar{\nu}_1)}$. Repeating the argument in a standard way over a countable collection of regular values gives the desired set Δ^* on which the property of linear independence holds.

Consider $(\bar{\omega}, \bar{\nu}_1) \in \Delta^*$ and pick $(\bar{\omega}, \nu_1)$ in a neighborhood of $(\bar{\omega}, \bar{\nu}_1)$ with $\nu_1 \neq \bar{\nu}_1$. We want to show that

$$\tilde{x}(P(\bar{\omega}, \nu_1), \bar{\omega}, \nu_1) \neq \tilde{x}(P(\bar{\omega}, \bar{\nu}_1), \bar{\omega}, \bar{\nu}_1) \tag{40}$$

so that for fixed $\bar{\omega}$, changing ν_1 changes the equilibrium allocation. Suppose that with $\nu_1 \neq \bar{\nu}_1$ equality holds in (40). Then from the first-order conditions for agent 1, $P = P(\bar{\omega}, \nu_1) = P(\bar{\omega}, \bar{\nu}_1) = \bar{P}$ so that

$$\langle (P_1 \,\square\, (x^i_1 - \omega^i_1))^{J+1}_{i=2} \rangle = \langle (\bar{P}_1 \,\square\, (\bar{x}^i_1 - \omega^i_1))^{J+1}_{i=2} \rangle . \tag{41}$$

Since the J vectors on the left and right side of (41) are linearly independent, we will have arrived at a contradiction if we can show that

$$\langle [\nu_1] N \rangle \neq \langle [\bar{\nu}_1] N \rangle . \tag{42}$$

Definition 10. An $S \times J$ matrix N with $J \leqslant S$ is in *general position* if every $J \times J$ submatrix of N has rank J.

Lemma 6. *Let N be an $S \times J$ matrix in general position with $J < S$. If δ, $\bar{\delta} \in \mathbb{R}^S_{++}$ satisfy $\langle [\delta] N \rangle = \langle [\bar{\delta}] N \rangle$ then there exists $a \in \mathbb{R}$ such that $\delta = a\bar{\delta}$.*

Proof. Without loss of generality let $\bar{\delta} = (1, \ldots, 1)$. Let $\delta \in \mathbb{R}^S_{++}$ satisfy $\langle [\delta] N \rangle = \langle N \rangle$. Thus each column of the matrix $[\delta]N$ can be written as a linear combination of the columns of the matrix N. There is thus a $J \times J$ matrix C such that $[\delta]N = NC$. Thus $C^T N_s = \delta_s N_s$, $s = 1, \ldots, S$ so that (δ_s, N_s) is an eigenvalue-eigenvector pair for C^T. We want to show that there exists $a \in \mathbb{R}$ such that $\delta = (\delta_1, \ldots, \delta_S) = a(1, \ldots, 1)$. Since the subspaces spanned by eigenvectors associated with distinct eigenvalues form a direct sum, unless all eigenvalues coincide, $\delta_1 = \cdots = \delta_S = a$, we contradict the general position of N.

Consider the projection $\bar{\pi} : \Omega \times \mathcal{N} \to \Omega$. Since the projection of a generic set is generic, $\Omega^* = \bar{\pi}(\Delta^*)$ is a generic subset of Ω. For each $\bar{\omega} \in \Omega^*$ there exists $\bar{\nu}_1 \in \mathcal{N}$ such that $(\bar{\omega}, \bar{\nu}_1) \in \Delta^*$. There is thus a neighborhood $\mathcal{N}_{\bar{\nu}_1}$ of $\bar{\nu}_1$ such that the equilibrium allocation map

$$x^* : \mathcal{N}_{\bar{\nu}_1} \to \mathbb{R}^{nI}, \quad x^*(\nu_1) = \bar{x}(P(\bar{\omega}, \nu_1), \bar{\omega}, \nu_1)$$

is $\mathscr{C}^1$ and injective. We have thus proved the following theorem.

Theorem 19. *Let $E'(\omega, N)$ be the set of equilibrium allocations of the nominal asset economy $\mathscr{E}'(\omega, N)$. If Assumption* 1 *holds and* (i) $0 < J < S$, (ii) $I > J$, (iii) *N is in general position, then there exists a generic set $\Omega^* \subset \Omega$ such that for each $\bar{\omega} \in \Omega^*$, $E'(\bar{\omega}, N)$ contains the image of a $\mathscr{C}^1$ injective map of an open set of dimension $S - 1$.*

Remark. If rank $N = S$ then the subspaces satisfy $[\nu_1]N = \mathbb{R}^S$ for all $\nu_1 \in \mathscr{S}^{S-1}_{++}$. The equilibria coincide with the GE equilibria and are thus generically finite and locally unique.

Remark. There is a close connection between Theorem 19 and the earlier Theorem 12: both assert that when markets are incomplete changing the asset structure changes the equilibrium allocation. In both cases changing the asset structure twists the subspace of date 1 income transfers so that some agent's date 1 net expenditures $(P_1 \mathbin{\square} (x^i_1 - \omega^i_1))$ are no longer affordable. Theorem 19 however considers a restricted set of subspace changes, namely $\langle N \rangle \to \langle [\nu_1]N \rangle$ with $\nu_1 \in \mathcal{N}$. It thus requires the additional assumption that if we exclude agent 1, there be enough agents $(I - 1 \geq J)$ so that generically their date 1 net expenditures span the subspace of income transfers. In this way *any* change in the subspace is sure to leave some agent's date 1 net expenditures out of the new subspace.

Remark. If N is not in general position or more generally if the returns matrix V consists of a mixture of real and nominal assets then not every change in ν_1 changes the subspace of income transfers. Thus the equilibrium set $E'(\omega, N)$ contains the image of an injective map of an open set which is typically of dimension less than $S - 1$: in most cases the dimension remains positive, Arrow securities which pay a unit of account in one state and nothing otherwise being an exception.

References

The GEI model with nominal assets first appears in Arrow (1953) where $N = I$ (the $S \times S$ identity matrix). It was extended to the case of a general returns matrix N by Cass (1984) and Werner (1985) who proved Theorem 18 [see also Werner (1989)]. The first example of indeterminacy with nominal assets was given by Cass (1985). Theorem 19 is due to Geanakoplos and Mas-Colell (1989) and Balasko and Cass (1989). The latter authors also show that if asset prices are exogenously fixed then there is still indeterminacy of dimension $S - J$. An important concept that we have not dealt with in this section is the idea of *restricted participation*; that is, not all agents may have full access to the asset markets. In the framework of the nominal asset model, Balasko, Cass and Siconolfi (1987) have shown that even if the returns matrix N has full rank if there is a subgroup of agents with restricted ability to participate on the asset markets then there is still indeterminacy of dimension $S - 1$.

Remark. The authors cited above interpret Theorem 19 as the assertion that *when markets are incomplete the equilibrium allocations that arise in an economy with nominal assets are seriously indeterminate*: *the dimension of indeterminacy is of the same magnitude as the degree of uncertainty about the future* $(S - 1)$.

The different equilibria whose existence is asserted by Theorem 19 arise by varying the purchasing power ν_1 of the unit of account across the states at date 1. As the proof of the theorem makes clear, a given equilibrium corresponds to a particular profile $\bar{\nu}_1$ of purchasing power; to correctly anticipate equilibrium prices $(\bar{q}, \bar{p})$ agents must correctly anticipate the future purchasing power $\bar{\nu}_1$ of the unit of account. *But there are no data in the model of the economy which indicate how the different profiles of purchasing power $\bar{\nu}_1 \in \mathcal{N}$ come to be chosen or are agreed upon by the agents*; the parameters $\nu_1 \in \mathcal{N}$ are simply *free variables*. What is needed is clear; the purchasing power of the unit of account must be determined by equilibrium equations just like any other variable in the model.

3.2. Monetary equilibrium and real effects of money

In the nominal asset model originally introduced by Cass (1984) and Werner (1985), the unit of account is typically viewed as the unit induced by money: the bonds for example pay off in dollars. But the money thus introduced only performs its first function, namely to act as a *unit of account*. Its second and third functions, namely to act as a *medium of exchange* and a *store of value* are left unmodelled.

Magill and Quinzii (1988) have presented a model which preserves the basic structure of the nominal asset economy but which adds a monetary framework in which all three functions of money can be analysed, albeit in a stylised way. They model the role of money as a medium of exchange via a cash-in-advance constraint. To separate the activities of sale and purchase of commodities in exchange for money they split each period into three subperiods. In the first subperiod agents sell their endowments to a central exchange receiving in return a money income. In the second subperiod they trade on the asset (bond) markets and decide how to allocate the resulting money holdings between *precautionary* balances ($z_0^i \geqslant 0$) to be used to date 1 and *transactions* balances. These latter balances are then used to purchase their commodity bundles from the central exchange. The same sequence is repeated in each state s at date 1, except that in the second subperiod, assets pay dividends and the precautionary balances are liquidated to finance the commodity purchases in the third subperiod.

The central exchange is run by the government which injects an amount of money $M = (M_0, M_1, \ldots, M_S)$ in the first subperiod of each state ($s = 0, 1, \ldots, S$) in exchange for the endowments. The statement that the transactions demand for money equals the supply in each state gives rise to a system of monetary equations

$$\sum_{i=1}^{I} p_s x_s^i = v_s M_s \,, \quad s = 0, 1, \ldots, S \tag{43}$$

akin to the *quantity theory equations*. The vector of velocities of circulation $v = (v_0, \ldots, v_S)$ is endogenously determined and depends on the precautionary holdings $(z_0^1, \ldots, z_0^I)$ of the agents. A *monetary equilibrium* is then essentially a nominal asset equilibrium to which are added the monetary equations (43). It is the latter system of equations which "closes" the model and enables the purchasing power of money to be endogenously determined.

How does the Walrasian test of counting non-redundant equations and unknowns apply to the concept of a monetary equilibrium? Briefly, excess demand on the spot and asset markets leads to a system of equations akin to (34). To this are adjoined the $S + 1$ monetary equations (43). Since equation (35) continues to be valid there are still $S + 1$ Walras' Laws. However the addition of the monetary equations (43) implies that there is now no homogeneity property in the prices. *The* $S + 1$ *equations* (43) *thus exactly compensate for the equations missing by virtue of the* $S + 1$ *Walras' Laws*. We would thus expect that generically there are a finite number of monetary equilibria (as is confirmed by the analysis of Magill and Quinzii).

The analysis of the agents precautionary demands for money is facilitated if it is assumed that $(1, \ldots, 1)^T \in \langle N \rangle$ or that the first asset is a riskless bond

$N^1 = (1, \ldots, 1)^T$. Its price then satisfies $q_1 = 1/(1 + r_1)$ where r_1 is the *riskless rate of interest*. With this assumption it can be shown that generically there are two types of equilibria: those in which $r_1 > 0$ and $v = (1, \ldots, 1)$ and those in which $r_1 = 0$ and $v \neq (1, \ldots, 1)$. In what follows we concentrate on a qualitative statement for the positive interest rate equilibria: in these the precautionary demand for money is zero since money is dominated by the riskless bond as a store of value.

For fixed N we let the economy be parametrised by the endowments and money supply

$$(\omega, M) \in \Omega \times \mathcal{M}\ , \quad \mathcal{M} = \mathbb{R}^{S+1}_{++}\ .$$

To factor out those monetary changes which are *neutral*, in a neighborhood $\mathcal{M}_{\bar{M}}$ of a monetary policy $\bar{M} \in \mathcal{M}$ we define the induced $(S-1)$-dimensional neighborhood

$$\tilde{\mathcal{M}}_{\bar{M}} = \left\{ M \in \mathcal{M}_{\bar{M}} \mid M_0 = \bar{M}_0, \sum_{s=1}^{S} M_s = \sum_{s=1}^{S} \bar{M}_s \right\}. \tag{44}$$

The following result regarding the neutrality or non-neutrality of monetary policy can then be derived [see Magill and Quinzii (1988)].

Theorem 20. *Let Assumption 1 hold. There is a generic set $\Delta \subset \Omega \times \mathcal{M}$ for which the monetary equilibria of the economy $\mathscr{E}(\omega, M; N)$ are regular.*

(a) *If rank $N = S$ any positive interest rate equilibrium allocation $x(\bar{\omega}, \bar{M})$ with $(\bar{\omega}, \bar{M}) \in \Delta$ satisfies $x(\bar{\omega}, M) = x(\bar{\omega}, \bar{M})$ for all M in a neighborhood of $\bar{M}$.*

(b) *If* (i) $(1, \ldots, 1) \in \langle N \rangle$, (ii) $0 < J < S$, (iii) $I > J$, (iv) *N is in general position, for any positive interest rate equilibrium allocation $x(\bar{\omega}, \bar{M})$ with $(\bar{\omega}, \bar{M}) \in \Delta$ there is an $(S-1)$-dimensional neighborhood $\tilde{\mathcal{M}}_{\bar{M}}$ of $\bar{M}$ [defined as in (44)] such that the image of the equilibrium allocation map $x(\bar{\omega}, \cdot) : \tilde{\mathcal{M}}_{\bar{M}} \to \mathbb{R}^{nI}$ is a submanifold of $\mathbb{R}^{nI}$ of dimension $S-1$.*

Remark. This result is closely related to the *policy effectiveness* debate of Sargent and Wallace (1975) and Fischer (1977). Theorem 20(a) may be viewed as a general equilibrium version of the Sargent–Wallace neutrality proposition: *with rational expectations monetary policy is locally neutral if* (*i*) *asset markets are complete and* (*ii*) *the velocity of circulation of money is locally independent of M.* Theorem 20(b) can be viewed as a general equilibrium version of the Fischer critique: *with rational expectations if* (*i*) *asset markets are incomplete and* (*ii*) *nominal asset returns and the velocity of circulation are locally independent of M, then generically monetary policy has real effects.* Of course for some types of contracts it may not be realistic to assume that nominal returns are fixed independently of anticipated monetary policy.

Remark. An important condition required for the validity of Theorems 19 and 20 is that there be sufficient diversity among agents in the economy. This diversity is twofold. First there must be enough agents ($I > J$). Second the agents must be distinct – more precisely genericity conditions are made to ensure that the agents have distinct endowments and hence distinct income profiles. The fact that the arguments depend in an essental way on diversity among the agents places these results in sharp contrast with an important strand of modern macroeconomics which is based on models of equilibrium with a single *representative* agent. *The redistributive income effects that lie behind the real effects of money supply changes are necessarily absent in all representative agent economies.*

4. Production and the stock market

In the previous sections we have shown how the traditional (GE) theory of an exchange economy can be extended to the framework of incomplete markets (GEI). The key feature in this transition is a change of emphasis from reliance on a system of markets for real goods to a division of roles between spot markets for allocating real goods and financial markets for redistributing income and sharing risks. Thus while GE theory views the economy as consisting solely of a *real sector*, the GEI theory provides a symmetric role for the *real* and *financial sectors* of the economy.

How is the traditional GE theory altered when we move to a production economy? What new phenomena enter? Is the resulting theory satisfactory? It will become clear in attempting to answer these questions that developing a satisfactory GEI theory of a production economy presents much greater challenges.

Two-period production economy

We consider the simplest two-period model of a production economy with uncertainty. To this end we adjoin to the exchange economy $\mathscr{E}(u, \omega)$ of Section 2.1 a finite number of firms $j = 1, \ldots, J$ each characterised by a *production set* $Y^j \subset \mathbb{R}^n$ and an initial endowment vector $\eta^j \in \mathbb{R}^n$. Each firm chooses a production plan $y^j \in Y^j$, $y^j = (y_0^j, y_1^j, \ldots, y_S^j)$ where $y_s^j = (y_{s1}^j, \ldots, y_{sL}^j)$ denotes the vector of goods produced in state s: if $y_{sl} < 0$ (>0) then good l is used in state s as an input (is produced in state as an output). The technical conditions that we imposed on the agent's characteristics (u, ω) in Section 2.1 are those that lead to a *smooth exchange economy*. The technical conditions that we now add regarding the firms' characteristics (Y^j, η^j) are those that lead to a *smooth production economy*. The reader should not be upset if these conditions seem a little tricky to express: the role of each assumption is in fact straightforward.

Fundamentally the production sets Y^j should be like the standard convex production sets of GE. However, to be able to use the machinery of differential topology in the qualitative analysis of equilibrium we need two additional properties:

(a) *each production set Y^j has a smooth boundary ∂Y^j,*
(b) *a convenient way of parametrising the decisions of firms.*

So that (a) does not imply that the production set Y^j involves *all* commodities, we say that Y^j is a full-dimensional submanifold of a linear subspace $E^j \subset \mathbb{R}^n$. However E^j cannot be an arbitrary subspace of $\mathbb{R}^n$ – it should involve some activity in each state (i.e. for any non-trivial production plan y^j, in each state some good is input or output). The initial endowments η^j are introduced to obtain property (b). So as not to be arbitrary, they should be compatible with the production sets Y^j in the sense of lying in the subspace E^j. Finally the production sets $Y = (Y^1, \ldots, Y^J)$ and endowment vectors $(\omega, \eta) = (\omega^1, \ldots, \omega^I, \eta^1, \ldots, \eta^J)$ must be related in such a way that it is not possible to produce an arbitrarily large amount of any commodity (aggregate output is bounded). More formally

Assumption 2 (*Firm characteristics*). (1) $Y^j \subset \mathbb{R}^n$ is closed, convex and $0 \in Y^j$.

(2) There exist linear subspaces $E_s^j \subset \mathbb{R}^L$, $s = 0, 1, \ldots, S$ with $\dim(E_s^j) > 0$ such that Y^j is a full-dimensional submanifold (with boundary) of $E^j = E_0 \times E_1 \times \cdots \times E_S$.

(3) Y^j satisfies free disposal relative to E^j.

(4) The boundary ∂Y^j is a $\mathscr{C}^2$ manifold with strictly positive Gaussian curvature at each point.

(5) There is a non-empty open set $\mathcal{O} \subset \mathbb{R}^{n(I+J)}$ such that if we define

$$\Omega = \left(\mathbb{R}^{nI}_{++} \times \sum_{j=1}^{J} E^j\right) \cap \mathcal{O}$$

then $\Sigma_{i=1}^{I} \omega^i + \Sigma_{j=1}^{J} \eta^j \in \mathbb{R}^n_{++}$, $\forall (\omega, \eta) \in \Omega$ and $(\Sigma_{i=1}^{I} \omega^i + \Sigma_{j=1}^{J} (Y^j + \eta^j)) \cap \mathbb{R}^n_+$ is compact $\forall (\omega, \eta) \in \Omega$.

To complete the description of the production economy we need a statement about the way the ownership of the J firms is distributed among the I consumers. Let

$$\zeta = \begin{bmatrix} \zeta_1^1 & \cdots & \zeta_1^I \\ \vdots & & \vdots \\ \zeta_J^1 & \cdots & \zeta_J^I \end{bmatrix}$$

denote the matrix of *initial ownership shares* where ζ^i_j is the ownership share of agent i in firm j. We assume

$$\zeta \in \mathbb{R}^{JI}_+ , \qquad \sum_{i=1}^{I} \zeta^i_j = 1 , \quad j = 1, \ldots , J . \tag{45}$$

If the agents' characteristics (u, ω) satisfy Assumption 1, firms' characteristics (Y, η) satisfy Assumption 2 and the ownership shares ζ satisfy (45) then we obtain a *production economy* $\mathscr{E}(u, Y, \zeta; \omega, \eta)$ which forms the basis for the analysis that follows. Whenever generic arguments are needed we parametrise the economy by the initial endowments

$$(\omega, \eta) \in \Omega = \left(\mathbb{R}^{nI}_{++} \times \sum_{j=1}^{J} E^j\right) \cap \mathscr{O} .$$

An *allocation* $(x, y) = (x^1, \ldots , x^I, y^1, \ldots , y^J)$ for the economy $\mathscr{E}(u, Y, \zeta; \omega, \eta)$ is a vector of consumption $x^i \in \mathbb{R}^n_+$ for each consumer $(i = 1, \ldots , I)$ and a production plan $y^j \in Y^j$ for each firm $(j = 1, \ldots , J)$. Equilibrium theory can be viewed as the qualitative study of the allocations that arise when we adjoin different market structures to the production economy $\mathscr{E}(u, Y, \zeta; \omega, \eta)$. As in the earlier sections we study two such market structures, that of classical GE (*contingent markets*) and that of GEI (*spot-security markets*).

Contingent markets (GE)

Contingent commodities and the vector of contingent prices $P \in \mathbb{R}^n$ were defined in Section 2.1. Let

$$y = \begin{bmatrix} y^1_0 & \cdots & y^J_0 \\ y^1_1 & \cdots & y^J_1 \\ \vdots & & \vdots \\ y^1_S & \cdots & y^J_S \end{bmatrix}$$

denote the $L(S + 1) \times J$ matrix whose columns are the J firms production plans. With contingent markets agent i's (GE) budget set becomes

$$B(P, y, \eta, \zeta^i, \omega^i) = \{x^i \in \mathbb{R}^n_+ \mid P(x^i - \omega^i - (y + \eta)\zeta^i) = 0\} .$$

The shareholders of each firm j $(j = 1, \ldots , J)$ are unanimous that the firm acts in their best interests (and more generally of all consumers) if it maximises the *present value of its profit* $P \cdot y^j$ over its production set Y^j. This leads to the following concept of equilibrium.

Definition 11. *A contingent market (CM) equilibrium* for the economy $\mathscr{E}(u, Y, \zeta; \omega, \eta)$ is a pair of actions and prices $((\bar{x}, \bar{y}), \bar{P})$ such that

(i) $\bar{x}^i$, $i = 1, \ldots, I$ satisfy

$$\bar{x}^i = \arg\max\{u^i(x^i) \mid x^i \in B(\bar{P}, \bar{y}, \eta, \xi^i, \omega^i)\}$$

(ii) $\bar{y}^j$, $j = 1, \ldots, J$ satisfy

$$\bar{y}^j = \arg\max\{\bar{P} \cdot y^j \mid y^j \in Y^j\}$$

(iii) $\Sigma_{i=1}^{I} (\bar{x}^i - \omega^i) = \Sigma_{j=1}^{J} (\bar{y}^j + \eta^j)$.

We also refer to such an equilibrium as a GE equilibrium.

Stock-market (GEI)

As we mentioned before, a system of contingent markets is not the type of market structure that we observe in an actual economy: it should be viewed as an *ideal* system of markets. A more realistic market structure is obtained by splitting the allocative role of markets between a system of real spot markets on the one hand and a system of financial markets on the other. In this section we restrict ourselves to the simplest class of financial contracts which allows us to describe the functioning of the GEI model of a production economy. We assume that the J financial assets consist of the J securities issued by the firms in which the agents hold the initial ownership shares defined by the matrix ζ. Real assets such as futures contracts can be included at the cost of some complication in the notation. A proper treatment of nominal assets such as bonds calls for an analysis along the lines of Section 3.2.

We arrive however at an awkward problem of modelling. If we look at the real world where time and uncertainty enter in an essential way then we must recognise two facts: first, in terms of the risks faced and the resources and ability to pay in all circumstances there are substantial differences between (small) individual consumers and (large) shareholder owned firms: thus loans will not be granted anonymously. Second, in practice not all consumers and firms deliver on their contracts in all contingencies: there is frequently *default*.

The highly idealised model that we consider below abstracts from these two crucial difficulties. Since we assume that consumers and firms have equal access to the financial markets and since there is no default, *under general assumptions regarding the behavior of firms, the equilibrium allocations that emerge do not depend on the financial policies chosen by the firms*. In short, to obtain determinate financial policies which influence the equilibrium allocation further imperfections need to be introduced.

Since modelling necessarily proceeds by steps, let us try to make this clear. Let $D^j = (D_0^j, D_1^j, \ldots, D_S^j)^T$ denote the vector of dividends paid by firm j (where D_0^j is paid after the security has been purchased) and let q_j denote its market price $(j = 1, \ldots, J)$. We allow all firms free access to the equity markets. This means that each firm can buy and sell the securities of all firms as it wants. Suppose firm j has chosen its production plan y^j and its vector of ownership shares in all firms $\xi^j = (\xi_1^j, \ldots, \xi_J^j) \in \mathbb{R}^J$. If we define the *matrix of stock market returns*

$$W(q, D) = \begin{bmatrix} D_0 - q \\ D_1 \end{bmatrix} = \begin{bmatrix} D_0^1 - q_1 & \cdots & D_0^J - q_J \\ D_1^1 & \cdots & D_1^J \\ \vdots & & \vdots \\ D_S^1 & \cdots & D_S^J \end{bmatrix}, \tag{46}$$

then each firm's dividend stream D^j satisfies

$$D^j = p \,\square\, (y^j + \eta^j) + W(q, D)\xi^j, \quad j = 1, \ldots, J. \tag{47}$$

If we define the matrix of *inter-firm shareholdings*

$$\xi = \begin{bmatrix} \xi_1^1 & \cdots & \xi_1^J \\ \vdots & & \vdots \\ \xi_J^1 & \cdots & \xi_J^J \end{bmatrix},$$

then the system of linear equations (47) can be solved to give

$$D = \left(p \,\square\, (y + \eta) - \begin{bmatrix} q\xi \\ 0 \end{bmatrix} \right) [I - \xi]^{-1}. \tag{48}$$

provided the matrix $(I - \xi)$ is non-singular (a sufficient condition is $\xi^j \in \mathbb{R}_+^J$, $\Sigma_{k=1}^J \xi_j^k < 1$, $j = 1, \ldots, J$). Equation (48) expresses the fact that *when firms are allowed to buy and sell shares in other firms then the dividends D^j of firm j depend not only its own production-portfolio decision (y^j, ξ^j) but on the production-portfolio decisions of all firms (y, ξ).*

Given the expression (48) for the dividends, the budget sets of the consumers can be defined. If agent i begins with the *initial* portfolio of ownership shares in the J firms $\zeta^i = (\zeta_1^i, \ldots, \zeta_J^i)$ and $z^i = (z_1^i, \ldots, z_J^i) \in \mathbb{R}^J$ denotes the *new* portfolio purchased, then his budget set is given by

$$\mathscr{B}(P, q, D; \zeta^i, \omega^i) = \{x^i \in \mathbb{R}_+^n \mid p \,\square\, (x^i - \omega^i) = q\zeta^i e_0 + Wz^i, z^i \in \mathbb{R}^J\}$$

where $e_0 = (1, 0, \ldots, 0) \in \mathbb{R}^{S+1}$. The following preliminary concept of equilibrium describes how the stock market values the plans $(y, \xi) = (y^1, \ldots, y^J, \xi^1, \ldots, \xi^J)$ chosen by the firms.

Definition 12. A *stock-market equilibrium with fixed producer plans* (y, ξ) is a pair $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}); (y, \xi))$ such that

(i) $(\bar{x}^i, \bar{z}^i)$, $i = 1, \ldots, I$ satisfy

$$\bar{x}^i = \arg\max\{u^i(x^i) \mid x^i \in \mathscr{B}(\bar{p}, \bar{q}, \bar{D}, \zeta^i, \omega^i)\},$$

$$\bar{p} \,\square\, (\bar{x}^i - \omega^i) = \bar{q}\zeta^i e_0 + \bar{W}\bar{z}^i,$$

(ii) (y^j, ξ^j), $j = 1, \ldots, J$ satisfy (a) $y^j \in Y^j$, (b) $(I - \xi)$ is invertible,
(iii) $\Sigma_{i=1}^I (\bar{x}^i - \omega^i) = \Sigma_{j=1}^J (y^j + \eta^j)$,
(iv) $\Sigma_{i=1}^I \bar{z}^i + \Sigma_{j=1}^J \xi^j = e$, $e = (1, \ldots, 1) \in \mathbb{R}^J$.

The following result shows that the equilibrium allocations $\bar{x}$ and the equilibrium prices $(\bar{p}, \bar{q})$ are independent of the firms financial policies ξ.

Proposition 21. *If $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}); (y, \xi))$ is a stock market equilibrium with fixed producer plans (y, ξ) then $((\bar{x}, \tilde{z}), (\bar{p}, \bar{q}); (y, \tilde{\xi}))$ is a stock market equilibrium with fixed producer plans $(y, \tilde{\xi})$ where $\tilde{z} = (I - \tilde{\xi})(I - \xi)^{-1}\bar{z}$.*

Proof. Consider the induced exchange economy $\mathscr{E}(u, \underset{\sim}{\omega}; A)$ with real asset structure $A = [y_1 + \eta_1]$ where agent i's endowment is given by $\underset{\sim}{\omega}_s^i = \omega_s^i + (y_s + \eta_s)\zeta^i$, $s = 0, 1, \ldots, S$, $i = 1, \ldots, I$. If $((\bar{x}, \bar{\theta}), (\bar{p}, \bar{q}))$ is an FM equilibrium for $\mathscr{E}(u, \underset{\sim}{\omega}; A)$ then $((\bar{x}, \bar{z}), (\bar{p}, \tilde{q}))$ is an equilibrium with fixed producer plans where

$$\tilde{q}_j = \bar{p}_0 y_0^j + \bar{q}_j, \qquad \bar{z}^i = [I - \xi](\bar{\theta}^i + \zeta^i).$$

The proof then follows from the fact that $\mathscr{E}(u, \underset{\sim}{\omega}; A)$ is independent of ξ.

Proposition 21 can readily be extended to the case where firms and consumers have access to K other real securities in zero net supply characterised by an $S \times K$ date 1 returns matrix R_1. In this more general setting Proposition 21 is in essence the Modigliani–Miller Theorem. In particular *if we let borrowing and lending be denoted by the numeraire asset which pays one unit of good* 1 *in each state at date* 1 *then we obtain the Modigliani–Miller proposition on the irrelevance of debt financing.*

Note that if we impose *short-sales* constraints on agents $(z^i \in \mathbb{R}_+^J)$ then the market values $\bar{q}$ may depend on the choice of financial policies ξ: for even if $\bar{\theta}^i + \zeta^i \in \mathbb{R}_+^J$, the matrix $[I - \xi]$ will not in general map $\mathbb{R}_+^J$ into $\mathbb{R}_+^J$. Similarly if we allow firms to have access to financial policies which alter the span of the financial markets then the market values $\bar{q}$ will be influenced by their financial policies ξ.

Firms objective functions

The above analysis suggests that there is a broad class of models, where even when markets are incomplete, while consumers view their own choices of portfolios z as being of great importance, as shareholders of the firms they do not view the firms' choices of financial policies as important. The choice of particular financial policies by firms is simply a matter of packaging: if consumers and firms have access to the same subspace of income transfers $\langle W \rangle$, a consumer can always repackage the income streams offered by firms. As shareholders however, the consumers view firms' choices of production plans (y) as a decision of great importance. *Do the spot and equity markets provide firms with enough price information to be able to deduce what the appropriate objective functions should be for making their choices of production plans?*

In the analysis that follows we restrict ourselves to the class of *linear* objective functions. Since there are spot markets available in each state and since the spot prices p_s guide the firm's decision within a state, the problem of determining an objective function for firm j reduces to determining the relative prices to be assigned to the states, namely the choice of a vector of present value prices

$$\beta^j = (1, \beta_1^j, \ldots, \beta_S^j) \in \mathbb{R}_{++}^{S+1}, \quad j = 1, \ldots, J.$$

Suppose for the moment β^j has been determined.

We assume that firm j's manager chooses the production financing decision $(y^j, \xi^j) \in Y^j \times \mathbb{R}^J$ which maximises the present value of its dividend stream

$$\beta^j D^j = \sum_{s=0}^{S} \beta_s^j D_s^j, \quad j = 1, \ldots, J \tag{49}$$

given the production-financing decisions (y^k, ξ^k) of all other firms $k \neq j$. Since the dividend stream D^j satisfies (47) we can write (49) as

$$\beta^j D^j = \beta^j \cdot (p \,\square\, (y^j + \eta^j)) + \beta^j W(q, D)\xi^j, \quad j = 1, \ldots, J.$$

Suppose $\beta^j W \neq 0$ then there exists a sequence of portfolios ξ_ν^j such that $\beta^j D^j \to \infty$ as $\nu \to \infty$. *Thus a necessary condition for each objective function* (49) *to attain a maximum is that* $\beta^j \in \mathbb{R}_{++}^{S+1}$ *satisfy*

$$\beta^j W = 0 \Leftrightarrow \beta^j \in \langle W \rangle^{\perp} \cap \mathbb{R}_{++}^{S+1}, \quad j = 1, \ldots, J \tag{50}$$

so that β^j *is a positive supporting state price to the attainable set* $\langle W \rangle$. But when this property holds

$$\beta^j D^j = \beta^j \cdot (p \,\square\, (y^j + \eta^j)) , \quad j = 1, \ldots, J ,$$

the present value of firm j's dividend stream equals the present value of its profit. *Thus with an objective function satisfying* (49) *and* (50) *firm j chooses its production plan y^j to maximise the present value of its profit and its financial policy ξ^j is irrelevant.* (The fact that each firms' objective function is independent of its financial policy can be viewed as the second part of the Modigliani–Miller Theorem. The first part is given by Proposition 21.)

With an objective function of the form (49) assigned to each firm the GEI model becomes closed. Since each firm has a criterion for evaluating its production-financing decision (y^j, ξ^j) the concept of a stock market equilibrium with fixed producer plans (y, ξ) can be replaced by the following concept.

Definition 13. *A stock-market equilibrium* for the economy $\mathscr{E}(u, Y, \zeta; \omega, \eta)$ is a pair $((\bar{x}, \bar{z}), (\bar{y}, \bar{\xi}), (\bar{p}, \bar{q}))$ such that the conditions of Definition 12 are satisfied with (ii)(a) replaced by

(ii)(a)′ there exist $\bar{\beta}^j \in \mathbb{R}^{S+1}_{++}$ satisfying (50) such that

$$\bar{y}^j = \arg\max\{\bar{\beta}^j \cdot (\bar{p} \,\square\, y^j) \mid y^j \in Y^j\} , \quad j = 1, \ldots, J . \tag{51}$$

Generically complete stock market $(J \geqslant S)$

In the context of Definition 13 the GEI "theory of the firm" is reduced to a theory of how β^j is determined. Consider the simplest case first where there are enough publicly traded firms for their equity contracts to span all possible contingencies $(J \geqslant S)$. In this case for generic $(\omega, \eta) \in \Omega$, for any stock market equilibrium, rank $\bar{p}_1 \,\square\, (\bar{y}_1 + \eta_1) = S$. Since the equilibrium does not depend on $\bar{\xi}$ we can set $\bar{\xi} = 0$. Thus

$$\bar{\beta}^j \bar{W} = 0 \Leftrightarrow \bar{\beta}^j_1(\bar{p}_1 \,\square\, (\bar{y}_1 + \eta_1)) = \bar{q} - \bar{p}_0 \bar{y}_0 , \quad j = 1, \ldots, J \tag{52}$$

has a unique (normalised) solution

$$\bar{\beta}^j = (1, \bar{\beta}^j_1) = (1, \bar{\pi}_1) = \bar{\pi} , \quad j = 1, \ldots, J . \tag{53}$$

With a complete stock market each firm can deduce its vector of present value prices $\bar{\beta}^j = \bar{\pi}$ from a knowledge of the spot and equity prices $(\bar{p}, \bar{q})$ and the outputs $(\bar{y} + \eta)$ of all firms [or more generally the dividend policies $\bar{D}$ defined by (48)]. Since each consumer's present value vector $\bar{\pi}^i$ satisfies (52) we obtain equality of the present value vectors of all consumers and firms

$$\bar{\pi}^i = \bar{\beta}^j = \bar{\pi} , \quad i = 1, \ldots, I, \; j = 1, \ldots, J . \tag{54}$$

The first-order conditions for consumers and firms on the spot markets then imply that their gradients satisfy

$$\left(\frac{1}{\bar{\lambda}_0^i}\right) D_{\bar{x}^i} u^i = \bar{\pi} \square \bar{p} \in N_{\bar{y}^j}\, \partial Y^j\,, \quad i = 1, \ldots, I,\ j = 1, \ldots, J \tag{55}$$

where $N_{\bar{y}^j}\, \partial Y^j$ denotes the set of normal vectors to the boundary ∂Y^j at $\bar{y}^j$. (55) are the standard first-order necessary conditions for Pareto optimality, which in view of Assumptions 1 and 2 are also sufficient.

The analysis of Section 2.2 can be extended to the production economy $\mathscr{E}(u, Y, \zeta; \omega, \eta)$. Let $E_C(\omega, \eta)$ denote the set of contingent market equilibrium allocations and let $E_Y(\omega, \eta)$ denote the set of stock market equilibrium allocations, then the following result can be established (see Theorem 5).

Theorem 22. *If Assumptions* 1 *and* 2 *hold and if* $J \geq S$ *then there exists a generic set* $\Omega^* \subset \Omega$ *such that*

$$E_Y(\omega, \eta) = E_C(\omega, \eta)\,, \quad \forall\, (\omega, \eta) \in \Omega^*\,.$$

There are a positive finite number of stock market equilibria each of which is Pareto optimal and locally a smooth function of the parameters (ω, η).

Remark. Three additional properties of the stock market equilibria of Theorem 22 should be noted. Since

$$\bar{q} = \bar{\pi}\bar{D} \Leftrightarrow \bar{q} = \bar{\pi} \cdot (\bar{p} \square (\bar{y} + \eta))$$

each firm maximises its market value under the standard competitive assumption that firms ignore the effect of changes in their production decisions on the prices $(\bar{\pi}, \bar{p})$. *Since* $\bar{\pi}^i = \bar{\pi}$, $i = 1, \ldots, I$ *all shareholders* (*and consumers*) *unanimously approve the production decisions* $\bar{y}$ *taken by the firms*. Since the stock market and contingent market equilibrium allocations coincide, *the stock market allocations do not depend on the financial policies* $\bar{\xi}$ *chosen by the firms* (which are therefore indeterminate).

Partial spanning

For simplicity we express the idea that follows for the case of a one good economy ($L = 1$). We say that the technology sets and initial endowments (Y, η) satisfy *partial spanning* if there exists a linear subspace $Z \subset \mathbb{R}^{S+1}$ of dimension $K \leq J$ such that

$$Y^j \subset Z\,, \quad \eta^j \in Z\,, \quad j = 1, \ldots, J\,.$$

If this condition is satisfied then generically in any stock market, equilibrium $\langle \bar{y} + \eta \rangle = Z$. For any firm j, any alternative production $y^j \in Y^j$ can be priced by no-arbitrage since this output is a combination of the outputs of all firms (securities) which are already priced in the market. Thus if $y^j = \Sigma_{k=1}^J \alpha_k \bar{y}^k$ then the objective function (51) is defined by

$$\bar{\beta}^j y^j = \sum_{s=0}^{S} \bar{\beta}_s^j \sum_{k=1}^{J} \alpha_k \bar{y}_s^k = \sum_{k=1}^{J} \alpha_k \sum_{s=0}^{S} \bar{\beta}_s^j \bar{y}_s^k = \sum_{k=1}^{J} \alpha_k \bar{q}_k .$$

Thus if the technology sets and initial endowments (Y, η) *satisfy partial spanning then even if the markets are incomplete* $(J < S)$, *the firms' objective functions* (51) *are generically uniquely defined by the stock market.* Furthermore it can be shown that generically the shareholders unanimously approve the production decisions $\bar{y}$ of the firms.

Incomplete stock market $(J < S)$

When the condition of partial spanning is not satisfied, in any stock market equilibrium each firm j will typically have access to dividend streams D^j satisfying

$$D_1^j \not\in \langle \bar{D}_1 \rangle$$

by changing its production plan. An incomplete stock market equilibrium differs in two important respects from the complete and partial spanning stock market equilibria.

(i) *With an incomplete stock market the set of normalised solutions of* (52) *is an affine subspace of dimension* $S - J > 0$. *The firms can therefore not use market observations on prices and dividends* $(\bar{q}, \bar{D})$ *to determine their present value vectors* $\bar{\beta}^j$. *Some extra-market information must be used to determine* $\bar{\beta}^j$.

(ii) *Whatever* $\bar{\beta}^j$ *vector is chosen, generically for all shareholders* $\bar{\pi}^i \neq \bar{\beta}^j$. *Shareholders will thus disagree with the production plan* $\bar{y}^j$ *chosen by the firm.*

In an incomplete stock market the decision problem faced by the manager of a firm is essentially a public goods problem for its constituency of shareholders. In view of (ii) whenever a firm's technology set permits it to consider production plans which lie outside the current span of the markets, the firm's manager cannot expect to obtain unanimous support for his choice of production plan. A standard way of resolving a problem of public choice when unanimity cannot be expected is to resort to the *Hicks–Kaldor criterion*. Let us see if applying this criterion can lead to a resolution of the firm's decision problem.

Let $((\bar{x}, \bar{z}), (\bar{y}, \bar{\xi}), (\bar{p}, \bar{q}))$ be a stock market equilibrium. Suppose the manager of firm j envisions a change in the firm's production plan

$$\bar{y}^j \rightarrow \bar{y}^j + \mathrm{d}y^j \, .$$

This changed production plan alters the equity contract that the firm places on the market. Suppose all agents have competitive perceptions in the sense that

$$\mathrm{d}p = 0 \, , \qquad \mathrm{d}q_k = 0 \, , \quad k \neq j \, .$$

The basic premise of the Hicks–Kaldor criterion is that the marginal utility of one unit of good 1 at date 0 is to be assigned the same value for all share holders. The idea that the gains of the winners (resulting from the change $\mathrm{d}y^j$) can be used to compensate the losers by means of a system of transfers at date 0 leads to the following criterion: *the change* $\mathrm{d}y^j \in T_{\bar{y}^j} \partial Y^j$ *is to be accepted (rejected) if* $\sum_{i=1}^{I} (1/\bar{\lambda}_0^i) \, \mathrm{d}u^i > 0$ (≤ 0).

Let $(\mathrm{d}q_j)^i$ denote agent i's perception of the change in the security price arising from the changed dividend stream $\mathrm{d}y^j$. From agent i's budget constraints

$$\bar{p} \,\square\, \mathrm{d}x^i = \begin{bmatrix} (\mathrm{d}q_j)^i(\zeta_j^i - \bar{z}_j^i) + \bar{p}_0 \, \mathrm{d}y_0^j \, \bar{z}_j^i + (\bar{p}_0(\bar{y}_0 + \eta_0) - \bar{q}) \, \mathrm{d}z^i \\ (\bar{p}_1 \,\square\, \mathrm{d}y_1^j)\bar{z}^i + \bar{p}_1(\bar{y}_1 + \eta_1) \, \mathrm{d}z^i \end{bmatrix} .$$

Since $(1/\bar{\lambda}_0^i) \, \mathrm{d}u^i = (1/\bar{\lambda}_0^i)(D_{\bar{x}^i} u^i) \, \mathrm{d}x^i = (\bar{\pi}^i \,\square\, \bar{p}) \, \mathrm{d}x^i = \bar{\pi}^i \cdot (\bar{p} \,\square\, \mathrm{d}x^i)$ and since $\bar{q} = \bar{\pi}^i \cdot (\bar{p} \,\square\, (\bar{y} + \eta))$ we obtain

$$\left(\frac{1}{\bar{\lambda}_0^i}\right) \mathrm{d}u^i = (\mathrm{d}q_j)^i(\zeta_j^i - \bar{z}_j^i) + \bar{\pi}^i \cdot (\bar{p} \,\square\, \mathrm{d}y^j)\bar{z}_j^i \, , \quad i = 1, \ldots, I \, .$$

Suppose agents' perceptions are competitive in the sense that the security price is assumed to adjust to the changed dividend stream, the present value of the changed dividend stream being evaluated with agent i's personal present value vector $\bar{\pi}^i$, then

$$(\mathrm{d}q_j)^i - \bar{\pi}^i(\bar{p} \,\square\, \mathrm{d}y^j) = 0 \, , \quad i = 1, \ldots, I$$

so that the Hicks–Kaldor sum reduces to

$$\sum_{i=1}^{I} \left(\frac{1}{\bar{\lambda}_0^i}\right) \mathrm{d}u^i = \sum_{i=1}^{I} (\mathrm{d}q_j)^i \zeta_j^i = \sum_{i=1}^{I} \zeta_j^i \bar{\pi}^i \cdot (\bar{p} \,\square\, \mathrm{d}y^j) \, .$$

This criterion, which was proposed by Grossman and Hart (1979), is equivalent to the firm having a criterion of the form (51) with present value vector $\bar{\beta}^j$ defined by

$$\bar{\beta}^j = \sum_{i=1}^{I} \zeta_j^i \bar{\pi}^i\,, \quad j = 1, \ldots, J\,. \tag{56}$$

We can argue that this case would seem natural *if the shareholders are perceived as monitoring the manager's production decision "before" the stock market meets.*

On the other hand *if the shareholders are perceived as monitoring the manager's decision "after" the stock market meets* then

$$(\mathrm{d}q_j)^i = 0\,, \quad i = 1, \ldots, I$$

since with no further security trading there can be no change in the equity's price. In this case the Hicks–Kaldor sum reduces to

$$\sum_{i=1}^{I} \left(\frac{1}{\bar{\lambda}_0^i}\right) \mathrm{d}u^i = \sum_{i=1}^{I} \bar{z}_j^i \bar{\pi}^i \cdot (\bar{p} \,\square\, \mathrm{d}y^j)\,.$$

This criterion, which was proposed by Drèze (1974), is equivalent to the firm having a present value vector $\bar{\beta}^j$ given by

$$\bar{\beta}^j = \sum_{i=1}^{I} \bar{z}_j^i \bar{\pi}^i\,, \quad j = 1, \ldots, J\,. \tag{57}$$

Since it is not economically meaningful to give negative weight to agents holding a short position in firm j, Drèze suggested that all agents be restricted to holding long positions in the equity contracts.

Definition 14. A stock market equilibrium $((\bar{x}, \bar{z}), (\bar{y}, \bar{\xi}), (\bar{p}, \bar{q}))$ in which $\bar{\beta}^j = \Sigma_{i=1}^{I} \zeta_j^i \bar{\pi}^i, j = 1, \ldots, J$ is called a *Grossman–Hart equilibrium*. If firms do not hold equity portfolios ($\xi \equiv 0$), if consumers are restricted to non-negative equity portfolios ($z \in \mathbb{R}_+^{JI}$) and if $\bar{\beta}^j = \Sigma_{i=1}^{I} \bar{z}_j^i \bar{\pi}^i$, $j = 1, \ldots, J$ then a stock market equilibrium $((\bar{x}, \bar{z}), (\bar{y}, 0), (\bar{p}, \bar{q}))$ is called a *Drèze equilibrium*.

A Grossman–Hart equilibrium does not always exist. We have however the following result ensuring the consistency of this equilibrium concept.

Theorem 23. *Under Assumptions* 1 *and* 2 *there is a generic set $\Omega' \subset \Omega$ such that for all $(\omega, \eta) \in \Omega'$ there is a positive finite number of Grossman–Hart equilibria, each of which is locally a smooth function of (ω, η).*

Remark. A similar result can be established for Drèze equilibria. The Grossman–Hart concept has the important property that it extends naturally to framework of a *stochastic production economy*. The reader can readily spell out the details using the framework developed in Section 2.4. As pointed out by Grossman and Hart (1979), the Drèze concept encounters problems in the multiperiod case.

For the two firm criteria defined by (56) and (57) the "extra-market information" referred to in (i) that is required to obtain a well-defined criterion (i.e. a determinate $\bar{\beta}^j$ vector for each firm) would have to be obtained from the *shareholders* of the firm. Both these criteria can thus be viewed as formalisations of the idea that *ownership implies control*. The competitive assumption that underlies the model however precludes shareholders from acting strategically in their purchase of firms securities. This is clearly a weakness of the model since there are important situations where it is most unrealistic to assume that shareholders do not take into account the effect that their security purchases will have on firms' production decisions.

Market value maximisation

All the preceding analysis has been based on the *competitive assumption* that consumers and firms do not take into account the effect of their commodity-portfolio decisions on the market prices (p, q). For consumers this seems a reasonable approximation since they are normally one of many buyers (sellers) on the commodity and security markets (modulo the proviso made above). For firms on the commodity markets where they are one of many buyers and sellers this may also yield a useful first approximation. But for firms on the equity markets the situation is quite different: since the firm is the sole supplier of its equity contract it can be argued that the firm should act strategically with regard to the equity contract that it markets.

We are thus led to a monopolistic concept of equilibrium which for simplicity we express for the case of a one good economy $(L = 1)$ in which $\xi \equiv 0$.

Let $((\bar{x}, \bar{z}), \bar{q}; y)$ denote a stock market equilibrium with fixed producer plans y for the economy $\mathscr{E}(u, Y, \xi; \omega, \eta)$ and let $(\bar{y}, \bar{\omega}, \bar{\eta})$ be a regular parameter value. Laying aside the difficulties posed by multiple equilibria (and proceeding informally), for each y in a neighborhood of $\bar{y}$ the market value $\bar{q}(y)$ is well defined. In order for $((\bar{x}, \bar{z}), \bar{q}(\cdot); \bar{y})$ to be a *market value maximising equilibrium* each firm's production plan $\bar{y}^j$ must maximise its market value $\bar{q}^j(y^j, \bar{y}_{-j})$ given the production plans $\bar{y}_{-j} = (\bar{y}^k)_{k \neq j}$ for all other firms. To our knowledge there is at present no theorem asserting the existence of such an equilibrium. Hart (1979) however has studied this concept and has argued that under assumptions ensuring that each firm is "negligible", shareholders will agree that market value maximisation is in their best interests.

Is it possible to define a competitive version of the above concept of equilibrium? Suppose firms make conjectures about the way the market values a dividend stream, i.e. they conjecture a present value vector $\beta^j \in \mathbb{R}^{S+1}$. If we require that firms have common conjectures then we are led to the following concept of equilibrium.

Definition 15. A stock market equilibrium in which $\bar{\beta}^j = \bar{\beta}$, $j = 1, \ldots, J$ is called a *competitive market value maximising equilibrium.*

Such equilibria exist generically, but are indeterminate.

Theorem 24. *If Assumptions* 1 *and* 2 *hold and if* $J < S$ *then there is a generic set* $\Omega' \subset \Omega$ *such that for all* $(\omega, \eta) \in \Omega'$ *the set of competitive market value maximising equilibrium allocations contains a set homeomorphic to a ball in* $\mathbb{R}^{S-J}$.

Remark. It is clear from the analysis of this section that the problem of formulating a consistent and satisfactory concept of equilibrium presents much greater challenges for a GEI production economy than for the GEI exchange economy analysed in Sections 2 and 3. In the section that follows we shall examine the efficiency properties of these GEI exchange and production equilibria.

References

The classic paper on stock market equilibrium with incomplete markets is due to Diamond (1967), who also proves a version of the Modigliani–Miller theorem. The concept of partial spanning was introduced by Ekern and Wilson (1974) and further analysed by Radner (1974). It was Drèze (1974) who first understood the public goods nature of the firm's decision problem when partial spanning no longer holds. He introduced the objective function defined by (57) and analysed the resulting concept of equilibrium (including existence). Grossman and Hart (1979) presented a systematic critique of the concept of a stock market equilibrium and introduced the criterion (56) which seems to offer a wider domain of applicability. A classic general equilibrium version of the Modigliani–Miller theorem was presented by Stiglitz (1974): it was extended to a wider array of assets and to the case of inter-firm shareholdings by Duffie and Shafer (1986b) and DeMarzo (1988a). Theorem 22 is due to Duffie and Shafer (1986b). Theorem 23 is a special case of a more general result on the existence of a monetary equilibrium (i.e. an extension of the equilibrium of Section 3.2) for a production economy given by Magill and Quinzii (1989). Theorem 24 is due to Duffie and Shafer (1986b).

5. Efficiency properties of markets

Under what conditions does a *market system* function satisfactorily? This question is given a precise answer by two basic theorems of GE under the assumptions of convexity of preferences and technology sets, absence of externalities, common information and price taking behavior. The Existence theorem and the First Welfare theorem assert that a GE market system "works" in the sense that for all economies

(i) *it has a solution* (existence),
(ii) *the resulting solution is unimprovable* (Pareto optimal).

A GEI market system works at least in the preliminary sense that (i) holds generically (Theorem 9); (ii) however is generically not satisfied; GEI allocations are not Pareto optimal (Theorem 10). Should this inefficiency property of GEI markets lead us to conclude that the GEI system is inadequate for solving the problem of resource allocation? It is clear that whenever a system of markets is incomplete the criterion of Pareto optimality is too demanding. Is there a less demanding criterion which respects the intrinsic incompleteness of the markets, with respect to which the GEI system can be judged as satisfactory? We will consider this question within the framework of an exchange economy and then within the framework of a production (stock market) economy.

5.1. *Inefficiency in exchange*

To simplify the analysis we restrict attention to the two-period model and assume that the financial contracts consist solely of the class that we have called *real assets*. We have therefore as the initial data an exchange economy $\mathscr{E}(u, \omega; A)$ with real asset structure A. We are interested in analysing the efficiency properties of the GEI equilibria of this economy. To this end it is important to understand the following property of a real asset contract. The purchaser (seller) of one unit of real asset j can take (make) delivery in state s at date 1 in one of two forms:

(a) as the *bundle of goods* $A_s^j \in \mathbb{R}^L$ or
(b) as the *income value* $p_s \cdot A_s^j \in \mathbb{R}$ of this bundle of goods.

If the commodity bundle A_s^j can be sold (purchased) freely on the spot markets at the price p_s or if there is only one good ($L = 1$), then each agent is indifferent between these two modes of delivery. In a GEI equilibrium, in view of the way they enter agents' budget sets, real assets are taken as financial instruments for redistributing income across the states: agents are thus viewed as accepting (making) delivery in the manner (b). If for some reason the bundles A_s^j cannot always be freely traded on the spot markets then the manner (a) of accepting (making) delivery makes the real asset contract much more restrictive. In the analysis that follows we shall see that real assets have a (weak) constrained efficiency property if and only if they are interpreted as goods delivery contracts (a). When agent i buys the portfolio $z^i = (z_1^i, \ldots, z_J^i)$ of the real assets then under the bundle of goods mode of delivery (a) he receives the bundle of goods

$$\sum_{j=1}^{J} A^j z_j^i = Az^i \in \mathbb{R}^{LS}$$

at date 1. We are thus led to the following definition. (*Notation*: for $\bar{x}^i \in \mathbb{R}^n$, $x_\sigma^i \in \mathbb{R}^L$ let $[(\bar{x}_s^i)_{s \neq \sigma}, x_\sigma^i] \in \mathbb{R}^n$ denote the vector which coincides with $\bar{x}^i$ except for the component σ which is x_σ^i.)

Definition 16. Let $\mathscr{E}_w(u)$ denote an economy with utility functions $u = (u^1, \ldots, u^I)$ and total resources $w = (w_0, w_1, \ldots, w_S) \in \mathbb{R}_{++}^{L(S+1)}$. An allocation $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^I)$ is *weakly constrained* (*WC*) *efficient* for $\mathscr{E}_w(u)$ if (i) $\Sigma_{i=1}^I \bar{x}_s^i = w_s$, $s = 0, 1, \ldots, S$; (ii) for each state $\sigma = 0, 1, \ldots, S$ there does not exist an alternative allocation $x_\sigma = (x_\sigma^1, \ldots, x_\sigma^I)$ satisfying $\Sigma_{i=1}^I x_\sigma^i = w_\sigma$ such that

$$u^i([(\bar{x}_s^i)_{s \neq \sigma}, x_\sigma^i]) > u^i(\bar{x}^i), \quad i = 1, \ldots, I;$$

(iii) there do not exist transfers of goods at date 0, $\tau_0 = (\tau_0^1, \ldots, \tau_0^I) \in \mathbb{R}^{LI}$ with $\Sigma_{i=1}^I \tau_0^i = 0$ and changes in the portfolios $\xi = (\xi^1, \ldots, \xi^I) \in \mathbb{R}^{JI}$ with $\Sigma_{i=1}^I \xi^i = 0$ such that

$$u^i(\bar{x}_0^i + \tau_0^i, \bar{x}_1^i + A\xi^i) > u^i(\bar{x}^i), \quad i = 1, \ldots, I.$$

The following result due to Grossman (1977) shows why the concept of WC efficiency is of interest.

Theorem 25. (i) *If* $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}))$ *is a GEI equilibrium then the allocation* $\bar{x}$ *is weakly constrained efficient.*

(ii) *If the allocation* $\bar{x}$ *is weakly constrained efficient for the economy* $\mathscr{E}_w(u)$ *and if* $\bar{x} \in \mathbb{R}_{++}^{nI}$ *then there exist a distribution of the goods* $(\omega^1, \ldots, \omega^I)$ *with* $\Sigma_{i=1}^I \omega^i = w$, *portfolios* $\bar{z} = (\bar{z}^1, \ldots, \bar{z}^I)$ *and prices* $(\bar{p}, \bar{q})$ *such that* $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}))$ *is a GEI equilibrium.*

Remark. Theorem 25 gives a characterisation of the efficiency properties satisfied by GEI equilibria. For the case of an exchange economy with one good ($L = 1$), it provides a natural extension of the two Welfare theorems to the case of incomplete markets when the concept of weak constrained efficiency is used instead of Pareto efficiency. However when there are two or more goods ($L \geqslant 2$) the theorem does not have such a natural interpretation, since it does not resolve the basic question of whether or not GEI markets are "efficient".

First, the WC concept does not deal properly with the case where the asset structure A is *regular* (the case of potentially complete markets) for in this case the GEI equilibrium allocations are generically (fully) Pareto efficient (Theorem 3). The definition of WC coincides with (full) Pareto efficiency only when A has column rank SL; when $L \geqslant 2$ this requires $J \geqslant SL$ rather than $J \geqslant S$ for regularity. Furthermore by insisting on a concept of efficiency which holds not generically but for *all* economies (i.e. all $\omega \in \Omega$) one is forced to make the concept sufficiently weak so that it applies to economies $\bar{\omega} \in \Omega$ which have equilibria that can be Pareto ordered [as in Example 5 of Hart (1975)].

Second, when the asset structure A is *not regular* (so that the markets are incomplete) we should stop looking for efficiency properties which hold for all equilibria of generic economies. Hart (1975) has given a robust example of an economy in which A is not regular in which there are equilibria which can be Pareto ordered. For such economies, even if one has a notion of efficiency which is only required to hold generically it must be sufficiently weak to permit the Pareto dominated equilibrium to be efficient. When markets are incomplete the focus should shift towards better understanding why GEI equilibria are inefficient.

To understand the reason why financial markets are inefficient it is helpful to examine the concept of WC efficiency in Definition 16 more carefully. Consider condition (iii): when the portfolios are reallocated $\bar{z} \rightarrow \bar{z} + \xi$ agents are not allowed to retrade on the spot markets; they must accept physical delivery of the entire bundle of date 1 goods implied by their changed asset position $A\xi^i$. Thus while in the equilibrium the assets are treated as instruments for allocating income, for the reallocation they are treated as instruments for delivering bundles of goods. The reason is clear: *if as a result of the portfolio changes, agents are permitted to retrade on the spot markets, then spot prices will change. This spillover effect from financial markets to the spot markets is precisely the effect that the next concept of constrained efficiency seeks to capture.* Note of course that in an economy with only one good ($L = 1$) there is no spillover effect to consider since there are no spot markets (spot prices).

In studying a concept of efficiency it is useful to introduce the idea of a fictional *planner*. The planner is viewed as having access to certain "feasible allocations": if by choosing one of these he can make agents better off then we say that the equilibrium allocation is inefficient. The problem is thus reduced to defining the "feasible allocations": choosing the standard set leads to the concept of *Pareto optimality* – but with incomplete markets this concept is irrelevant: we are giving the planner much more freedom to allocate resources across states than is provided by the system of spot and financial markets. For an economy with two or more goods the appropriate concept has been introduced by Stiglitz (1982) and extended to the GEI model by Geanakoplos and Polemarchakis (1986). The key idea is to subject the planner to constraints which mimic those implicit in the system of financial markets. The planner can thus choose a pair (γ^i, z^i) consisting of a *fee* γ^i (payable at date 0) and a *portfolio* z^i for each agent $i = 1, \ldots, I$. The consumption allocation $x = (x^1, \ldots, x^I)$ is then determined through spot markets at an appropriate market clearing price (p). Let $(\gamma, z) = (\gamma^1, \ldots, \gamma^I, z^1, \ldots, z^I)$, then we define the feasible plans $((\gamma, z), (x, p))$ as follows.

Definition 17. A plan $((\bar{\gamma}, \bar{z}), (\bar{x}, \bar{p}))$ is *constrained feasible* for the exchange economy $\mathscr{E}(u, \omega; A)$ if

(i) $\Sigma_{i=1}^{I} \bar{\gamma}^i = 0$
(ii) $\Sigma_{i=1}^{I} \bar{z}^i = 0$
(iii) $(\bar{x}, \bar{p})$ satisfy $\Sigma_{i=1}^{I} (\bar{x}^i - \omega^i) = 0$ and for $i = 1, \ldots, I$

$$\bar{x}^i = \arg\max u^i(x^i) \text{ subject to}$$

$$\bar{p}_0(x_0^i - \omega_0^i) = \gamma^i \bar{p}_0 e_{01}, \quad e_{01} = (1, 0, \ldots, 0) \in \mathbb{R}^L,$$

$$\bar{p}_1 \square (x_1^i - \omega_1^i) = \bar{p}_1 A \bar{z}^i .$$

A plan $((\bar{\gamma}, \bar{z}), (\bar{x}, \bar{p}))$ is *constrained efficient* if it is constrained feasible and there does not exist a constrained feasible plan $((\gamma, z), (x, p))$ such that $u^i(x^i) > u^i(\bar{x}^i)$, $i = 1, \ldots, I$.

Remark. For convenience we assume that the fee is paid in units of good 1. If we define the *virtual* endowments

$$\underset{\sim}{\omega}^i = (\omega_0^i - \bar{\gamma}^i e_{01}, \omega_1^i + A\bar{z}^i), \quad i = 1, \ldots, I,$$

then in (iii), $(\bar{x}, \bar{p})$ is an equilibrium of the (virtual) exchange economy $\mathscr{E}(u, \underset{\sim}{\omega})$. Note also that if $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}))$ is a GEI equilibrium, then $((\bar{\gamma}, \bar{z}), (\bar{x}, \bar{p}))$ with $\bar{\gamma} = \bar{q}(\bar{z}^1, \ldots, \bar{z}^I)$, is a constrained feasible plan.

In the one good case constrained efficiency and weak constrained efficiency are essentially equivalent concepts. However when there are two or more goods they are quite different. In Definition 16 real assets are viewed as *goods* delivery assets: in Definition 17 they are viewed as *income* delivery assets. Do the price effects present in the latter definition create distortions which make it impossible for a price taking equilibrium like a GEI equilibrium to be constrained efficient? For a marginal change in the consumers' portfolios (z) will change demand and supply on the spot markets and hence the relative prices. If markets are incomplete agents evaluations of rates of substitution for income across the states are different and then such relative price changes may have an effect on welfare. A planner who takes into account these price changes thus has an additional instrument for redistributing income across the states which is not available to the more myopic competitive system. This key intuition was formalized by Stiglitz (1982) in the context of a particular example: he stressed however the possibility that this is a general phenomenon. That this is indeed the case was confirmed by Geanakoplos and Polemarchakis (1986).

They consider a numeraire asset exchange economy (Example 2) in which there is no consumption at date 0: it will be convenient to translate their result into the context of the standard model we have been considering, in which there is consumption at date 0. The following assumption is then useful.

Assumption 3 (*separability*). There exist differentiably strictly concave utility functions $u_0^i : \mathbb{R}_+^L \to \mathbb{R}$, $u_1^i : \mathbb{R}_+^{SL} \to \mathbb{R}$ such that

$$u^i(x) = u_0^i(x_0) + u_1^i(x_1) \,, \quad \forall\, x \in \mathbb{R}_+^n,\ i = 1, \ldots, I \,.$$

It is also convenient to express the parametrisation of preferences that is needed a little differently as follows. Let $\mathcal{U}$ denote the space of utility functions satisfying Assumption 1 endowed with the $\mathcal{C}^2$ compact open topology. For a numeraire asset economy we let

$$A = \begin{bmatrix} A_{11}^1 & \cdots & A_{11}^J \\ \vdots & & \vdots \\ A_{S1} & \cdots & A_{S1}^J \end{bmatrix} \in \mathbb{R}^{SJ}$$

denote the asset returns matrix. The Geanakoplos–Polemarchakis result can then be stated as follows.

Theorem 26. *Let $\mathscr{E}(u, \omega; A)$ be a numeraire asset economy in which the agents' characteristics (u, ω) satisfy Assumptions* 1 *and* 3. *If* (i) *the returns matrix A is in general position,* (ii) *there exist $z \in \mathbb{R}^J$ with $Az \geqslant 0$ and $z' \in \mathbb{R}^J$ with $A_s z' \neq 0$, $s = 1, \ldots, S$,* (iii) $2 \leqslant J < S$, (iv) $0 < 2(L-1) \leqslant I < S(L-1)$, *then there exists an open dense set $\Delta \subset \mathcal{U} \times \Omega$ such that for $(\bar{u}, \bar{\omega}) \in \Delta$ every GEI equilibrium $((\bar{x}, \bar{z}), (\bar{p}, \bar{z}))$ can be improved upon by a constrained feasible plan $((\gamma, z), (x, p))$* satisfying $\gamma^i = \bar{q} z^i$, $i = 1, \ldots, I$.

Remark. To bring out the striking feature of this theorem the following clarification is useful. We have seen that a GEI equilibrium yields a constrained feasible plan $((\bar{\gamma}, \bar{z}), (\bar{x}, \bar{p}))$ in which the fees charged to the agents satisfy

$\bar{\gamma}^i = \bar{q}\bar{z}^i$, $i = 1, \ldots, I$. In comparing such an equilibrium with any alternative constrained feasible plan $((\gamma, z), (x, p))$ it is useful to decompose the fee as

$$\gamma^i = \bar{q}z^i + \tau^i, \quad i = 1, \ldots, I, \tag{58}$$

referring to the component τ^i as a *transfer* to agent i. With the decomposition (58) the vector of transfers $\tau = (\tau^1, \ldots, \tau^I)$ must be chosen to lie in the *space of transfers*

$$\mathcal{T} = \left\{ t \in \mathbb{R}^I \;\middle|\; \sum_{i=1}^{I} t^i = 0 \right\}.$$

We say that the alternative plan $((\bar{q} \square z + \tau, z), (x, p))$ is constrained feasible *with* (*without*) *transfers* if $\tau \neq 0$ $(= 0)$. Theorem 26 asserts that *the welfare of each agent can be improved without resorting to transfers provided there are not too many agents* ($I < S(L-1)$). Since the welfare of the agents is changed by inducing changes in the $(S+1)(L-1)$ relative prices, it is clear that there must be a bound on the number of agents. Indeed Mas-Colell (1987) has given an example showing that Theorem 26 is not valid if the upper bound on I is removed.

If the planner is free to choose not only the portfolios (z) but also any vector of transfers $\tau \in \mathcal{T}$ then we are resorting to the *Hicks–Kaldor criterion*, namely the idea that welfare is improved if the gains of the winners are sufficient to compensate for the losses of the losers. In this case there are $I - 1$ additional control variables at the disposition of the planner and the welfare of an arbitrary number of agents can be improved after the payment of appropriate transfers. In the following theorem we do not restrict ourselves to numeraire assets, but consider rather the general class of real asset structures.

Theorem 27. *If $\mathscr{E}(u, \omega; A)$ be a real asset exchange economy in which agents characteristics (u, ω) satisfy Assumptions* 1 *and* 3. *If* (i) $0 < J < S$, (ii) $L > 1$, (iii) $I > (L-1)S$, *then there exists an open dense set $\Delta \subset \mathscr{U} \times \Omega \times \mathscr{A}$ such that for $(u, \omega, A) \in \Delta$ every GEI equilibrium allocation is constrained inefficient with transfers.*

Remark. If A is restricted to being a numeraire asset structure and if we assume $A \in \mathbb{R}^{SJ}$ is in general position then the genericity with respect to A can be omitted (i.e. $\Delta \subset \mathscr{U} \times \Omega$).

Proof. We decompose the proof into two parts. *Step* 1: derive the first order conditions for constrained efficiency. *Step* 2: show that there is an open dense set Δ such that these conditions are not satisfied at any equilibrium of an economy with parameters $(u, \omega, A) \in \Delta$.

Step 1: When the planner chooses a fee-portfolio pair (γ, z) for each of the I agents in the economy he in essence assigns a *virtual endowment*

$$\underset{\sim}{\omega}^i = (\omega_0^i - \gamma^i e_{01}, \omega_1^i + Az^i), \quad i = 1, \dots, I \tag{59}$$

to each of the I agents. The plan (γ, z) thus leads to the *virtual exchange economy* $\mathscr{E}(u, \underset{\sim}{\omega})$. An allocation is then induced as a (pure) *spot market equilibrium* of $\mathscr{E}(u, \underset{\sim}{\omega})$. Thus each agent's demand function is given by

$$x^i(p, p \,\square\, \underset{\sim}{\omega}^i) = \arg\max\{u^i(x^i) \mid x^i \in \mathscr{B}(p, p \,\square\, \underset{\sim}{\omega}^i)\}, \tag{60}$$

$$\mathscr{B}(p, m^i) = \{x^i \in \mathbb{R}_+^n \mid p \,\square\, x^i = m^i\}, \quad m^i \in \mathbb{R}_{++}^{S+1},$$

and a spot market equilibrium price $p \in \mathbb{R}_{++}^n$ is a solution of the system of excess demand equations

$$F(p, \underset{\sim}{\omega}) = \sum_{i=1}^{I} (x^i(p, p \,\square\, \underset{\sim}{\omega}^i) - \underset{\sim}{\omega}^i) = 0. \tag{61}$$

Since the budget sets $\mathscr{B}(p, p \,\square\, \underset{\sim}{\omega}^i)$ are independent of the levels of the spot prices, we normalise the spot prices so that $p_{s1} = 1$, $s = 0, 1, \dots, S$.

If $\bar{\underset{\sim}{\omega}}$ is a regular parameter value for the economy $\mathscr{E}(u, \underset{\sim}{\omega})$ then any equilibrium price can be written as a smooth function $p(\underset{\sim}{\omega})$ in a neighborhood of $\bar{\underset{\sim}{\omega}}$. A marginal change $(\mathrm{d}\gamma, \mathrm{d}z)$ in the planner's decision induces a marginal change in the virtual endowments

$$(\mathrm{d}\gamma, \mathrm{d}z) \to \mathrm{d}\underset{\sim}{\omega}^i = (\mathrm{d}\underset{\sim}{\omega}_0^i, \mathrm{d}\underset{\sim}{\omega}_1^i) = (-\mathrm{d}\gamma^i e_{01}, A\,\mathrm{d}z^i), \quad i = 1, \dots, I \tag{62}$$

where we assume that $(\mathrm{d}\gamma, \mathrm{d}z)$ satisfy

$$\sum_{i=1}^{I} \mathrm{d}\gamma^i = 0, \qquad \sum_{i=1}^{I} \mathrm{d}z^i = 0. \tag{63}$$

As the economy moves to a neighboring virtual exchange economy

$$\mathscr{E}(u, \bar{\underset{\sim}{\omega}}) \to \mathscr{E}(u, \bar{\underset{\sim}{\omega}} + \mathrm{d}\underset{\sim}{\omega}),$$

each equilibrium changes

$$(\bar{x}, \bar{p}) \to (\bar{x} + \mathrm{d}x, \bar{p} + \mathrm{d}p)$$

where $\bar{p} + \mathrm{d}p = p(\bar{\underset{\sim}{\omega}} + \mathrm{d}\underset{\sim}{\omega})$. Each consumer i adjusts consumption so as to satisfy the changed budget constraints

$$\bar{p} \,\square\, \mathrm{d}x^i = \bar{p} \,\square\, \mathrm{d}\underset{\sim}{\omega}^i - \mathrm{d}p \,\square\, (\bar{x}^i - \underset{\sim}{\bar{\omega}}^i)\,, \quad i = 1, \ldots, I\,. \tag{64}$$

The first-order conditions at the spot market equilibrium $(\bar{x}, \bar{p})$ imply that for each agent i there exist $\bar{\lambda}^i \in \mathbb{R}^{S+I}_{++}$ such that

$$D^i_{\bar{x}^i} u^i = \bar{\lambda}^i \,\square\, \bar{p}\,, \quad i = 1, \ldots, I\,. \tag{65}$$

The change in utility for agent i, $\mathrm{d}u^i = (D_{\bar{x}^i} u^i)\, \mathrm{d}x^i$ can thus be written as

$$\mathrm{d}u^i = \bar{\lambda}^i \cdot (\bar{p} \,\square\, \mathrm{d}\underset{\sim}{\omega}^i - \mathrm{d}p \,\square\, (\bar{x}^i - \underset{\sim}{\bar{\omega}}^i))\,, \quad i = 1, \ldots, I\,. \tag{66}$$

Let $\lambda^i = (\lambda^i_0, \ldots, \lambda^i_S)$ and let $\pi^i = (1, \pi^i_1) = (1, \lambda^i_1/\lambda^i_0, \ldots, \lambda^i_S/\lambda^i_0)$. Using (62), (64), (65) and Assumption 3 ($\mathrm{d}u^i = \mathrm{d}u^i_0 + \mathrm{d}u^i_1$), (66) can be written as

$$\begin{aligned} &\left(\frac{1}{\bar{\lambda}^i_0}\right) \mathrm{d}u^i_0 = -\mathrm{d}p_0(\bar{x}^i_0 - \underset{\sim}{\bar{\omega}}^i_0) - \mathrm{d}\gamma^i\,, \quad i = 1, \ldots, I\,, \\ &\left(\frac{1}{\bar{\lambda}^i_0}\right) \mathrm{d}u^i_1 = \bar{\pi}^i_1 V(\bar{p}_1, A)\, \mathrm{d}z^i - \bar{\pi}^i_1(\mathrm{d}p_1 \,\square\, (\bar{x}^i_1 - \bar{\omega}^i_1))\,. \end{aligned} \tag{67}$$

Let us again make use of Assumption 3. Suppose we can find a change in the portfolios $\mathrm{d}z$ such that

$$\sum_{i=1}^{I} \left(\frac{1}{\bar{\lambda}^i_0}\right) \mathrm{d}u^i_1 > 0\,. \tag{68}$$

Since the period 0 economy $\mathscr{E}(u_0, \bar{\omega}_0)$ is a self-contained GE economy we can generate any profile of date 0 utility changes $\mathrm{d}u_0 = (\mathrm{d}u^1_0, \ldots, \mathrm{d}u^I_0)$ satisfying $\Sigma^I_{i=1} (1/\bar{\lambda}^i_0)\, \mathrm{d}u^i_0 = 0$, by an appropriate choice of fees $\mathrm{d}\gamma = (\mathrm{d}\gamma^1, \ldots, \mathrm{d}\gamma^I)$. Thus if (68) holds then we can find $\mathrm{d}u_0$ such that

$$\mathrm{d}u^i = \mathrm{d}u^i_0 + \mathrm{d}u^i_1 > 0\,, \quad i = 1, \ldots, I\,.$$

Hence a necessary condition for constrained efficiency is that

$$\sum_{i=1}^{I} \left(\frac{1}{\bar{\lambda}^i_0}\right) \mathrm{d}u^i_1 = 0 \quad \textit{for all } \mathrm{d}z \in \mathbb{R}^{JI} \textit{ satisfying } \sum_{i=1}^{I} \mathrm{d}z^i = 0\,. \tag{69}$$

Assumption 3 implies that the virtual economy splits up into a date 0 and a date 1 economy

$$\mathscr{E}(u, \underset{\sim}{\omega}) = (\mathscr{E}(u_0, \underset{\sim}{\omega}_0), \mathscr{E}(u_1, \underset{\sim}{\omega}_1))$$

with excess demand equations (61) written as

$$F_0(p_0, \underset{\sim}{\omega}_0) = 0\,, \tag{70a}$$

$$F_1(p_1, \underset{\sim}{\omega}_1) = 0\,. \tag{70b}$$

The spot price function of a regular economy can thus be decomposed as

$$p(\underset{\sim}{\omega}) = (p_0(\underset{\sim}{\omega}_0), p_1(\underset{\sim}{\omega}_1))\,.$$

Consider a GEI equilibrium $((\bar{x}, \bar{z}, (\bar{p}, \bar{q}))$ for which the induced virtual endowment $\bar{\underset{\sim}{\omega}}$ is regular. The first-order conditions for agent i's portfolio choice imply $\underset{\sim}{\bar{\pi}}{}^i_1 V(\bar{p}_1, A) = \bar{q}$, $i = 1, \ldots, I$. Thus using (67), the necessary condition (69) becomes

$$\sum_{i=1}^{I} \left(\frac{1}{\bar{\lambda}^i_0}\right) \mathrm{d}u^i_1 = \sum_{i=1}^{I} \bar{\pi}^i_1(\mathrm{d}p_1 \,\square\, (\bar{x}^i_1 - \bar{\underset{\sim}{\omega}}{}^i_1)) = 0 \tag{71}$$

for all price changes $\mathrm{d}\hat{p}_1$ achievable by the planner, namely those satisfying

$$\mathrm{d}\hat{p}_1 = \sum_{i=1}^{I} \left[\frac{\partial \hat{p}_1}{\partial \underset{\sim}{\omega}{}^i_1}\right] \mathrm{d}\underset{\sim}{\omega}{}^i_1\,, \qquad \sum_{i=1}^{I} \mathrm{d}\underset{\sim}{\omega}{}^i_1 = 0\,, \quad \mathrm{d}\underset{\sim}{\omega}{}^i_1 \in \langle A \rangle,\ i = 1, \ldots, I \tag{72}$$

where $\hat{p}_1$ denotes the truncated system of prices obtained by omitting the price of good 1 in each state (recall $\mathrm{d}p_{s1} = 0$, $s = 1, \ldots, S$). Let $(\hat{x}^i, \hat{\omega}^i)$ and $\hat{F}_1$ denote the truncations of $(\bar{x}^i, \underset{\sim}{\omega}^i)$ and F_1. Define the $(L-1)S \times S$ matrix of differences in the income effects between agent α and agent I (truncated with respect to good 1 in each state)

$$Q_\alpha = \left[\frac{\partial \hat{x}^\alpha_1}{\partial m^\alpha_1} - \frac{\partial \hat{x}^I_1}{\partial m^I_1} \quad \cdots \quad \frac{\partial \hat{x}^\alpha_1}{\partial m^\alpha_S} - \frac{\partial \hat{x}^I_1}{\partial m^I_S}\right], \quad \alpha = 1, \ldots, I-1\,. \tag{73}$$

Differentiating the equilibrium equations (70b) and noting that $\partial \hat{F}_1 / \partial \hat{p}_1$ is non-singular at a regular value $\underset{\sim}{\omega}_1$ gives

$$\mathrm{d}\hat{p}_1 = -\left[\frac{\partial \hat{F}_1}{\partial \hat{p}_1}\right]^{-1} \sum_{\alpha=1}^{I-1} Q_\alpha V(\bar{p}_1)\, \mathrm{d}z^\alpha\,.$$

Thus if we define the weighted net trade vector (at the equilibrium)

$$\bar{\xi} = \sum_{i=1}^{I} \bar{\pi}^i_1 \,\square\, (\hat{x}^i_1 - \hat{\underset{\sim}{\omega}}{}^i_1) \in \mathbb{R}^{(L-1)S} \tag{74}$$

and let $(\cdot, \cdot)$ denote the inner product on $\mathbb{R}^{(L-1)S}$ then the efficiency condition (71), (72) reduces to the orthogonality condition

$$\left(-\left[\frac{\partial \hat{F}_1}{\partial \hat{p}_1}\right]^{-1} \sum_{\alpha=1}^{I-1} Q_\alpha V(\bar{p}_1)\,\mathrm{d}z^\alpha, \bar{\xi}\right) = 0\,, \quad \forall\, (\mathrm{d}z^1, \ldots, \mathrm{d}z^{I-1}) \in \mathbb{R}^{J(I-1)}\,. \tag{75}$$

If we can show that the $(L-1)S \times J(I-1)$ matrix

$$M = [Q_1 V(\bar{p}_1) \quad \cdots \quad Q_{I-1} V(\bar{p}_1)] \tag{76}$$

has rank$(L-1)S$ then the only vector $\bar{\xi}$ that can solve (75) is $\bar{\xi} = 0$. Since the markets are incomplete the vectors $(\bar{\pi}_1^i)_{i=1}^I$ are generically distinct. This can be used to show that generically $\bar{\xi}$ in (74) is not zero, so that the orthogonality condition (75) is generically not satisfied in a GEI equilibrium.

Step 2: To complete the proof it suffices to show that there is an open dense set $\Delta \subset \mathcal{U} \times \Omega \times \mathcal{A}$ such that for every $(u, \omega, A) \in \Delta$ there are a finite number of equilibria at each of which:

(a) *the induced virtual exchange economy* $\mathscr{E}(u_1, \underset{\sim}{\omega}_1^i)$ *is regular or equivalently* $|\partial \hat{F}_1 / \partial \hat{p}_1| \neq 0$;

(b) $\bar{\xi} \neq 0$;

(c) *for some column* $V^j(\bar{p}_1)$ of the matrix $V(\bar{p}_1)$ the vectors $\{Q_1 V^j(\bar{p}_1), \ldots, Q_{L(S-1)} V^j(\bar{p}_1)\}$ *are linearly independent.*

Since the negation of each of the statements (a), (b) and (c) can be written as an equation or system of equations which is added to the existing GEI equilibrium equations, to prove the result we need to show that in each case we obtain a system of equations $(h = 0)$ with more equations than unknowns which can be controlled $(h \pitchfork 0)$. A transversality argument then concludes the proof.

To prove (a) and (b) we fix $u \in \mathcal{U}$ and apply genericity arguments with respect to (ω, A). Thus we add the equation $|\partial \hat{F}_1 / \partial \hat{p}_1| = 0$ $(\bar{\xi} = 0)$ to the GEI equilibrium equations and show that the resulting system of equations can be controlled. The argument can be repeated for a countable dense collection of utility functions $\{u_n\}_{n=1}^\infty = \{u_n^1, \ldots, u_n^I\}_{n=1}^\infty$. Since the resulting property is open, we obtain an open dense set $\Delta' \subset \mathcal{U} \times \Omega \times \mathcal{A}$ at which (a) and (b) hold.

Showing that (c) is not true is equivalent to showing that the system of equations

$$\sum_{\alpha=1}^{L(S-1)} b_\alpha Q_\alpha V^j(\bar{p}_1) = 0 \quad \text{for some } b \in \mathscr{S}^{(L-1)S-1} \tag{77}$$

has a solution (where $\mathscr{S}^{(L-1)S-1}$ is the $[(L-1)S-1]$-dimensional unit sphere). Note that adjoining (77) to the equilibrium equations involves adding $(L-1) \times S-1$ new variables (b) and $(L-1)S$ equations. To prove that the system of equations (77) can be "controlled" without affecting the equilibrium equations, we note that if (77) is satisfied then $b \in \mathscr{S}^{(L-1)S-1}$ implies $b_i \neq 0$ for some i: "controlling" the equations then amounts to showing that it is possible to make an arbitrary infinitesimal change in the matrix $Q_i V^j(\bar{p}_1)$ by perturbing agent i's utility function u^i, the perturbation being effected in such a way that the gradient $D_{x_1^i} u_1^i(\bar{x}_1^i)$ remains unchanged, so that the equilibrium equations are unaffected. The date 1 matrix of income effects for the problem in (60) [the truncation of which appears in (73)] is given by

$$K^i = \left[\frac{\partial x_1^i}{\partial m_1^i} \right] = (u_{xx}^i)^{-1} [\bar{p}_1]^T ([\bar{p}_1](u_{xx}^i)^{-1} [\bar{p}_1]^T)^{-1} ,$$

$$[\bar{p}_1] = \begin{bmatrix} \bar{p}_1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & \bar{p}_S \end{bmatrix}$$

where $u_{xx}^i \equiv D^2_{x_1^i x_1^i} u^i(\bar{x}_1^i)$ denotes the matrix of second derivatives of u_1^i evaluated at $\bar{x}_1^i$. Let $u_x^i = D_{x_1^i} u_1^i(\bar{x}_1^i)$ denote the gradient of u_1^i at $\bar{x}_1^i$. The vector of utility functions $u \in \mathscr{U}$ can now be perturbed, $u \rightarrow \tilde{u} \in \mathscr{U}$ in such a way that u^α is unchanged for $\alpha \neq i$, and $\tilde{u}^i$ satisfies Assumption 1 and

$$\tilde{u}_x^i = u_x^i , \qquad (\tilde{u}_{xx}^i)^{-1} = (u_{xx}^i)^{-1} + C^i . \tag{78}$$

For such a change

$$\mathrm{d}Q_i V^j(\bar{p}_1) = \mathrm{d}\hat{K}^i V^j(\bar{p}_1) .$$

Pick any vector $c \in \mathbb{R}^{(L-1)S}$ with $\|c\| < \epsilon$ for ϵ sufficiently small. We need to show that there is a matrix C^i satisfying (78) such that $\mathrm{d}\hat{K}^i V^j(\bar{p}_1) = c$. We leave it to the reader to check that C^i can be chosen so that

$$[\bar{p}_1] C^i [\bar{p}_1]^T = 0 \quad \text{and} \quad \hat{C}^i [p_1]^T v = c$$

where $v = ([\bar{p}_1](u_{xx}^i)^{-1} [\bar{p}_1]^T)^{-1} V^j(\bar{p}_1) \neq 0$, showing that the system of equations (77) can be controlled.

This perturbation argument shows that there is a dense subset $\Delta \subset \Delta'$ such that property (c) holds. Since this property is open Δ can be taken to be an open dense set and the proof is complete.

Remark. A final comment on Theorem 27 is in order. If $J = 0$ (spot markets only) then $\mathrm{d}\underset{\sim}{\omega}_1^i = 0$ in (72) so that (71) holds; *a GEI market structure consisting* *only of spot markets is constrained efficient.* If $J \geqslant S$ then generically the asset structure A is regular so that generically $\bar{\pi}_1^i = \bar{\pi}_1$, $i = 1, \ldots, I$; thus not only does (71) hold but in addition we have Pareto optimality (recall Theorem 3). With only one good ($L = 1$) there are no price effects ($\mathrm{d}p_1 \equiv 0$) so that (71) always holds (see Theorem 25). The two special cases where (71) is satisfied, namely when there is no net-trade in equilibrium $\bar{x}_1 - \underset{\sim}{\omega}_1^i = 0$ (which arises if the initial endowment ω is Pareto optimal) or when the income matrices satisfy $Q_\alpha = 0$, $\alpha = 1, \ldots, I-1$ (which arises if the utility functions u_1^i are additively separable and identical homothetic within each state) are eliminated by the choice of the set Δ.

5.2. Inefficiency in production

In the previous section we have shown that in an exchange economy $\mathscr{E}(u, \omega; A)$, a knowledgeable planner can in principle exploit differences in agents' income effects in a GEI equilibrium to induce an improved allocation of the portfolios $z^1, \ldots, z^I$. In Section 4 we defined the concept of a stock market (GEI) equilibrium for a production economy. Are there new sources of inefficiency that arise when we consider a GEI equilibrium for a production economy? This question is important since the stock market is one of the major institutions on which society's risks in the activity of production are shared among agents in the economy and which influences the production decisions of firms. *If we recognise the fact that the structure of markets is incomplete, can the stock market be expected to perform its role of exchanging risks and allocating investment efficiently?*

To answer this question we need to extend the concept of constrained efficiency to a production economy. The planner is now viewed as choosing not only the fee and portfolio (γ^i, z^i) for each consumer but also the production plan y^j for each producer. The consumption allocation (x) is then determined as before through spot markets at an appropriate market clearing price (p).

Definition 18. A plan $((\bar{\gamma}, \bar{z}, \bar{y}), (\bar{x}, \bar{p}))$ is *constrained feasible* for the production economy $\mathscr{E}(u, Y, \zeta; \omega, \eta)$ (constrained feasible with no short sales) if

(i) $\Sigma_{i=1}^I \bar{\gamma}^i = 0$
(ii) $\Sigma_{i=1}^I (\bar{z}^i - \zeta^i) = 0$, $\bar{z} \in \mathbb{R}_+^{JI}$
(iii) $\bar{y}^j \in Y^j$, $j = 1, \ldots, J$
(iv) $(\bar{x}, \bar{p})$ satisfy $\Sigma_{i=1}^I (\bar{x}^i - \omega^i) = \Sigma_{j=1}^J (\bar{y}^j + \eta^j)$, and for $i = 1, \ldots, I$

$$\bar{x}^i = \arg\max u^i(x^i) \text{ subject to}$$

$$\bar{p}_0(\bar{x}_0^i - \omega_0^i) = \bar{p}_0(-\gamma^i e_{01} + (\bar{y}_0 + \eta_0)\bar{z}^i)$$

$$\bar{p}_1 \,\square\, (\bar{x}_1^i - \omega_1^i) = \bar{p}_1(\bar{y}_1 + \eta_1)\bar{z}^i .$$

A plan $((\bar{\gamma}, \bar{z}, \bar{y}), (\bar{x}, \bar{p}))$ is *constrained efficient* if it is constrained feasible and there does not exist a constrained feasible plan $((\gamma, z, y), (x, p))$ such that $u^i(x^i) > u^i(\bar{x}^i)$, $i = 1, \ldots, I$.

Remark. Diamond (1967) showed that *if there is only one good and if firms' production functions exhibit multiplicative uncertainty then every stock market equilibrium allocation is constrained efficient.* Drèze (1974) showed that in the one good case, if firms have general neoclassical production sets, then a necessary condition for constrained efficiency is that firm j uses the objective function

$$v^j(y^j) = \beta^j \cdot y^j \text{ with } \beta^j = \sum_{i=1}^{I} z^i_j \pi^i \, .$$

However as Drèze pointed out, since the constrained feasible plans of a production economy are *non-convex*, the necessary conditions are not sufficient. In fact he gave examples of stock market equilibria which are constrained inefficient when $L = 1$. Recently Geanakoplos, Magill, Quinzii and Drèze (1987) have shown that *if markets are incomplete and if there are two or more goods then generically every Drèze equilibrium allocation is constrained inefficient* (with no short sales). While their argument is carried out for the case of a Drèze equilibrium their construction indicates that the result will surely hold for any objective function implying price taking behavior on the part of the firms.

First-order conditions for efficiency

When a planner chooses a triple (γ, z, y) this is equivalent to choosing a *virtual* endowment of goods

$$\underset{\sim}{\omega}^i = (\omega^i_0 - \gamma^i e_{01} + (y_0 + \eta_0)z^i, \omega^i_1 + (y_1 + \eta_1)z^i) \, , \quad i = 1, \ldots, I \tag{78}$$

for each consumer. The consumption allocation and price (x, p) are then a spot market equilibrium of the virtual exchange economy $\mathscr{E}(u, \underset{\sim}{\omega})$ defined by equations (70). Let $((\bar{\gamma}, \bar{z}, \bar{y}), (\bar{x}, \bar{p}))$ be a constrained feasible plan for which the induced virtual endowment $\bar{\underset{\sim}{\omega}}$ defined by (78) is regular. A marginal change $(\mathrm{d}\gamma, \mathrm{d}z, \mathrm{d}y)$ in the planner's decision, which must satisfy the conditions of local feasibility

$$\sum_{i=1}^{I} \mathrm{d}\gamma^i = 0 \, , \qquad \sum_{i=1}^{I} \mathrm{d}z^i = 0 \, , \quad \mathrm{d}z^i_j \geq 0 \text{ if } \bar{z}^i_j = 0, \; \mathrm{d}y^j \in T_{\bar{y}^j} \, \partial Y^j, \quad j = 1, \ldots, J \tag{79}$$

induces a marginal change in the virtual endowment

$$(\mathrm{d}\gamma, \mathrm{d}z, \mathrm{d}y) \to \mathrm{d}\underset{\sim}{\omega}^i = -\mathrm{d}\gamma^i e_0 + (\bar{y} + \eta)\,\mathrm{d}z^i + \mathrm{d}y\,\bar{z}^i\,, \quad i = 1, \ldots, I \tag{80}$$

where $e_0 = (1, 0, \ldots, 0) \in \mathbb{R}^{LS}$. The resulting change in utility for each agent i is given by (66). By the same argument as in the previous section, under Assumption 3 a necessary condition for constrained efficiency is that

$$\sum_{i=1}^{I} (1/\bar{\lambda}_0^i)\,\mathrm{d}u = 0 \quad \text{for all } (\mathrm{d}\gamma, \mathrm{d}z\ \mathrm{d}y) \text{ given by (79)}\,.$$

Dividing (66) by $\bar{\lambda}_0^i$ and summing over i gives the *marginal change in social welfare* arising from the change $(\mathrm{d}\gamma, \mathrm{d}z, \mathrm{d}y)$

$$\sum_{i=1}^{I} \left(\frac{1}{\bar{\lambda}_0^i}\right) \mathrm{d}u^i = \sum_{i=1}^{I} \bar{\pi}^i \cdot (\bar{p} \,\square\, (\bar{y} + \bar{\eta}))\,\mathrm{d}z^i + \sum_{i=1}^{I} \bar{\pi}^i \cdot (\bar{p} \,\square\, \mathrm{d}y)\bar{z}^i$$
$$- \sum_{i=1}^{I} \bar{\pi}_1^i \cdot [\mathrm{d}p_1 \,\square\, (\bar{x}_1^i - \bar{\underset{\sim}{\omega}}_1^i)] \tag{81}$$

where $\Sigma_{i=1}^{I} \bar{\pi}_0^i\,\mathrm{d}p_0\,(\bar{x}_0^i - \bar{\underset{\sim}{\omega}}_0^i) = \mathrm{d}p_0\,\Sigma_{i=1}^{I}(\bar{x}_0^i - \bar{\underset{\sim}{\omega}}_0^i) = 0$ since spot markets clear in the virtual equilibrium. The first two terms in (81) represent the *direct income effect* of the change $(\mathrm{d}z, \mathrm{d}y)$, the last term is the *indirect price effect*.

Let $((\bar{x}, \bar{z}, \bar{y}), (\bar{p}, \bar{q}, \bar{\pi}))$ be a Drèze equilibrium (Definition 14). It can be shown that there is a generic set $\tilde{\Omega}$ such that for every economy $\mathscr{E}(u, Y, \zeta; \omega, \eta)$ with $(\omega, \eta) \in \tilde{\Omega}$, in each stock market equilibrium $((\bar{x}, \bar{z}, \bar{y}), (\bar{p}, \bar{q}, \bar{\pi}))$ the induced endowment ω defined by (78) with $\bar{\gamma} = (\zeta - \bar{z})\bar{q}$ is *regular* for the spot market economy $\mathscr{E}(u, \omega)$. Thus we can evaluate the marginal change in social welfare arising from a change $(\mathrm{d}\gamma, \mathrm{d}z, \mathrm{d}y)$ in the neighborhood of the stock market equilibrium. The first order conditions for the *portfolio choice* $\bar{z}^i$ of agent i imply that there exist $\bar{\rho}^{ij} \geqq 0$ such that

$$\bar{\pi}^i \cdot (\bar{p} \,\square\, (\bar{y}^j + \bar{\eta}^j)) = \bar{q}^j - \bar{\rho}^{ij}\,, \quad \rho^{ij} = 0 \text{ if } \bar{z}_j^i > 0\,.$$

Multiplying by $\mathrm{d}z_j^i$ and summing over i and j gives

$$\sum_{i=1}^{I} \bar{\pi}^i \cdot (\bar{p} \,\square\, (\bar{y} + \bar{\eta}))\,\mathrm{d}z^i = -\sum_{i=1}^{I}\sum_{j=1}^{J} \bar{\rho}^{ij}\,\mathrm{d}z_j^i = -\bar{\rho}\,\mathrm{d}z\,. \tag{82}$$

The first-order condition for *profit maximising* by firm j implies

$$\sum_{i=1}^{I} \bar{z}_j^i \bar{\pi}^i \cdot (\bar{p} \,\square\, \mathrm{d}y^j) = 0\,, \quad \forall\, \mathrm{d}y^j \in T_{\bar{y}^j}\,\partial Y^j\,. \tag{83}$$

Thus in a stock market equilibrium the marginal change in social welfare reduces to

$$\sum_{i=1}^{I}\left(\frac{1}{\bar{\lambda}_0^i}\right)\mathrm{d}u^i = -\bar{\rho}\,\mathrm{d}z - \sum_{i=1}^{I}\bar{\pi}_1^i\cdot[\mathrm{d}\tilde{p}_1 \,\square\, (\bar{x}_1^i - \underset{\sim}{\omega}_1^i)]\,.$$

The first term represents the cost of the no-short sales constraints $z_j^i \geqq 0$ and this term is zero in an equilibrium where $\bar{z}_j^i > 0$, for all i, j. *The second term is the effect on welfare of the induced changes in spot prices*; it is this term which is crucial to our analysis.

$\tilde{p}(\cdot)$ is a function of ω, which in turn is a function of the planner's action (γ, z, y). We indicate this by writing

$$(\gamma, z, y) \rightarrow \omega \rightarrow \tilde{p}\,.$$

Let $\partial\tilde{p}_1/\partial z_j^i$ and $\partial\tilde{p}_1/\partial y_{sl}^j$ denote the partial derivatives of the vector valued function $\tilde{p}_1$ with respect to z^{ij} and y_{sl}^j, respectively, written as column vectors. Thus

$$\frac{\partial\tilde{p}_1}{\partial y_1^j} = \left[\frac{\partial\tilde{p}_1}{\partial y_1^j} \quad \cdots \quad \frac{\partial\tilde{p}_1}{\partial y_S^j}\right]$$

is an $SL \times SL$ matrix. We thus have the following necessary conditions for constrained efficiency.

Proposition 28 (*efficiency conditions*). *Under Assumption* 3, *if a Drèze equilibrium* $((\bar{x}, \bar{z}, \bar{y}), (\bar{p}, \bar{q}, \bar{\pi}))$ *is constrained efficient then*

$$\text{(i)}\ \sum_{i=1}^{I}\bar{\pi}_1^i\cdot\left[\left(\frac{\partial\tilde{p}_1}{\partial z^{kj}} - \frac{\partial\tilde{p}_1}{\partial z^{k'j}}\right)\square\,(\bar{x}_1^i - \underset{\sim}{\omega}_1^i)\right] = 0\,, \quad j = 1, \ldots, J$$

for all k *and* k' *such that* $\bar{z}_j^k > 0$, $\bar{z}_j^{k'} > 0$,

$$\text{(ii)}\ \sum_{i=1}^{I}\bar{\pi}_1^i\cdot\left[\left(\frac{\partial\tilde{p}_1}{\partial y_1^j}\,\mathrm{d}y_1^j\right)\square\,(\bar{x}_1^i - \underset{\sim}{\omega}_1^i)\right] = 0\,, \quad \textit{for all}\ \mathrm{d}y^j \in T_{\bar{y}^j}\,\partial Y^j,$$

$$j = 1, \ldots, J\,.$$

Remark. We call (i) the *portfolio efficiency* condition and (ii) the *production efficiency* condition; (i) is the same as the efficiency condition (71), (72) of the previous section. Consider the following cases for which (i) and (ii) hold.

(a) *There is one good* ($L = 1$). (i) and (ii) hold since the price effects vanish. This explains the result of Diamond (1967), for with *multiplicative* uncertainty the set of feasible allocations is *convex* and the first-order conditions are sufficient. For the general (one good) case studied by Drèze (1974) the set of feasible allocations is *non-convex* and the necessary conditions are not sufficient. As mentioned above Drèze gave examples with $L = 1$ which are not constrained efficient.

(b) *All agents' present value vectors coincide*. This happens if the asset markets are complete and the portfolio constraints $z^i_j \geqslant 0$ are not binding.

(c) *There is zero net trade* ($\bar{x}^i_1 - \underset{\sim}{\bar{\omega}}^i_1 = 0$, $i = 1, \ldots, I$) *in the induced virtual equilibrium*. This occurs in the rather exceptional case where the induced virtual endowment is Pareto optimal.

Case (c) is clearly exceptional; (a) and (b) suggest the possibility that if there are at least two goods in each state ($L \geqslant 2$) and if markets are incomplete ($J < S$) then Drèze equilibria are generically constrained inefficient. That this is indeed the case was proved by Geanakoplos, Magill, Quinzii and Drèze (1987) who established the following result.

Theorem 29 (generic inefficiency of stock market equilibrium). *Let* $\mathscr{E}(u, Y, \zeta; \omega, \eta)$ *be a production economy satisfying Assumptions* 1–3. *If* (i) $I \geqslant 2$, (ii) $L \geqslant 2$, (iii) $I + J \leqslant S + 1$, (iv) $E^j = \mathbb{R}^n$ *for some firm* $j \in \{1, \ldots, J\}$, *then there exists a generic set* $\Omega^* \subset \Omega$ *such that for every* $(\omega, \eta) \in \Omega^*$ *each Drèze equilibrium allocation is constrained inefficient with transfers.*

Proof. The idea is to write the system of equations satisfied by an equilibrium and to show that any solution of these equations will generically not satisfy the efficiency conditions (ii) of Proposition 28. Modulo some technical preliminaries involved in showing that generically equilibria are of full rank and locally smooth functions of the parameters, the problem reduces to the analysis of the local behavior of the spot market equilibrium price $\tilde{p}(\underset{\sim}{\omega})$ of the induced virtual economy $\mathscr{E}(u, \underset{\sim}{\omega})$.

Let

$$\hat{Z}(p, \underset{\sim}{\omega}) = \sum_{i=1}^{I} (\tilde{x}^i(p, p \,\square\, \underset{\sim}{\omega}^i) - \underset{\sim}{\omega}^i) = 0$$

denote the system of equations defining $\tilde{p}(\underset{\sim}{\omega})$. The efficiency condition (ii) can be written as the inner product condition

$$\left(\left[\frac{\partial \tilde{p}_1}{\partial y^j_1} \right] \mathrm{d}y^j_1, \sum_{i=1}^{I} \bar{\pi}^i_1 \,\square\, (\bar{x}^i_1 - \underset{\sim}{\bar{\omega}}^i_1) \right) = 0 , \quad \forall\, \mathrm{d}y^j_1 \in \mathbb{R}^{LS} . \tag{84}$$

In view of the normalisation of spot prices, $\mathrm{d}\tilde{p}_{s1} = 0$, $s = 0, 1, \ldots, S$. Thus if we let

$$Q = \left[\frac{\partial \hat{\tilde{p}}_1}{\partial y^j_1} \right] : \mathbb{R}^{LS} \to \mathbb{R}^{(L-1)S} , \qquad Q^T : \mathbb{R}^{(L-1)S} \to \mathbb{R}^{LS} ,$$

$$u = \mathrm{d}y^j_1 , \qquad v = \sum_{i=1}^{I} \bar{\pi}^i_1 \,\square\, (\hat{\bar{x}}^i_1 - \underset{\sim}{\hat{\bar{\omega}}}^i_1)$$

then (84) reduces to

$$(Qu, v) = (u, Q^T v) = 0\,, \quad \forall u \in \mathbb{R}^{LS} \Leftrightarrow Q^T v = 0\,. \tag{85}$$

If we can show $\operatorname{rank}(Q^T) = (L-1)S$ then $v = 0$ is the only solution of (85). Since it can be shown that $\bar{\pi}^1, \ldots, \bar{\pi}^I$ are distinct, $v \neq 0$ generically.

We show rank $Q^T = (L-1)S$. In view of the separability assumption A3, the equation $\hat{Z}(p, \underset{\sim}{\omega}) = 0$ splits into a pair of equations $\hat{Z}_0(p_0, \underset{\sim}{\omega}_0) = 0$, $\hat{Z}_1(p_1, \underset{\sim}{\omega}_1^i) = 0$. Differentiating the latter at $\underset{\sim}{\omega} = \underset{\sim}{\bar{\omega}}$ and using the fact that $\tilde{p}_1(\underset{\sim}{\bar{\omega}}) = \bar{p}_1$ gives

$$\left[\frac{\partial \hat{Z}_1}{\partial \hat{p}_1}\right] \frac{\partial \hat{\bar{p}}_1}{\partial y^j_{sl}} = -\sum_{i=1}^{I} \bar{z}^{ij} \frac{\partial \hat{x}^i_1}{\partial m^i_s} \bar{p}_{sl} + e^{sl}_1\,, \quad s = 1, \ldots, S,\ l = 1, \ldots, L \tag{86}$$

where $e^{sl}_1 \in \mathbb{R}^{LS}$ is the vector whose component (s, l) is 1 and whose other components are zero. Since $\bar{\omega}$ is regular the matrix $B = [\partial \hat{Z}_1 / \partial \hat{p}_1]$ has rank $(L-1)S$, so that B^{-1} is well-defined. Thus (86) can be written as

$$Q = B^{-1} C$$

where the matrix C is given by (recall $\bar{p}_{s1} = 1$, $s = 1, \ldots, S$)

$$C = \begin{bmatrix} -\sum_{i=1}^{I} \bar{z}^i_j \dfrac{\partial \tilde{x}^i_{12}}{\partial m^i_1} & 1 - \bar{p}_{12} \sum_{i=1}^{I} \bar{z}^i_j \dfrac{\partial \tilde{x}^i_{12}}{\partial m^i_1} & \cdots & -\bar{p}_{SL} \sum_{i=1}^{I} \bar{z}^i_j \dfrac{\partial \tilde{x}^i_{12}}{\partial m^i_S} \\ \vdots & \vdots & & \vdots \\ -\sum_{i=1}^{I} \bar{z}^i_j \dfrac{\partial \tilde{x}^i_{SL}}{\partial m^i_1} & \bar{p}_{12} \sum_{i=1}^{I} \bar{z}^i_j \dfrac{\partial \tilde{x}^i_{SL}}{\partial m^i_1} & \cdots & 1 - \bar{p}_{SL} \sum_{i=1}^{I} \bar{z}^i_j \dfrac{\partial \tilde{x}^i_{SL}}{\partial m^i_S} \end{bmatrix}.$$

C is a matrix with SL columns and $(L-1)S$ rows. To prove that rank $Q = (L-1)S$ it suffices to show that rank $C = (L-1)S$. Let C_{sl} denote column (s, l) of C. If we subtract from each column C_{sl}, $l \geq 2$, the multiple $\bar{p}_{sl} C_{s1}$ of column C_{s1}, $s = 1, \ldots, S$ then we obtain a new matrix $D = [\cdots \ C_{s1} \ \ C_{s2} - \bar{p}_{s2} C_{s1} \ \cdots \ C_{sL} - \bar{p}_{sL} C_{s1} \ \cdots]$ with the same rank as C

$$D = \begin{bmatrix} -\sum_{i=1}^{I} \bar{z}^i_j \dfrac{\partial \tilde{x}^i_1}{\partial m^i_1} & I & -\sum_{i=1}^{I} \bar{z}^i_j \dfrac{\partial \tilde{x}^i_1}{\partial m^i_2} & 0 & \cdots & -\sum_{i=1}^{I} \bar{z}^i_j \dfrac{\partial \tilde{x}^i_1}{\partial m^i_S} & 0 \\ -\sum_{i=1}^{I} \bar{z}^i_j \dfrac{\partial \tilde{x}^i_2}{\partial m^i_1} & 0 & -\sum_{i=1}^{I} \bar{z}^i_j \dfrac{\partial \tilde{x}^i_2}{\partial m^i_2} & I & \cdots & -\sum_{i=1}^{I} \bar{z}^i_j \dfrac{\partial \tilde{x}^i_2}{\partial m^i_S} & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ -\sum_{i=1}^{I} \bar{z}^i_j \dfrac{\partial \tilde{x}^i_S}{\partial m^i_1} & 0 & -\sum_{i=1}^{I} \bar{z}^i_j \dfrac{\partial \tilde{x}^i_S}{\partial m^i_2} & 0 & \cdots & -\sum_{i=1}^{I} \bar{z}^i_j \dfrac{\partial \tilde{x}^i_S}{\partial m^i_S} & I \end{bmatrix}$$

where I is an $(L-1) \times (L-1)$ identity matrix. Clearly rank $D = (L-1)S$.

References

The need to formulate an appropriate concept of constrained efficiency in a model with incomplete markets was first recognised by Diamond (1967). Theorem 25 was proved by Grossman (1977). One of the earliest attempts to formalise the constrained inefficiency of GEI is due to Stiglitz (1982). The first fully articulated general equilibrium version of this result is due to Geanakoplos and Polemarchakis (1986). Theorem 29 is one of several inefficiency results obtained by Geanakoplos, Magill, Quinzii and Drèze (1987).

6. Concluding remarks

6.1. Interface with finance

A key idea that emerges from the GEI model is the interdependence between the real and financial sectors (markets) of the economy. In this survey we have concentrated on the qualitative properties of the GEI equilibrium allocations with real and nominal assets and with production. We have not stressed or explored the *qualitative properties of the asset prices* in such equilibria. Such an analysis leads us to the domain of finance. The one good model can be viewed as the basic equilibrium model of finance. Under the assumption of quadratic utilities it leads to the classical *capital asset pricing model* (CAPM). The no-arbitrage pricing formula

$$q_j = \sum_{s=1}^{S} \beta_s V_s^j , \quad j = 1, \ldots, J \tag{87}$$

is used as the point of departure for exploring the relation between asset prices and risk characteristics of the economy. In the CAPM model it leads to the famous *beta pricing formula* relating asset prices to their volatility relative to the market portfolio. The *principle of no-arbitrage* which underlies (87) forms the basis for a rich and varied analysis in the theory of finance – indeed it can be viewed as the central principle of modern finance. The Black–Scholes theory of derivative asset pricing is one of the most striking applications. For this and related issues in the theory of finance we refer the reader to Chapter 31.

6.2. Secondary assets

An important family of securities are the various secondary (derivative) assets, in particular options. Ross (1976) was the first to point out that introducing a sufficient number of option contracts might in principle provide a relatively low transaction cost way of achieving full spanning. Friesen (1979) has described in detail how to implement any complete markets equilibrium in a multiperiod model by constructing options on stocks. McManus (1986) has shown that in a real asset model with enough options to potentially span, equilibria exist generically.

When the financial markets (including options) are incomplete, the presence of options causes difficulties. It is useful to distinguish two cases. Those in which the striking prices are denominated in a numeraire commodity (or commodity bundle) and those in which the striking prices are denominated in nominal terms. In the first case Polemarchakis and Ku (1986) have exhibited a robust counterexample to existence of equilibrium using European options. In such a model, pseudo-equilibria always exist under standard assumptions; the difficulty is that it may not be possible generically to perturb the parameters of the model to force the pseudo-equilibria to become true equilibria. In a model which includes Polemarchakis–Ku type counterexamples Krasa (1987) has shown, that in a precise sense the "likelihood" of non-existence is smaller the more variable the aggregate endowment vector. Kahn and Krasa (1990) have exhibited robust examples of non-existence with American options. These counterexamples only require $L = 1$ and do not appear amenable to the analysis of Krasa (1987).

In the case where the options have nominal strike prices, Krasa and Werner (1989) have shown that equilibria always exist, and that the dimension of the set of equilibrium allocations may in some cases be equal to the number of states S, rather than $S - 1$ as in the nominal asset case of Section 3; thus, absolute price levels may matter as well as relative price levels across states. Even if there are enough assets (including options) to span all states, not only are complete market allocations achievable, but also many inefficient equilibrium allocations will be present. Kahn and Krasa (1990) have shown that with American options with nominal strike prices, even if there are enough options to potentially span, only inefficient equilibria may rise. The basic difficulty with American options is that an agent, with the choice of early exercise of the option, can affect the span of markets. Clearly much research remains to be done to properly integrate options into the GEI model.

6.3. Endogenous asset formation

This survey has concentrated on models in which the asset structure is taken as exogenous (with the exception of firms' equity contracts). It is essential to the continuing study of GEI models to obtain an understanding of the types of assets that are likely to be introduced and successfully traded. On the empirical side a useful survey of innovation in publicly traded security markets is given by Miller (1986). On the theoretical side there is a paucity of research; Silber (1981) and Duffie and Jackson (1988) have examined the problem of designing and marketing futures contracts; Allen and Gale (1988) have analysed a GEI model in which firms design optimal securities in the presence of transactions costs; Cuny (1989) has studied a strategic model of exchanges designing securities to maximize their brokerage fees.

Related to the issue of endogenous asset formation and whether or not it will lead to complete markets, is the problem of demonstrating that "almost" complete markets will lead to "almost" Pareto efficient allocations. Consider a model in which the number of states is countably infinite and there are only a finite number of securities. With less assets than states, the markets can in general never be complete, but one can ask whether equilibrium allocations approach Pareto efficiency as the number of traded securities approaches infinity. The initial research by Green and Spear (1987) has been generalized by Zame (1988). Zame has shown that in a precise sense generically, Pareto efficiency will fail in the limit as the number of securities approaches infinity if there is no provision for default.

6.4. Bankruptcy

Bankruptcy and default like limited liability can be viewed as contractual arrangements designed to augment the span of markets. When properly formulated they should play a central role in the GEI model. Although bankruptcy has been studied in the context of temporary equilibrium models [Green (1973), Stahl (1985a, b)], there have so far been only a few studies [Dubey, Geanakoplos and Shubik (1989), Dubey and Geanakoplos (1989b)] in the framework of the GEI model. The difficulty lies in satisfactorily modelling the phenomenon of default without breaking the basic GEI equilibrium concept in the process. By introducing the idea of default penalties and an equilibrium default rate on each contract the above authors have shown how the concept of a GEI equilibrium can be extended to include the phenomenon of default. An interesting result which makes use of this default-GEI equilibrium has recently been obtained by Zame (1989). He shows in a model with an infinite state space that equilibrium allocations are approximately Pareto efficient if the default penalty is large and the assets "almost span" all uncertainty, a conclusion which is false if default is not permitted.

6.5. Alternative approaches to firm behavior

In Section 4 we examined only a few approaches to modelling the problem of decision making by firms. An approach we did not discuss, but which is important in practice is to incorporate into the model the voting process by which corporate firms are typically controlled, that is shareholders vote for a board of directors and the board hires a manager. Initial studies of this corporate voting mechanism as a pure majority voting problem are those of Gevers (1974), Benninga and Muller (1979, 1981), Winter (1981) and Sadanand and Williamson (1988). Drèze (1985) has developed a model in which the board of directors has veto power and demonstrates existence of equilibrium. DeMarzo (1988b) has studied the relation between voting mechanisms and value maximization: he shows that any production equilibrium which satisfies the Pareto criterion with respect to the shareholders must be a stock market equilibrium (as in Definition 13), with each firms' β^j being a convex combination of shareholder present value vectors π^i (as in the Drèze and Grossman–Hart criteria).

An understanding of the phenomenon of takeovers must play an important role in a more complete theory of firm behavior. Hart (1977) has formulated a GEI model in which takeover bids are possible and examines whether the possibility of takeovers leads to value maximization. A related problem has been studied in the framework of incomplete contracts and asymmetric information by Grossman and Hart (1988) and Harris and Raviv (1987), who analyse the one share – one vote rule in the context of corporate takeovers.

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