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EXISTENCE OF EQUILIBRIUM WITH INCOMPLETE MARKETS*

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This paper proves a new *fixed-point theorem* for establishing generic existence of equilibrium with incomplete markets. The theorem can be stated in two equivalent forms: first as a fixed-point theorem on the Grassmanian of k-dimensional subspaces of R^n: second as a generalisation of the Borsuk–Ulam theorem. The proof relies on the methods of algebraic topology: geometrically existence follows from the global twisting in the fibres of a naturally induced vector bundle.

1. Introduction

This paper lays out a general approach to the problem of establishing the generic existence of equilibrium in an economy with incomplete markets. The general equilibrium model that we study is the model of simultaneous equilibrium on a system of real spot markets and financial markets for assets studied in Magill and Shafer (1990). The basic model admits a rich variety of financial market structures and includes the Arrow–Debreu model with complete contingent markets [Debreu (1959, ch. 7)] as a special case.

*The first draft of this paper appeared in February 1986. A revised draft was presented at the Conference 'Incomplete Markets and Asymmetric Information' at the University of Paris-Dauphine, June 1986. These first drafts contained proofs of theorems (B**, B*, B′, B) and were based on the reduction of the equilibrium existence problem to the vector bundle problem indicated in Lemma 3. The subsequent work of Hirsch et al. (1990) prompted the formulation in terms of theorems (A**, A*, A′, A). As Theorem D′ indicates these two sets of theorems lead to the 'same' underlying vector bundle problem so that existence depends on the same topological property (non-zero Euler class) of the 'same' vector bundle.

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In an elegant paper Duffie and Shafer (1985) have recently established existence of equilibrium in the same model using mod 2 degree theory and an extension of Balasko's (1975) existence argument based on the projection from the equilibrium manifold onto the space of endowment–asset structure pairs. Our work, which was undertaken simultaneously, was motivated by an attempt to find a *general fixed-point argument* for existence which would reduce to the standard Brouwer fixed-point approach in the case of complete markets. In the analysis which follows we prove a general *fixed-point theorem* (in several eqivalent forms) from which the generic existence of equilibrium follows directly. This existence theorem turns out to generalise a classical theorem of topology and contains the Brouwer Theorem as a special case.

The basic idea behind the generic existence proof can be broken down into three steps. The first is to introduce the definition of a *proper pseudo-equilibrium* and to show that the problem of proving the existence of equilibrium is equivalent to proving the existence of a proper pseudo-equilibrium. This concept first appears in section 6 in Magill and Shafer (1990). The basic step in arriving at a pseudo-equilibrium is an *arbitrage* argument which allows asset prices and asset trades to be eliminated so as to redefine each agent's budget constraint in such a way that the basic cause for discontinuity in the budget correspondence, which can lead to non-existence, is removed. The definition of a pseudo-equilibrium brings with it a *trial subspace* to which agents must confine their income transfers arising from trade on spot markets. More precisely, a *pseudo-equilibrium* consists of a price-subspace pair for which the price clears the spot market, while the (trial) subspace *includes* the actual subspace of income transfers achievable by trading in assets. In a *proper* pseudo-equilibrium the trial subspace is required to coincide with the actual subspace of income transfers.

The second and fundamental step consists in showing that a pseudo-equilibrium exists. This leads to a new type of *fixed-point problem*. In the basic model there are two time periods ($t=0,1$) and n states of nature at date 1. If there are k assets, then generically trading in assets allows agents to transfer income in a k-dimensional subspace L of the Euclidean space R^n. Thus the fixed-point problem is posed in the space of prices (the simplex) and k-dimensional subspaces of R^n.

There are several ways of parametrising the set of all k-dimensional subspaces of R^n. The first is to consider directly the *Grassmanian manifold* $G^{n,k}$ of k-dimensional subspaces of R^n: a point L in $G^{n,k}$ is simply a k-dimensional subspace: this is the approach of Duffie and Shafer (1985). We show that the problem of finding an equilibrium subspace reduces to a fixed-point problem on the Grassmanian which is similar to the standard fixed-point problem for an equilibrium price on the simplex. In fact the basic *Grassmanian fixed-point theorem* (A and A′) can be viewed as an analogue for the Grassmanian of Brouwer's theorem for the simplex in that it 'almost' asserts that every mapping of the Grassmanian into itself has a fixed-point (see Remark 6). A fixed-point of the induced *asset subspace return function* is then an *equilibrium subspace*.

The second approach uses the *Stiefel manifold* $O^{n,n-k}$ of $n-k$ orthonormal vectors in R^n: an element Q of $O^{n,n-k}$ is simply an $(n-k)\times n$ orthonormal matrix: this is the approach introduced in Magill and Shafer (1990). Since the span of the rows of Q is an $n-k$-dimensional subspace, its orthogonal complement is a k-dimensional subspace of R^n. In this way by letting Q vary over the Stiefel manifold $O^{n,n-k}$ we can generate all k-dimensional subspaces of R^n. In this case the map that needs to be zero to obtain an equilibrium subspace is the projection onto the orthogonal subspace of the columns of the basic asset returns matrix (see p. 51). Since the $n-k$ row vectors of Q can be rotated within the subspace that they span without altering this subspace, the map needs to satisfy an *equivariance property* under the action of the orthogonal group O_{n-k}. In this way the subspace equilibrium problem on the Stiefel manifold leads naturally to a theorem (B and B′) which generalises the classical Borsuk–Ulam Theorem [Guillemin and Pollack (1974, pp. 91–93), Spanier (1966, p. 266)].

In fact we show that Theorems A and B (A′ and B′) are equivalent: either can be used to derive the other. We do so by showing that *the subspace fixed-point problem is equivalent to the property* (*Theorem C′*) *that a certain canonical vector bundle admits no non-zero section.* The problem is thus reduced to the simplest topological property of a naturally induced vector bundle.

The topology of vector bundles is well-known and is the subject of an extensive theory known as the *theory of characteristic classes* [see Husemoller (1975), Milnor and Stasheff (1974), Osborn (1982)]. Intuitively the characteristic classes of a vector bundle are cohomology classes which measure the *global twisting of the fibers in the bundle. The top class is the Euler class: when the Euler class is not zero there is a twisting of the fiber as a point completely traverses the zero section which prevents any section from being pulled apart from the zero section: a non-zero Euler class is an obstruction to any attempt to pull a section apart from the zero section. It is this property that ensures the existence of an equilibrium subspace and forms the basis for the proofs of all the theorems.*

In this paper we establish the existence result using the methods of *algebraic topology*: we use *cohomology theory* to show that the Euler class of the canonical induced vector bundle is non-zero. An alternative geometric approach which is exploited in Hirsch et al. (1990) involves proving directly that the *self-intersection number* of the zero section is non-zero. Thus in the latter paper cohomology theory is replaced by *intersection theory*.

Having established the existence of a pseudo-equilibrium, the third step is the most straightforward and reduces to an application of Transversality

Theory: it is shown that generically in the space of endowments and asset structure pairs every pseudo-equilibrium is a *proper* pseudo-equilibrium.

The plan of the paper is as follows. In section 2 we outline the basic economic model of equilibrium, reducing the problem of establishing the existence of equilibrium to the problem of proving the existence of a proper pseudo-equilibrium. In section 3 we show that the existence of a pseudo-equilibrium can be formulated as a fixed-point problem either on a Grassmanian or a Stiefel manifold and we state the fixed-point theorems (A and B) which imply existence of a pseudo-equilibrium. Section 4 studies the equilibrium subspace problem and shows how it can be reduced to a vector bundle problem. Sections 5 and 6 prove the basic fixed-point theorems.

2. Reduction to pseudo-equilibrium

In this section we will lay out the basic general equilibrium model of an economy with real and financial markets in which the asset structure is incomplete.[1] We introduce the concept of *equilibrium* for such an economy: in this original form the concept is difficult to work with. By a sequence of steps we transform this concept into an equivalent concept of equilibrium which we call a *proper pseudo-equilibrium.* This leads us directly to the general fixed-point formulation of equilibrium that we seek.

Consider an economy over two time periods ($t=0,1$). To reflect uncertainty about the future let there be n possible states ($s=1,\dots,n$) that can occur at date 1: at $t=0$ it is not known which state will occur and at $t=1$ 'nature' selects some state s. For notational convenience we can let $t=0$ denote state $s=0$. There are m agents ($i=1,\dots,m$) and l goods ($h=1,\dots,l$) in each state $s=0,\dots,n$; we let $r=l(1+n)$ denote the total number of goods. Each agent i has an *initial endowment* $w^i=(w^i(0),w^i(1),\dots,w^i(n))\in R^r_{++}$ and chooses a vector of consumption $x^i=(x^i(0),x^i(1),\dots,x^i(n))\in R^r_+$. Agent i's preferences are represented by a *utility function* $u^i\colon R^r_+\to R$, $i=1,\dots,m$.

Assumption U (utility function). Each utility function u^i, $i=1,\dots,m$ satisfies:

(i) $u^i\in\mathscr{C}(R^r_+)$, $u^i\in\mathscr{C}^2(R^r_{++})$;

(ii) $Du^i(x)\in R^r_{++}\ \forall x\in R^r_{++}$;

(iii) $h^T D^2u^i(x)h<0\ \forall h\neq 0$ such that $Du^i(x)h=0$, $\quad\forall x\in R^r_{++}$;

(iv) If $U^i(\xi)=\{x\in R^r_+\,|\,u^i(x)\geqq u^i(\xi)\}$, then $\overline{U^i(\xi)}\subset R^r_{++}$, $\quad\forall\xi\in R^r_{++}$.

The market structure is as follows. There is a spot market for each of the l goods at $t=0$ and in each state $s=1,\dots,n$ at $t=1$: let $p=(p(0),p(1),\dots,p(n))\in R^r_+$ denote the induced vector of *spot prices.* Sometimes

[1] For a fuller analysis of this model with applications to various different types of financial markets like *futures markets* and *security markets* for equity of firms, see Magill and Shafer (1990).

it will be convenient to decompose the price vector p into two components, for $t=0,1$ respectively: $p=(p_0,p_1)=(p(0),(p(s))_{s=1}^n)$. A similar decomposition will be used for the endowment and consumption vectors $w^i=(w_0^i,w_1^i)$, $x^i=(x_0^i,x_1^i)$. There are k asset markets with a vector of *asset prices* $q=(q_1,\ldots,q_k)\in R^k$. One unit of asset j $(j=1,\ldots,k)$ is a contract promising delivery of $a_h^j(s)$ units of good h in state s, $h=1,\ldots,l$, $s=1,\ldots,n$. Let $A(s)=(a_h^j(s),\ j=1,\ldots,k,\ h=1,\ldots,l)$ denote an $l\times k$ matrix for each $s=1,\ldots,n$ and let $A=(A(1),\ldots,A(n))\in R^v$, where $v=lkn$. Thus A summarises the *asset structure* of the economy. Define the $n\times k$ matrix $V(p;A)=p\,\Box\,A=(p(s)A(s))_{s=1}^n$. Let $p_1\,\Box\,x_1=(p(s)x(s))_{s=1}^n$, then the opportunity set of agent i who buys $z^i=(z_1^i,\ldots,z_k^i)\in R^k$ units of the k assets is given by

$$\mathscr{B}_{z^i}(p,q;w^i)=\left\{x\in R_+^r\ \middle|\ \begin{matrix} p(0)(x(0)-w^i(0))=-qz^i \\ p_1\,\Box\,(x_1-w_1^i)=V(p_1;A)z^i \end{matrix}\right\}, \tag{1}$$

and $\mathscr{B}(p,q;w^i)=\bigcup_{z^i\in R^k}\mathscr{B}_{z^i}(p,q;w^i)$ is the budget set of agent i. Let $\mathscr{E}((u^i,w^i),A)$ denote the resulting economy in which agent i has utility function–endowment pair (u^i,w^i) and the asset structure is A.

Definition 1. An *equilibrium* for the economy $\mathscr{E}((u^i,w^i),A)$ is a pair $((\bar{x}^i,\bar{z}^i),(\bar{p},\bar{q}))\in R_+^{mr}\times R^{mk}\times R_+^r\times R^k$ such that

(i) $\quad \bar{x}^i=\underset{x^i\in\mathscr{B}(p,q;w^i)}{\operatorname{argmax}}\ u^i(x^i)$ and $\bar{x}^i\in\mathscr{B}_{\bar{z}^i}(p,q;w^i)$, $\quad i=1,\ldots,m$

(ii) $\quad \sum_{i=1}^m(\bar{x}^i-w^i)=0,$

(iii) $\quad \sum_{i=1}^m\bar{z}^i=0.$

Remark 1. Let $\omega=(w^1,\ldots,w^m)$. It is known that (u^i) satisfying Assumption U and $(\omega,A)\in R_{++}^{mr}\times R^v$ can be chosen such that *no equilibrium exists* [see Hart (1975) and Magill and Shafer (1990)]. Our object is to show that, for any fixed choice of (u^i) satisfying Assumption U, an equilibrium exists for most $(\omega,A)\in R_{++}^{mr}\times R^v$.

The strategy of the reduction scheme to a pseudo-equilibrium can be explained as follows. Non-existence of equilibrium arises from the *non-compactness* of the portfolio trades (z^i) and the *discontinuity* of the budget correspondence $p\mapsto\mathscr{B}(p,q;w^i)$: the latter arises from changes in the rank of the matrix $V(p_1;A)$ in eq. (2) as date 1 spot prices p_1 vary, reflecting changes in the dimension of the subspace of income transfers spanned by the columns of the matrix $V(p)$.[2] These two difficulties will be circumvented in two steps.

First, *we eliminate the portfolio trades and asset prices* $((z^i),q)$: the idea here

[2]Sometimes it is convenient to replace p_1 by p in the expression $V(\cdot)$ with the understanding that it does not depend on p_0.

is that since assets promise to deliver goods at date 1, their prices must be related to the spot prices in such a way as to present *no arbitrage opportunities* to agents. This natural economic condition allows asset prices as well as asset trades to be eliminated from the budget equations [the analysis here follows that in Magill and Shafer (1990)]. This first step leads us to the concept of a *no-arbitrage equilibrium.*

Second, *we replace the* actual subspace *of income transfers* $\langle V(p)\rangle$ *made possible by trading in the assets, when spot prices are p, by a fixed trial* subspace L *which is independent of p.* Clearly such a trial subspace remains fictitious unless in equilibrium $\langle V(p)\rangle = L$. This second step leads to the concept of a *regular pseudo-equilibrium.*

For any vector y, let $y \geq 0$ denote $y_i \geqq 0\, \forall i$ and $y_j > 0$ for some j. The right-hand side of eq. (1) can be written more simply in terms of the matrix $W(p,q) = \left[\begin{smallmatrix} -q \\ V(p) \end{smallmatrix}\right]$.

Definition 2. q is a *no-arbitrage asset price* relative to p if there does not exist a portfolio trade $z \in R^k$ which generates a portfolio with a semipositive return $W(p,q)z \geq 0$.

Lemma 1. If q is a no-arbitrage asset price relative to p, then there exists $\beta \in R^{n+1}_{++}$ such that $q = \sum_{s=1}^{n} \hat{\beta}_s p(s) A(s)$, $\hat{\beta} = (1/\beta_0)\beta$.

Proof. This is an immediate consequence of the separation theorem [Gale (1960, Cor. 2, p. 49)] which asserts that for any $(n+1) \times k$ matrix W exactly one of the following holds. Either there exists $z \in R^k$ such that $Wz \geq 0$ or there exists $\beta \in R^{n+1}_{++}$ such that $\beta W = 0$. $\square$

Remark 2. If $((\bar{x}^i, \bar{z}^i), (\bar{p}, \bar{q}))$ is an equilibrium and if we define $p = \hat{\beta} \,\square\, \bar{p}$, then $((\bar{x}^i, \bar{z}^i), (p, \bar{q}))$ is an equilibrium: this simply expresses the fact that period 1 spot prices can be *rescaled* without affecting the equilibrium.

It is clear, however, that in an equilibrium $((\bar{x}^i, \bar{z}^i), (\bar{p}, \bar{q}))$ the asset price $\bar{q}$ must be a no-arbitrage price relative to $\bar{p}$. This suggests using a $\bar{\beta}$ defined by Lemma 1 to rescale spot prices. If we do this note that the period 0 budget constraint becomes

$$p(0)(x(0) - w^i(0)) = -\bar{q}z^i = -\sum_{i=1}^{n} p(s)A(s)z^i = -\sum_{s=1}^{n} p(s)(x(s) - w^i(s))$$

which is equivalent to $p(x - w^i) = 0$. The period 1 budget constraint can then be written as $p_1 \,\square\, (x_1 - w_1^i) \in \langle V(p)\rangle$, where $\langle V(p)\rangle$ denotes the subspace of R^n

spanned by the k columns of $V(p)$. Each agent i's budget set has thus been reduced to the form

$$\mathscr{B}(p;w^i)=\{x\in R^r_+ \,|\, p(x-w^i)=0,\ p_1 \Box (x_1-w_1^i)\in\langle V(p)\rangle\} \qquad i=1,\dots,m. \tag{2}$$

This leads to the following simplified concept of equilibrium.

Definition 3. *A no-arbitrage equilibrium* for the economy $\mathscr{E}((u^i,w^i),A)$ is a pair $((\bar{x}^i),p)$ such that

(i) $\bar{x}^i=\arg\max_{x^i\in\mathscr{B}(p;w^i)} u^i(x^i), \qquad i=1,\dots,m,$

(ii) $\sum_{i=1}^{m}(\bar{x}^i-w^i)=0.$

Remark 3. Thus given an equilibrium $((\bar{x}^i,\bar{z}^i),(\bar{p},\bar{q}))$ there exists $\hat{\beta}$ such that if $p=\hat{\beta}\Box\bar{p}$, then $((\bar{x}^i),p)$ is a no-arbitrage equilibrium. Conversely if $((\bar{x}^i),p)$ is a no-arbitrage equilibrium and if we define $((\bar{z}^i),q)$ by

$$V(p)\bar{z}^i=p_1\Box(\bar{x}_1^i-w_1^i), \qquad i=2,\dots,m,$$

$$\bar{z}^1=-\sum_{i=2}^{m}\bar{z}^i, \qquad q=\sum_{s=1}^{n}p(s)A(s),$$

then $((\bar{x}^i,\bar{z}^i),(p,q))$ is an equilibrium. Thus the problem of establishing the existence of an equilibrium has been reduced to the problem of establishing the existence of a *no-arbitrage equilibrium.*

Remark 4. There is still some freedom in the choice of $\hat{\beta}$ which it is convenient to exploit. Let $((\bar{x}^i,\bar{z}^i),(\bar{p},\bar{q}))$ be an equilibrium and consider the constrained maximum problem solved by each agent i. From the first-order conditions for the Lagrangean

$$L^i(x^i,z^i,\lambda^i)=u^i(x^i)-\lambda_0^i[\bar{p}(0)(x^i(0)-w^i(0))+\bar{q}z^i]$$
$$-\sum_{s=1}^{n}\lambda_s^i[\bar{p}(s)(x^i(s)-w^i(s))-\bar{p}(s)A(s)].$$

We note that if we choose $\hat{\beta}=\hat{\lambda}^i$ where $\hat{\lambda}_i=(1/\lambda_0^i)\lambda^i$, then

$$\bar{x}^i=\arg\max_{x^i\in B(p;w^i)} u^i(x^i) \quad \text{where} \quad B(p;w^i)=\{x\in R^r_+\,|\,p(x-w^i)=0\} \tag{3}$$

so that *agent i can choose his consumption vector as if he faced complete contingent markets.*

If we choose $\hat{\beta}=\hat{\lambda}^1$, then we obtain the following concept of equilibrium.

Definition 4. *A normalised no-arbitrage equilibrium* for the economy $\mathscr{E}((u^i,w^i),A)$ is a pair $((\bar{x}^i),p)$ such that

(i) $\bar{x}^1=\underset{x^1\in B(p;w^1)}{\arg\max}\ u^1(x^1),\quad \bar{x}^i=\underset{x^i\in \mathscr{B}(p;w^i)}{\arg\max}\ u^i(x^i),\qquad i=2,\dots,m,$

(ii) $\sum_{i=1}^{m}(\bar{x}^i-w^i)=0.$

Our final step consists in replacing the *actual subspace* of income transfers $\langle V(p)\rangle$ achievable by trading in assets, with a *trial subspace L*. When there are k assets, then generically[3] the actual subspace $\langle V(p)\rangle$ will be a k-dimensional subspace of R^n.

If we are to find an appropriate subspace we will need to have at our disposal a way of parametrising such subspaces so as to generate a sufficiently rich family of subspaces from which to seek out an equilibrium one. We consider two ways of doing this. The first is to use the approach of Duffie and Shafer (1985). This consists in considering the *Grassmanian manifold* $G^{n,k}$ of all k-dimensional subspaces of R^n. A point $L\in G^{n,k}$ is simply a k-dimensional subspace.

If we examine the budget sets (2) and (3) of agents in a normalised no-arbitrage equilibrium, we note that we can choose one more normalisation of spot prices. Let us do this in the standard way of placing p in the non-negative simplex $\Delta_+^{r-1}=\{p\in R_+^r\,|\sum_{i=1}^r p_i=1\}$, letting $\Delta_{++}^{r-1}=\{p\in R_{++}^r\,|\sum_{i=1}^n p_i=1\}$. We are now ready to complete the final step in the derivation of a pseudo-equilibrium. Replacing the subspace $\langle V(p)\rangle$ in the budget set (2) by the subspace $L\in G^{n,k}$, we obtain a *budget correspondence* B: $\Delta_+^{r-1}\times G^{n,k}\times R_{++}^r\to R_+^r$,

$$B(p,L;w^i)=\{x\in R_+^r\,|\,p(x-w^i)=0,p_1\,\Box\,(x_1-w_1^i)\in L\},\qquad i=2,\dots,m. \quad (4)$$

Definition 5. *A pseudo-equilibrium* for the economy $\mathscr{E}((u^i,w^i),A)$ over the *Grassmanian* is a pair $((\bar{x}^i),(\bar{p},\bar{L}))\in R_+^{mr}\times\Delta_{++}^{r-1}\times G^{n,k}$ such that

(i) $\bar{x}^1=\underset{x^1\in B(\bar{p};w^1)}{\arg\max}\ u^1(x),\quad \bar{x}^i=\underset{x^i\in B(\bar{p},\bar{L};w^i)}{\arg\max}\ u^i(x^i),\qquad i=2,\dots,m,$

(ii) $\sum_{i=1}^{m}(\bar{x}^i-w^i)=0,$

[3]In the space of spot prices and asset structures (p,A).

(iii) $\langle V(\bar{p})\rangle \subset \bar{L}$.

A pseudo-equilibrium is *proper* if

(iii′) $\langle V(\bar{p})\rangle = \bar{L}$.

The second way of parametrising k-dimensional subspaces of R^n is perhaps less intuitive, but intimately related. Suppose we choose an $(n-k)\times n$ orthonormal matrix Q, then $\langle Q^T\rangle$, the span of the columns of the transpose of Q, is an $n-k$-dimensional subspace of R^n. The orthogonal decomposition $R^n=\langle Q^T\rangle \oplus \langle Q^T\rangle^{\perp}$ leads to a k-dimensional subspace $\langle Q^T\rangle^{\perp}$. We can generate all k-dimensional subspaces of R^n in this way by letting Q vary over the *Stiefel manifold* $O^{n,n-k}=\{Q\in R^{(n-k)n} \mid QQ^T=I\}$ of all $(n-k)\times n$ orthonormal matrices. Clearly there are many points $Q\in O^{n,n-k}$ which generate the same subspace. In fact if we let O_{n-k} denote the *orthogonal group* of $(n-k)\times(n-k)$ orthogonal matrices, then

$$\langle Q^T\rangle = \langle (gQ)^T\rangle \quad \forall g\in O_{n-k}. \tag{5}$$

We will need to add this condition as an extra restriction whenever Q appears in the analysis, to be sure that the mathematical formulation depends only on the subspace in question and not on its particular representation.

If we use the Stiefel manifold representation for subspaces, then when we replace the subspace $\langle V(p)\rangle$ in (2) by a subspace $\langle Q^T\rangle^{\perp}$ with $Q\in O^{n,n-k}$, we obtain a budget correspondence $b\colon \Delta_+^{r-1}\times O^{n,n-k}\times R^r_{++}\to R^r_+$,

$$b(p,Q;w^i)=\{x\in R^r_+ \mid p(x-w^i)=0,\ \ p_1\,\square\,(x_1-w^i_1)\in\langle Q^T\rangle^{\perp}\},$$

$$i=2,\dots,m, \tag{6}$$

where in view of (5), $b(p,\cdot\,;w^i)$ is O_{n-k} invariant:

$$b(p,gQ;w^i)=b(p,Q;w^i),\quad \forall g\in O_{n-k}. \tag{7}$$

Definition 6. A *pseudo-equilibrium* for the economy $\mathscr{E}((u^i,w^i),A)$ over the *Stiefel manifold* is a pair $((\bar{x}^i),(\bar{p},\bar{Q}))\in R^{mr}_+\times\Delta^{r-1}_{++}\times O^{n,n-k}$ such that

(i) $\bar{x}^1=\operatorname*{arg\,max}_{x^1\in B(\bar{p};w^1)} u^1(x^1),\quad \bar{x}^i=\operatorname*{arg\,max}_{x^i\in b(\bar{p},\bar{Q};w^i)} u^i(x^i),\qquad i=2,\dots,m,$

(ii) $\sum_{i=1}^{m}(\bar{x}^i-w^i)=0,$

(iii) $\langle V(\bar{p})\rangle \subset \langle \bar{Q}^T\rangle^{\perp}$.

A pseudo-equilibrium is *proper* if

(iii′) $\langle V(\bar{p})\rangle = \langle \bar{Q}^T\rangle^{\perp}$.

Lemma 2. If $((\bar{x}^i),(\bar{p},\bar{L}))$ *or* $((\bar{x}^i),(\bar{p},\bar{Q}))$ *is a proper pseudo-equilibrium, then there exists* $((\bar{z}^i),\bar{q}) \in R^{mk} \times R^k$ *such that* $((\bar{x}^i,\bar{z}^i),(\bar{p},\bar{q}))$ *is an equilibrium.*

Proof. The result follows from Remark 3, noting that a proper pseudo-equilibrium is a normalised no-arbitrage equilibrium $((\bar{x}^i),\bar{p})$, since the budget sets in (2) and (4) or (6) coincide. □

We have thus reduced the problem of establishing the existence of an equilibrium to the problem of establishing the existence of a proper pseudo-equilibrium.

3. Existence of equilibrium

Consider an economy $\mathscr{E}((u^i,w^i),A)$ for which the preferences of agents embodied in (u^i) are fixed, then the economy can be parametrised by the *endowment–asset structure* pair $(\omega,A)=(w^1,\ldots,w^m,A)\in R^{mr}_{++}\times R^{\nu}$. In this section we will show that the following existence result is a consequence of a general fixed-point theorem for incomplete markets (Theorem A or B below).

Theorem 1 (existence of pseudo-equilibrium). Let the utility functions (u^i) *satisfy Assumption U, then for every* $(\omega,A)\in R^{mr}\times R^{\nu}$ *the economy* $\mathscr{E}((u^i,w^i),A)$ *has a pseudo-equilibrium.*

By a transversality argument, which is by now familiar [see Duffie and Shafer (1985, p. 297)], one then establishes that generically pseudo-equilibria are proper. More precisely we leave it to the reader to establish the following result: *there exists an open set* $\Omega\subset R^{mr}_{++}\times R^{\nu}$ *whose complement is null such that for every* $(\omega,A)\in\Omega$ *every pseudo-equilibrium is proper.* This leads to the basic existence theorem.

Theorem 2 (generic existence of equilibrium). Let the utility functions (u^i) *satisfy Assumption U, then there exists an open set* Ω *in the space of endowment–asset structure pairs* $R^{mr}_{++}\times R^{\nu}$*, whose complement is null, such that for every* $(\omega,A)\in\Omega$ *the economy* $\mathscr{E}((u^i,w^i),A)$ *has an equilibrium.*

Our problem thus reduces to establishing the existence of a pseudo-equilibrium. Consider first the formulation of a pseudo-equilibrium over the Grassmanian (Definition 5). Under Assumption U the solution of each agent's utility maximising problem in Definition 5(i) exists, is unique and leads to *individual demand* functions on the *spot markets*:

$$F^1: \Delta^{r-1}_{++} \times R_+ \to R^r_+, \quad F^i: \Delta^{r-1}_{++} \times G^{n,k} \times R^r_{++} \to R^r_+, \qquad i=2,\dots,m,$$

$$F^1(p; pw^1) = \underset{x^1 \in B(\bar{p}; w^1)}{\arg\max}\ u^1(x^1), \quad F^i(p, L; w^i) = \underset{x^i \in B(p, L; w^i)}{\arg\max}\ u^i(x^i),$$

$$i=2,\dots,m \tag{8}$$

Remark 5. It is readily shown that F^i, $i=1,\dots,m$ *are* $\mathscr{C}^1$ *functions.*

The functions in (8) lead naturally to the *aggregate excess demand* function on the system of spot markets, $Z: \Delta^{r-1}_{++} \times G^{n,k} \times R^{mr}_{++} \to R^r$,

$$Z(p, L; \omega) = F^1(p; pw^1) - w^1 + \sum_{i=2}^{m} (F^i(p, L; w^i) - w^i). \tag{9}$$

The *asset markets* are characterised by the *asset return* function $\Psi: \Delta^{r-1}_+ \times G^{n,k} \times R^v \to R^{nk}$ defined by

$$\Psi(p, L; A) = V(p; A). \tag{10}$$

The conditions (i)–(iii) in Definition 5 thus reduce to

$$Z(\bar{p}, \bar{L}; \omega) = 0, \qquad \langle \Psi(\bar{p}, \bar{L}; A \rangle \subset \bar{L}. \tag{11}$$

It follows from (10) that Ψ is $\mathscr{C}^1$ and from Remark 5 that Z is $\mathscr{C}^1$. It is convenient for the rest of this section to omit the explicit dependence of (Z, Ψ) on the parameters (ω, A).

Let $E^{r-1} = \{z \in R^r | \sum_{i=1}^r z_i = 1\}$ denote the $r-1$-dimensional affine subspace containing Δ^{r-1}_+. To show that (11) has a solution we consider the *price adjustment* function $M: \Delta^{r-1}_{++} \times G^{n,k} \to E^{r-1}$ defined by

$$M(p, L) = p + p \,\Box\, Z(p, L),$$

whose fixed points coincide with the zeros of Z. By a standard argument it follows from the fact that Z is bounded below and that $p^m \in \Delta^{r-1}_+$ such that $p^m \to p \in \partial\Delta^{r-1}_+$ implies $\|Z(p^m, L)\| \to \infty$, that M is 'essentially' inward pointing on $\partial\Delta^{r-1}_+$. This property needs to be made more precise. In Lemma 4 of the appendix we extend the aproach of Dierker (1974, p. 79) by showing that there is a function $\delta: \Delta^{r-1}_+ \times G^{n,k} \to [0,1]$ such that the *modified price adjustment* map $\Phi: \Delta^{r-1}_+ \times G^{n,k} \to E^{r-1}$ defined by

$$\Phi(p,L)=\delta(p,L)M(p,L)+(1-\delta(p,L))u,$$

where $u=(1/r,\dots,1/r)$ is *inward* pointing on $\partial\Delta_+^{r-1}$

$$\Phi(p,L)\in\Delta_+^{r-1}\quad\forall p\in\partial\Delta_+^{r-1},\quad\forall L\in G^{n,k},$$

and its fixed points coincide with the zeros of Z

$$\Phi(p,L)=p\quad\text{if and only if}\quad Z(p,L)=0.$$

Thus if we use the price adjustment and asset return functions (Φ,Ψ), the existence of a pseudo-equilibrium reduces to the following

Pseudo-equilibrium fixed-point problem A. Find $(\bar{p},\bar{L})\in\Delta_+^{r-1}\times G^{n,k}$ such that $\Phi(\bar{p},\bar{L})=\bar{p}$, $\langle\Psi(\bar{p},\bar{L})\rangle\subset\bar{L}$

If we let $m=r-1$, $C=\Delta_+^{r-1}$, $H^m=E^{r-1}$, then the following theorem provides the solution.

A. Grassmanian fixed-point theorem. Let H^m be an m-dimensional affine subspace, $C\subset H^m$ a compact convex subset with non-empty relative interior. Let (Φ,Ψ) be continuous functions $\Phi:C\times G^{n,k}\to H^m$, $\Psi:C\times G^{n,k}\to R^{nk}$ such that $\Phi(\partial C,L)\subset C,\forall L\in G^{n,k}$, then there exists $(\bar{p},\bar{L})\in G^{n,k}$ such that $\Phi(\bar{p},\bar{L})=\bar{p},\langle\Psi(\bar{p},\bar{L})\rangle\subset\bar{L}$.

Proof. See section 6.

This theorem contains as a special case the standard theorem [Dierker (1974, Theorem 8.2, p. 77)] which is used to establish existence in the case of *complete markets* [Dierker (1974, Theorem 8.3, p. 78)]. The component of the theorem which is new is the *subspace fixed-point* part. This property, which is examined in section 4, can be given several equivalent formulations, which we call Theorems A′, B′ and C′ respectively. Theorem B′ will be related to the Stiefel manifold approach induced by Definition 6, to which we now turn.

The reduction of the pseudo-equilibrium problem over the Stiefel manifold to a system of equations follows the same procedure as above. Demand functions $f^i(p,Q;w^i)$, $i=2,\dots,m$, which in view of (7) are O_{n-k} invariant, $f^i(p,gQ;w^i)=f^i(p,Q;w^i)\,\forall g\in O_{n-k}$, and the *aggregate excess demand* function $z(p,Q;\omega)=F^1(p;pw^1)-w^1+\sum_{i=2}^m(f^i(p,Q;w^i)-w^i)$ are introduced. If $m(p,Q)=p+p\,\square\,z(p,Q)$, then again by Lemma 4 in the appendix there is a function $\delta:\Delta_+^{r-1}\times O^{n,n-k}\to[0,1]$ such that the modified price adjustment function

$\phi(p,Q)=\delta(p,Q)m(p,Q)+(1-\delta(p,Q))u$ is inward pointing on $\partial\Delta_{+}^{r-1}$ and its fixed points coincide with the zeros of z. δ and hence ϕ is O_{n-k}-invariant.

The *asset market* function is derived as follows. Let $V_j(p)$ denote column j $(j=1,\ldots,k)$ of the matrix $V(p)$. To show property (iii) in Definition 6, namely $\langle V(\bar{p})\rangle\subset\langle\bar{Q}^T\rangle^{\perp}$, we need to show $V_j(\bar{p})\in\langle\bar{Q}^T\rangle^{\perp}$, $j=1,\ldots,k$. This is equivalent to showing that for each j the projection of $V_j(\bar{p})$ onto $\langle\bar{Q}^T\rangle$ is zero. Let Q_i^T, $i=1,\ldots,n-k$ denote the columns of Q^T, then (Q_i^T) is a basis for $\langle Q^T\rangle$ and

$$\pi_{\langle\bar{Q}^T\rangle}V_j(\bar{p})=\sum_{i=1}^{n-k}(\bar{Q}_iV_j(\bar{p}))\bar{Q}_i^T=0,\qquad j=1,\ldots,k,$$

which is equivalent to $\bar{Q}_iV_j(\bar{p})=0$, $i=1,\ldots,n-k$, $j=1,\ldots,k$, or in matrix form

$$\psi(p,Q)=QV(p)=0$$

where $\psi\colon\Delta_{+}^{r-1}\times O^{n,n-k}\to(R^{n-k})^k$ is the desired *asset market* function. Note that $\psi(p,gQ)=g\psi(p,Q),\forall g\in O_{n-k}$.

Pseudo-equilibrium fixed-point problem B.[4] Find $(\bar{p},\bar{Q})\in\Delta_{+}^{r-1}\times O^{n,n-k}$ such that $\phi(\bar{p},\bar{Q})=\bar{p}$, $\psi(\bar{p},\bar{Q})=0$.

The following theorem, which will be shown to be equivalent to Theorem A above, provides the solution.

B. Stiefel manifold fixed-point theorem. Let H^m, C be as in Theorem A and let (ϕ,ψ) be continuous functions $\phi\colon C\times O^{n,n-k}\to H^m$, $\psi\colon C\times O^{n,n-k}\to(R^{n-k})^k$ such that $\phi(\partial C,Q)\subset C\,\forall Q\in O^{n,n-k}$, $\phi(p,gQ)=\phi(p,Q)$, $\psi(p,gQ)=g\psi(p,Q)$ $\forall g\in O_{n-k}$, $\forall(p,Q)\in C\times O^{n,n-k}$, then there exists $(\bar{p},\bar{Q})\in C\times O^{n,n-k}$ such that $\phi(\bar{p},\bar{Q})=\bar{p}$, $\psi(\bar{p},\bar{Q})=0$.

Proof. See section 6.

4. Vector bundles and the subspace fixed-point property

The proof of Theorems A and B is complicated by the fact that there is a simultaneous fixed-point problem in *prices* (on the simplex) and a fixed-point problem in *subspaces* (on the Grassmanian or Stiefel manifold). For this reason, and to bring out the essentially new ideas involved in the equilibrium

[4]For convenience we refer to both an equation of the form $f(\bar{x})=\bar{x}$ and an equation of the form $f(\bar{x})=0$ as a '*fixed-point*' problem.

subspace problem, we first consider this component by itself: this leads us to Theorems A′ and B′ below which are shown to be equivalent to a *canonical vector bundle problem*, Theorem C′. *This vector bundle problem has the merit of focusing attention directly on the essential topological problem involved in finding an equilibrium subspace.* In the next section the mathematical approach that we use allows us to prove a more general version of these theorems (A*, B*, C*), where B* has the important property of being the natural generalisation of the well-known Borsuk–Ulam Theorem. The idea that underlies the proof of B* (non-zero Euler class) also forms the kernel for the proof of Theorems A and B in section 6. We hope that in decomposing the analysis of proof of the general theorem in this way that we succeed in achieving two objectives: first to explain the new ideas involved in the equilibrium subspace problem and second to show how an economic problem can lead to an interesting generalisation of a classical theorem of topology.

Consider the equations for a pseudo-equilibrium over the Grassmanian

$$\Phi(\bar{p}, \bar{L}) = \bar{p}, \qquad \langle V(\bar{p}) \rangle \subset \bar{L}. \tag{12}$$

An economist would argue, at least initially, as follows. If we impose a subspace L on the agents in their spot market trades, then the spot markets will generate an equilibrium price which depends on L, $p = p(L)$. This leads to a matrix of actual asset returns $\Psi(L) = V(p(L))$. The equations for a pseudo-equilibrium reduce to

$$\langle \Psi(\bar{L}) \rangle \subset \bar{L}. \tag{13}$$

Equilibrium is reduced to a *subspace equilibrium* on the asset markets. This involves finding a subspace $\bar{L}$ such that after determining the equilibrium spot price implied $p(\bar{L})$ the asset return matrix $\Psi(\bar{L}) = V(p(\bar{L}))$ generates a *subspace of actual income transfers consistent with the imposed subspace* $\bar{L}$. The economic intuition behind this approach leads to the essence of the subspace problem: however, we cannot hope in general to solve for spot prices p as a continuous function of the subspace L – a priori there may be several spot price equilibria associated with a given subspace.

The problem (13) will be called the *subspace fixed-point problem A′*. Theorem A restricted to the Grassmanian asserts that a solution to (13) exists.

Theorem A′ (Grassmanian). If $\Psi: G^{n,k} \to R^{nk}$ is a continuous matrix-valued function, then there exists $\bar{L} \in G^{n,k}$ such that $\langle \Psi(\bar{L}) \rangle \subset \bar{L}$ or in component form: if $\Psi_i: G^{n,k} \to R^n$, $i = 1, \ldots, k$ are continuous functions, then there exists $\bar{L} \in G^{n,k}$ such that $\Psi_i(\bar{L}) \in \bar{L}$, $i = 1, \ldots, k$.

Remark 6. This theorem is 'almost' a Brouwer theorem for the Grassmanian: *it 'almost' asserts that every mapping of the Grassmanian $G^{n,k}$ into itself has a fixed point.* The map $L \mapsto \langle \Psi(L) \rangle$ associates with each k-dimensional subspace a new subspace: this new subspace need not be k-dimensional – but 'generically' it will be. Since $\langle \Psi(L) \rangle$ can be 'smaller" than a k-plane, we define a fixed point by $\langle \Psi(L) \subset L$, but when it is a k-plane we have $\Psi(\bar{L}) \rangle = \bar{L}$: in this sense we can say that 'generically' we have an analogue of the Brouwer theorem for the Grassmanian.

We can analyse the equations for a pseudo-equilibrium over the Stiefel manifold in a similar way. Thus given the equations

$$\phi(\bar{p}, \bar{Q}) = \bar{p}, \qquad \bar{Q}V(\bar{p}) = 0, \tag{14}$$

if we impose the subspace $\langle Q^T \rangle^{\perp}$ on the agents in their spot market trades, then the spot markets will generate an equilibrium price which depends on $Q, p = p(Q)$. Since this price depends only on the k-dimensional subspace and not on its representation, $p(gQ) = p(Q)\, \forall g \in O_{n-k}$. Let $\psi(Q) = QV(p(Q))$, then

$$\psi(gQ) = gQV(p(gQ)) = gQV(p(Q)) = g\psi(Q) \quad \forall g \in O_{n-k} \forall Q \in O^{n,n-k}. \tag{15}$$

Thus the equations for a pseudo-equilibrium reduce to

$$\psi(\bar{Q}) = 0,$$

where ψ is O_{n-k} *equivariant* [eq. (15)]. This will be called the *subspace fixed-point problem B′*. Restricting Theorem B to the Stiefel manifold gives:

Theorem B′ (Stiefel manifold). If $\psi: O^{n,n-k} \to (R^{n-k})^k$ is a continuous function satisfying $\psi(gQ) = g\psi(Q)\, \forall g \in O_{n-k}$, $\forall Q \in O^{n,n-k}$, then there exists $\bar{Q} \in O^{n,n-k}$ such that $\psi(\bar{Q}) = 0$.

We want to show that Theorems A′ and B′ are equivalent: either can be used to derive the other. We do this by showing that they are equivalent to a third theorem, C′, where this latter theorem reveals most clearly the *topological property* in which the three fixed-point theorems share their origin. To exhibit this equivalence we need the concept of a vector bundle.[5]

Definition 7. An n-dimensional *vector bundle* $\xi = (E, M, \pi)$ is a triple such that (i) E and M are topological spaces, (ii) $\pi: E \to M$ is a surjective map,[6]

[5]See Husemoller (1975), Milnor and Stasheff (1974), Osborn (1982).
[6]Note that throughout the paper a *map* between topological spaces mean a *continuous* function.

(iii) $\pi^{-1}(x)=E_x$ is an n-dimensional vector space, (iv) for each $x\in M$ there exists a neighborhood in M and a homeomorphism (U,h), $h: U\times R^n\to\pi^{-1}(U)$ such that the restriction $h_x: x\times R^n\to E_x$ is a vector space isomorphism for each $x\in U$. M is called the *base* space, E the *total* space, π the *projection* and E_x the *fibre* over $x\in M$.

Definition 8. A *section* of a vector bundle ξ is a map $\sigma: M\to E$ such that $\sigma(x)\in E_x$, $\forall x\in M$. The *zero-section* $\sigma_0: M\to E$ satisfies $\sigma_0(x)=0\in E_x$, $\forall x\in M$. σ is a *non-zero section* if $\sigma\cap\sigma_0=\varnothing$.

The fundamental question that interests us is the following

Vector bundle problem. Does a vector bundle ξ admit a non-zero section?

This is a special case of a whole class of topological questions that have been studied for vector bundles which leads to the theory of *characteristic classes.*[7] The question posed leads to one such cohomology class of ξ, called the *Euler class* $e(\xi)$. The following property is the basis for the proofs of Theorems A and B in section 6.

Theorem D. If the Euler class of a vector bundle is not zero, then the vector bundle admits no non-zero section, i.e. if $e(\xi)\neq 0$ and $\sigma: M\to E$ is a section, then $\sigma\cap\sigma_0\neq\varnothing$.

Remark 7. Mas-Colell (1985, pp. 188–214) has emphasised the importance of the Poincaré–Hopf theorem in the analysis of price equilibria. Note that there is a closely related result here for subspace equilibria. Let $\xi=(E,M,\pi)$ be an n-dimensional vector bundle over a compact, connected n-dimensional manifold M, let $e(\xi)$ denote its *Euler class* (taken over Z if ξ is orientable and over Z_2 otherwise) and let μ_M denote the *fundamental homology class* of M. If $\#(\sigma_0\cdot\sigma_0)$ denotes the *self-intersection number* of the zero section (over Z if ξ is oriented, over Z_2 otherwise) and let $\langle\cdot,\cdot\rangle: H^n(M)\times H_n(M)\to Z(Z_2)$ denote the *Kronecker index*, then

$$\langle e(\xi),\mu_M\rangle=\#(\sigma_0\cdot\sigma_0)=\#(\sigma\cdot\sigma_0) \tag{16}$$

for any section $\sigma: M\to E$. Thus the 'integral' of the Euler class over the manifold, the *Euler number*, which is a basic invariant of the vector bundle, equals the algebraic number of intersections of *any* section with the zero section. If $\xi=\tau_M$ the *tangent bundle* of M, then $\langle e(\tau_M),\mu_M\rangle=\chi(M)$, where

[7]An example familiar to the reader is the following: *is there a vector field on the sphere S^{r-1} which is non-zero at every point on the sphere?* In this case $\xi=\tau S^{r-1}$, the *tangent bundle* of the sphere. The answer is yes when $r-1$ is odd and no when it is even. The theory of characteristic classes is studied in Husemoller (1975), Milnor and Stasheff (1974), Osborn (1982).

$\chi(M)$ denotes the *Euler characterisic* of M. In this case (16) reduces to the Poincaré–Hopf theorem. Note that if we use *Poincaré duality theory*, then (16) can be stated more generally as follows: the locus of intersection of any section with the zero section is the *Poincaré dual* of the Euler class of the vector bundle. The Euler class is thus a basic invariant of a vector bundle by means of which one can describe the zero locus of *any* section of the vector bundle [Bott and Tu (1982, pp. 122–135)]. Before making explicit use of the Euler class let us study the vector bundles induced by problems A′ and B′.

Let us first reduce problem B′ to a vector bundle problem. Let X and Y be topological spaces on which a group G acts, i.e., $g: X \to X$, $g: Y \to Y \, \forall g \in G$ and let $f: X \to Y$ be a map such that $f(gx) = gf(x) \, \forall g \in G$, $\forall x \in X$. Such a map f is called a *G-map*. Consider also the *quotient map* (projection) $\pi_G: X \to X/G$ defined by $\pi_G(x) = \{gx \mid \forall g \in G\} = [x]$ where $[x]$ denotes the equivalence class of x. We are interested in the case where

$$X = O^{n,n-k}, \quad Y = R^{k(n-k)}, \quad G = O_{n-k}. \tag{17}$$

In this case G acts *freely* on X (i.e. $\{g \in G \mid gx = x\} = I$, $\forall x \in X$ where I is the identity of G) so that X/G is a manifold. We may thus consider the vector bundle $\xi = (E, M, \pi) = (X \times_G Y, X/G, \pi)$ where $X \times_G Y = (X \times Y)/G$. *Every G-map f defines a section of ξ, $\sigma_f: X/G \to X \times_G Y$* by $\sigma_f[x] = [x, f(x)]$ where the latter denotes the equivalence class of $(x, f(x))$ in $X \times_G Y$. Conversely, *every section $\sigma: X/G \to X \times_G Y$ defines a G-map $f_\sigma: X \to Y$* by letting $\sigma[x] = [x, f_\sigma(x)]$. This simple observation leads to the following.

Lemma 3. Let X and Y be G-spaces with X, Y, G given by (17), *then the following two properties are equivalent:*

Property (α). *The vector bundle $\xi = (X \times_G Y, X/G, \pi)$ admits no non-zero section.*

Property (β). *If f: $X \to Y$ is a G-map, then $f^{-1}(0) \neq \varnothing$.*

Thus the Borsuk–Ulam property (β) has been transformed into an equivalent vector bundle property. Since under the map $[Q] \mapsto \langle Q^T \rangle$ the quotient space $O^{n,n-k}/O_{n-k}$ is identified with the Grassmanian $G^{n,n-k}$ of $n-k$-dimensional subspaces of R^n, our next vector bundle problem should seem most natural. Consider the canonical vector bundle $\gamma^{n,n-k} = (\Gamma^{n,n-k}, G^{n,n-k}, \pi')$ with total space $\Gamma^{n,n-k} = \{(L, v) \in G^{n,n-k} \times R^n \mid v \in L\}$ and the k-fold Whitney sum

$$\gamma_k^{n,n-k} = \gamma^{n,n-k} \oplus \cdots \oplus \gamma^{n,n-k} = (\Gamma_k^{n,n-k}, G^{n,n-k}, \pi')$$

where

$$\Gamma_k^{n,n-k} = \{(L, v) \in G^{n,n-k} \times R^{nk} \,|\, v = (v_1, \ldots, v_k), v_i \in L, i = 1, \ldots, k\}.$$

Theorem C′ (vector bundle). The vector bundle $\gamma_k^{n,n-k}$ admits no non-zero section.

Theorem D′. Theorems A′, B′ and *C′ are equivalent.*

Proof. (*i*) (A′⇒C′). Let $h: G^{n,k} \to G^{n,n-k}$ be the homeomorphism $h(L) = L^{\perp}$ and let $\sigma_{\tilde{\psi}}: G^{n,n-k} \to \Gamma_k^{n,n-k}$ be a section of $\gamma_k^{n,n-k}$, $\sigma_{\tilde{\psi}}(L) = (L, \tilde{\psi}(L))$ with $\tilde{\psi}(L) = (\tilde{\psi}_1(L), \ldots, \tilde{\psi}_k(L))$. The section $\sigma_{\tilde{\psi}}$ induces a function $\psi = (\psi_1, \ldots, \psi_k)$, $\psi_i: G^{n,k} \to R^n$, $i = 1, \ldots, k$ defined by $\psi_i = \tilde{\psi}_i \circ h$. By Theorem A′ there exists $\bar{L} \in G^{n,k}$ such that $\psi_i(\bar{L}) \in \bar{L}$, $i = 1, \ldots, k$. Since $\psi_i(\bar{L}) \in \bar{L}^{\perp} \Rightarrow \psi_i(\bar{L}) = 0 \Rightarrow \tilde{\psi}_i(\bar{L}^{\perp}) = 0$, $i = 1, \ldots, k$, $\Rightarrow \sigma_{\tilde{\psi}} \cap \sigma_0 \neq \varnothing$. (*C′*⇒A′). The map $\psi: G^{n,k} \to R^{nk}$ induces a section $\sigma_{\tilde{\psi}}: G^{n,n-k} \to \Gamma_k^{n,n-k}$ by defining $\tilde{\psi}_i = \pi_L \circ \psi_i \circ h^{-1}$, $i = 1, \ldots, k$, $\tilde{\psi} = (\tilde{\psi}_1, \ldots, \tilde{\psi}_k)$, $\sigma_{\tilde{\psi}}(L) = (L, \tilde{\psi}(L))$. By Theorem C′ there exists $\bar{L}$ such that $\tilde{\psi}(\bar{L}) = 0 \Rightarrow \psi_i(\bar{L}^{\perp}) \in \bar{L}^{\perp}$, $i = 1, \ldots, k$.

(ii) (B′⇔C′). By Lemma 3 it suffices to note that the vector bundles

$$\xi = (O^{n,n-k} \times_{O_{n-k}} R^{k(n-k)}, O^{n,n-k}/O_{n-k}, \pi) \quad \text{and}$$

$$\gamma_k^{n,n-k} = [\Gamma_k^{n,n-k}, G^{n,n-k}, \pi')$$

are isomorphic. The isomorphism is constructed in the obvious way:

$$\begin{array}{ccc} O^{n,n-k} \times_{O_{n-k}} R^{k(n-k)} & \xrightarrow{\quad g \quad} & \Gamma_k^{n,n-k} \\ & \pi \searrow \quad \swarrow \pi' & \\ & G^{n,n-k} & \end{array}$$

For any $[Q] \in O^{n,n-k}/O_{n-k}$ there exists a unique $L = \langle Q^T \rangle \in G^{n,n-k}$. For any $v \in R^{k(n-k)}$ and $L \in G^{n,n-k}$ there exists a unique $v' \in L \oplus \cdots \oplus L$ (k-fold). Let $g: O^{n,n-k} \times_{O_{n-k}} R^{k(n-k)} \to \Gamma_k^{n,n-k}$ be defined by $g([Q, v]) = (\langle Q^T \rangle, v')$. g is an isomorphism of ξ and $\gamma_k^{n,n-k}$. Thus if ξ admits no non-zero section, then $\gamma_k^{n,n-k}$ admits no non-zero section and conversely. □

The subspace existence property can thus be thought of as reducing to the topological property that the k-fold Whitney sum of canonical vector bundles $\gamma_k^{n,n-k}$ admits no non-zero sections.

5. Proof of subspace fixed-point property

In this section we prove the subspace Theorems A′, B′ and C′. The approach we use allows us to prove a more general version of these theorems: this generalisation is most readily explained in terms of the Borsuk–Ulam property (β) of Lemma 3. If we replace the orthogonal group O_{n-k} in eq. (17) by the subgroup T_{n-k} of unit diagonal matrices

$$T_{n-k} = \left\{ \begin{bmatrix} \tau_1 & & 0 \\ & \ddots & \\ 0 & & \tau_{n-k} \end{bmatrix} \middle| \; \tau_i \in \{1, -1\}, \quad i=1,\dots,k \right\},$$

then Lemma 3 remains valid. The quotient map $\pi_G: O^{n,n-k} \to O^{n,n-k}/G$ which with $G=O_{n-k}$ leads to the Grassmanian $G^{n,n-k}$, under the group $G=T_{n-k}$ leads to the *flag manifold* $F^{n,n-k}$ of all $n-k$ mutually orthogonal one-dimensional subspaces (lines through the origin) in R^n,

$$F^{n,n-k} = \{l \mid l=(l_1,\dots,l_{n-k}), \quad l_i \text{ a line through 0 in } R^n, \quad l_i \perp l_j, \; i \neq j\}.$$

An equivalence class $[Q] \in O^{n,n-k}/T_{n-k}$ generates an $n-k$ flag $l=(l_1,\dots,l_{n-k}) \in F^{n,n-k}$ as follows. Write $Q^T=(Q_i^T)_{i=1}^{n-k}$ as $n-k$ column vectors $Q_i^T \in R^n$, then each column generates a line through the origin in R^n, $l_i = \langle Q_i^T \rangle$ and since the columns $(Q_i^T)_{i=1}^{n-k}$ are mutually orthogonal the map $[Q] \mapsto (\langle Q_1^T \rangle, \dots, \langle Q_{n-k}^T \rangle)$ identifies the quotient space $O^{n,n-k}/T_{n-k}$ with the flag manifold $F^{n,n-k}$ of $n-k$ flags in R^n. For any k flag $k \in F^{n,k}$ let $\langle l \rangle$ denote the k-dimensional subspace spanned by l, then Theorems A′ and B′ generalise as follows.

Theorem A (flag manifold). If $\Psi: F^{n,k} \to R^{nk}$ is a continuous function, then there exists $\bar{l} \in F^{n,k}$ such that $\langle \Psi(\bar{l}) \rangle \subset \langle \bar{l} \rangle$.*

Theorem B (general Borsuk–Ulam). If $\psi: O^{n,n-k} \to (R^{n-k})^k$ is a T_{n-k} map, then $\psi^{-1}(0) \neq \varnothing$.*

Theorem B* is the true generalisation of the classical Borsuk–Ulam theorem [Guillemin and Pollack (1974, pp. 91–93)]. When $n-k=1$, $O^{n,1}=S^{n-1}$, the $n-1$-dimensional sphere, Theorem B* reduces to the statement that *if $\psi: S^{n-1} \to R^{n-1}$ is a continuous function satisfying $\psi(-Q) = -\psi(Q)$ for all $Q \in S^{n-1}$, then there exists $\bar{Q} \in S^{n-1}$ such that $\psi(\bar{Q})=0$*, which is the Bursuk–Ulam theorem.

Consider the vector bundle $\varepsilon_k^{n,k} = (E_k^{n,k}, F^{n,k}, \pi)$ with total space

$$E_k^{n,k} = \{(l, v) \in F^{n,k} \times R^{nk} \mid v=(v_1,\dots,v_k) \quad v_i \in \langle l \rangle^{\perp}, \quad \mathrm{i}=1,\dots,k\}.$$

Theorem C (flag vector bundle). The vector bundle $\varepsilon_k^{n,k}$ admits no non-zero section.*

We leave it as an exercise for the reader to show

Theorem D. Theorems A*, B* and C* are equivalent.*

Remark 8. We shall prove Theorem B* using the methods of algebraic topology. The reader should recall the following basic facts from *cohomology theory.*[8] Let M be a manifold of dimension m (more generally a topological space) and let $S^k(M;Z_2)$ denote the space of singular cochains of dimension k on M with coefficients in Z_2. A boundary operator $\delta_k: S^k \to S^{k+1}$ is defined satisfying $\delta_{k+1} \circ \delta_k = 0$. The modules of cocyles $Z^k(M;Z_2) = \ker \delta_k = \{w \in S^k | \delta_k w = 0\}$ and coboundaries $B^k(M;Z^2) = \operatorname{Im} \delta_{k-1} = \{w \in S^k | w = \delta_{k-1} b, b \in S^{k-1}\}$ lead to the kth *cohomology module*

$$H^k(M;Z_2) = Z^k(M;Z_2)/B^k(M;Z_2) \quad \text{and} \quad H^*(M) = \bigoplus_{k=0}^{m} H^k(M),$$

where H^* is the graded module formed from the direct sum of the modules H^k. Introducing the *cup product* of cochains $c \cup c'$ from $S^k(M) \times S^q(M) \to S^{k+q}(M)$ leads to a multiplicative structure on $H^*(M)$ which makes it into a graded ring called the *cohomology ring* of M over Z_2. The proof depends explicitly on the additional structure induced in $H^*(M)$ by the cup product of cochains.

Proof of Theorem B.* By Lemma 3 this is equivalent to proving that the vector bundle $\xi = (O^{n,n-k} \times_{T_{n-k}} R^{k(n-k)}, O^{n,n-k}/T_{n-k}, \pi)$ admits no non-zero section. Suppose not, namely suppose ξ admits a non-zero section σ_ψ with $\sigma_\psi[Q] = [Q, \psi(Q)]$. Then σ_ψ induces a section $\sigma[Q] = [Q, (\psi(Q)/\|\psi(Q)\|)]$ of the sphere bundle $\zeta = (O^{n,n-k} \times_{T_{n-k}} S^{\mu-1}, O^{n,n-k}/T_{n-k}, \pi)$ with $\mu = k(n-k)$.

Consider the universal bundle for T_{n-k}, $(ET_{n-k}, BT_{n-k}, P_k)$ where $ET_{n-k} = O^{\infty,n-k}$, $BT_{n-k} = F^{\infty,n-k}$ are the infinite Stiefel and flag manifolds respectively.[9] Since $T_{n-k} = O_1 \times \cdots \times O_1$ ($n-k$ times), $BT_{n-k} \cong BO_1 \times \cdots \times BO_1$ so that $H^*(BT_{n-k}; Z_2) \cong H^*(BO_1; Z_2) \otimes \cdots \otimes H^*(BO_1; Z_2) = Z_2[t_1, \ldots, t_{n-k}]$ where the latter denotes the ring of polynomials in $n-k$ letters with coefficients in Z_2. This is the universal example from which we construct the other cohomology rings.

Consider the embeddings i': $O^{n,n-k} \to O^{\infty,n-k}$, $i: F^{n,n-k} \to F^{\infty,n-k}$ and the

[8]Good introductions are Dieudonné (1982), Dold (1972), Munkres (1984); the standard reference is Spanier (1966); an excellent intuitive survey, especially for a geometric treatment of the Euler class, is Bott and Tu (1982).

[9]$O^{\infty,m}$ is topologised as the direct limit of the sequence $O^{n,m} \subset O^{n+1,m} \subset \ldots$. Thus a subset of $O^{\infty,m}$ is open if and only if its intersection with $O^{n,m}$ is open as a subset of $O^{n,m}$ for $n = 1, 2, \ldots$: similarly for $F^{\infty,m}$ [see Milnor and Stasheff (1974, p.63)].

sphere bundle $\tau_\infty = (O^{\infty,n-k} \times_{T_{n-k}} S^{\mu-1}, F^{\infty,k}, \pi_\infty)$, then we have the commutative diagram

$$\text{(D)}\qquad \begin{array}{ccc} O^{n,n-k} \times_{T_{n-k}} S^{\mu-1} & \xrightarrow{i'} & O^{\infty,n-k} \times_{T_{n-k}} S^{\mu-1} \\ \pi \downarrow & & \downarrow \pi_\infty \\ F^{n,n-k} & \xrightarrow{i} & F^{\infty,n-k} \end{array}$$

Applying the cohomology functor gives the cummutative diagram

$$\text{(D*)}\qquad \begin{array}{ccc} H^*(O^{n,n-k} \times_{T_{n-k}} S^{\mu-1}) & \xleftarrow{i'^*} & H^*(O^{\infty,n-k} \times_{T_{n-k}} S^{\mu-1}) \\ \pi^* \uparrow & & \uparrow \pi_\infty^* \\ H^*(F^{n,n-k}) & \xleftarrow{i^*} & H^*(F^{\infty,n-k}) \end{array}$$

We use this diagram to show that the existence of a section for the sphere bundle ζ leads to a contradiction. We know the nature of the cohomology ring $H^*(F\infty,n-k)$: we want to find the algebraic way of writing the cohomology rings $H^*(O^{\infty,n-k} \times_{T_{n-k}} S^{\mu-1})$ and $H^*(F^{n,n-k})$. The next two lemmas provide the answer: the proofs are given below.

Lemma (a). The projection $\pi_\infty: O^{\infty,n-k} \times_{T_{n-k}} S^{\mu-1} \to F^{\infty,n-k}$ induces an isomorphism*

$$Z_2[t_1,\dots,t_{n-k}]/\mathcal{M} \cong H^*(O^{\infty,n-k} \times_{T_{n-k}} S^{\mu-1})$$

of algebras, where $\mathcal{M}$ is the ideal generated by $(t_1,\dots,t_{n-k})^k$.

Lemma (b). The embedding $i: F^{n,n-k} \to F^{\infty,n-k}$ induces an isomorphism*

$$Z_2[t_1,\dots,t_{n-k}]/\mathcal{N} \cong H^*(F^{n,n-k})$$

where $\mathcal{N}$ is the ideal generated by the polynomials $p_1,\dots,p_{n-k}$ where

$$p_i = t_i^{n-i+1} + a_{i,n-i} t^{n-i} + \cdots + a_{i,0}, \tag{18}$$

each a_{ij} being a polynomial in $t_1,\dots,t_{i-1}$ with $a_{i,j}t^j$ being of degree $n-i+1$.

Lemmas (a*) and (b*) imply the following two properties

(i) $(t_1,\dots,t_{n-k})^k \in \ker \pi_\infty^*$

(ii) $(t_1,\dots,t_{n-k})^k \notin \ker i^*$

The existence of a section $\sigma\colon F^{n,n-k}\to O^{n,n-k}\times_{T_{n-k}} S^{\mu-1}$ implies

(iii) *the homomorphism* π^* *is injective.*

The commutativity of (D*) together with (iii) show that (i) and (ii) contradict each other. □

Proof of Lemma (*a*). τ_∞ is the sphere bundle derived from the vector bundle $\xi_\infty=(O^{\infty,n-k}\times_{T_{n-k}} R^\mu, F^{\infty n,n-k},\pi)$. This vector bundle can be written as the k-fold Whitney sum of the bundle $\eta=(O^{\infty,n-k}\times_{T_{n-k}} R^{n-k}, F^{\infty,n-k},\pi')$, $\xi_\infty=\eta\oplus\cdots\oplus\eta$. The Euler class $e(\eta)$ of η is the element of $H^{n-k}(F^{\infty,n-k})$ given by $e(\eta)=t_1,\dots,t_{n-k}$. Since the Euler class of a Whitney sum is the cup product of the Euler classes $e(\xi_\infty)=e(\eta)\cup\cdots\cup e(\eta)=(t_1,\dots,t_{n-k})^k$. Consider the Thom–Gysin exact sequence [Spanier (1966, p. 260)] for the sphere bundle τ_∞

$$\cdots\to H^i(F^{\infty,n-k})\xrightarrow{\cup e(\xi_\infty)} H^{i+n-k}(F^{\infty,n-k})\to H^{i+n-k}(O^{\infty,n-k}\times_{T_{n-k}} S^{\mu-1})$$
$$\to H^{i+1}(F^{\infty,n-k})\to\cdots \tag{19}$$

Since the polynomial ring $Z_2[t_1,\dots,t_{n-k}]$ has no zero divisors (i.e., there do not exist $f,g\in Z_2[t_1,\dots,t_{n-k}]$, $f\neq 0, g\neq 0$ such that $fg=0$) the homomorphism $g\mapsto g\cup e(\xi_\infty)$ is injective: thus the long exact sequence (19) breaks into short exact sequences

$$0\to H^i(F^{\infty,n-k})\xrightarrow{\cup e(\xi_\infty)} H^{i+n-k}(F^{\infty,n-k})$$
$$\to H^{i+n-k}(O^{\infty,n-k}\times_{T_{n-k}} S^{\mu-1})\to 0.$$

It follows that $H^*(O^{\infty,n-k}\times_{T_{n-k}} S^{\mu-1})\cong H^*(F^{\infty,n-k})/\mathcal{M}$ where $\mathcal{M}$ is the ideal generated by $(t_1,\dots,t_{n-k})^k$. □

Proof of Lemma (*b*). We prove the lemma by induction on $n-k$. If $n-k=1$, since $F^{n,1}=\mathbb{P}^{n-1}$, real $n-1$-dimensional projective space, by a standard result [Spanier (1966, p. 264)] $H^*(\mathbb{P}^{n-1};Z_2)\cong Z_2[t_1]/(p_1)$ where (p_1) is the ideal generated by t_1^n. Suppose the result holds for $n-k-1$ so that

$$H^*(F^{n,n-k-1};Z_2)\cong Z_2[t_1,\dots,t_{n-k-1}]/(p_1,\dots,p_{n-k-1}).$$

Consider the fibration $\pi_k\colon F^{n,n-k}\to F^{n,n-k-1}$ induced by projection on the last

$n-k-1$ factors, $(l_1,\dots,l_{n-k})\to(l_2,\dots,l_{n-k})$. The fiber is $F^{k+1,1}=\mathbb{P}^k$. Since $\mathbb{P}^k$ is universally totally non-homologous to 0, the Leray–Hirsch theorem [Spanier (1966, p. 258)] applies and we have the isomorphism of $H^*(F^{n,n-k-1})$-modules:

$$H^*(F^{n,n-k})\cong H^*(F^{n,n-k-1})\otimes H^*(\mathbb{P}^k).$$

Note that $t_{n-k}\in H^1(F^{\infty,n-k})$ gives rise to an element in $H^1(F^{n,n-k})$ also denoted by t_{n-k} and $t_{n-k}|_{\mathbb{P}^k}$ generates $H^*(\mathbb{P}^k)$. This implies that t_{n-k} satisfies the relation

$$p_{n-k}(t_{n-k})=t_{n-k}^{k+1}+a_{n-k,k}(t_1,\dots,t_{n-k-1})t^k+\cdots+a_{n-k,0}(t_1,\dots,t_{n-k-1})$$

where each $a_{n-k,j}$ is a polynomial[10] in $t_1,\dots,t_{n-k-1}$ of degree $k+1-j$, which completes the proof. □

6. Proof of fixed-point theorem

The fixed-point Theorems A and B can be generalised as in section 5 by replacing the Grassmanian $G^{n,k}$ by the flag manifold $F^{n,k}$ in Theorem A and the orthogonal group O_{n-k} by the unit diagonal group T_{n-k} in Theorem B. We leave it as an exercise to prove that the resulting two theorems are equivalent.

*A** Flag manifold fixed-point theorem. Let H^m be an m-dimensional affine subspace, $C\subset H^m$ a compact convex subset with non-empty relative interior. Let (Φ,Ψ) be continuous functions $\Phi: C\times F^{n,k}\to H^m$, $\Psi: C\times F^{n,k}\to R^{nk}$ such that $\Phi(\partial C,l)\subset C\ \forall l\in F^{n,k}$, then there exist $(\bar p,\bar l)\in C\times F^{n,k}$ such that $\Phi(\bar p,\bar l)=\bar p$, $\langle\Phi(\bar p,\bar l)\rangle\subset\langle l\rangle$.*

*B** General Borsuk–Ulam theorem. Let H^m, C be as in A** and let (ϕ,ψ) be continuous functions $\phi: C\times O^{n,n-k}\to H^m$, $\psi: C\times O^{n,n-k}\to(R^{n-k})^k$ such that $\phi(\partial C,Q)\subset C\ \forall Q\in O^{n,n-k}$, $\phi(p,gQ)=\phi(p,Q)$, $\psi(p,gQ)=g\psi(p,Q)\ \forall g\in T_{n-k}$, $\forall(p,Q)\in C\times O^{n,n-k}$, then there exists $(\bar p,\bar Q)\in C\times O^{n,n-k}$ such that $\phi(\bar p,\bar Q)=\bar p$, $\psi=(\bar p,\bar Q)=0$.*

*Proof of Theorem B**. Step 1.* We establish the result first in the simpler case where $H^m=R^m$, $C=D^m=B(0,1)$ and ϕ satisfies

$$\phi(\partial B(0,1),Q)\subset B(0,1-\varepsilon)\quad \forall Q\in O^{n,n-k} \tag{20}$$

[10] The polynomials $a_{n-k,j}$ are the *Stiefel–Whitney classes* of the vector bundle $\varepsilon_1^{n,n-k-1}$.

for some $\varepsilon > 0$. Suppose therefore that the conclusion is false, namely

$$(\phi(p,Q),\psi(p,Q)) \neq (p,0) \quad \forall (p,Q) \in C \times O^{n,n-k}. \tag{21}$$

Whenever $\phi(p,Q) \neq p$ the line segment $\lambda\phi(p,Q)+(1-\lambda)p$ joining ϕ and p is well-defined. Thus on the set $W = \{(p,Q) \in B(0,1) \times O^{n,n-k} \mid \phi(p,Q) \neq p\}$ we can define $\phi_0(p,Q) = \rho$ where ρ is the unique point on the segment $\lambda\phi(p,Q)+(1-\lambda)p$, $\lambda \leqq 0$, satisfying $\rho \in \partial B(0,1)$. To arrive at a function defined on all of $B(0,1) \times O^{n,n-k}$ we proceed as follows. Let

$$\delta = \inf\{\|p-\phi(p,Q)\| + \|\psi(p,Q)\| \mid (p,Q) \in B(0,1) \times O^{n,n-k}\},$$

then (21) implies $\delta > 0$. Let $\theta: R_+ \to [0,1]$ be any continuous function satisfying $\theta(s) = 0$, $s \in [0,\delta_0]$, $\theta(s) = 1$, $s \in [2\delta_0, \infty)$ where $\delta_0 = \frac{1}{4}\inf(\varepsilon,\delta)$ so that $\theta(\varepsilon) = \theta(\delta) = 1$. Then the function

$$f(p,Q) = \theta(\|p-\phi(p,Q)\|)\phi_0(p,Q)$$

is defined and continuous on all of $B(0,1) \times O^{n,n-k}$ and has values in R^m. We note that f has the two properties

(a) $f(p,Q) = p \; \forall p \in \partial B(0,1), \quad \forall Q \in O^{n,n-k}$,

(b) $f(p,gQ) = f(p,Q) \; \forall g \in O^{n-k}, \forall (p,Q) \in B(0,1) \times O^{n,n-k}$,

where (a) follows from the fact that $p \in \partial B(0,1) \Rightarrow \|p-\phi(p,Q)\| \geqq \varepsilon \Rightarrow \theta(\varepsilon) = 1$ and $\phi_0(p,Q) = p$, $\forall p \in B(0,1)$. The pair of functions (f,ψ) satisfy the conditions of Theorem E below. Thus there exists $(\bar{p},\bar{Q}) \in B(0,1) \times O^{n,n-k}$ such that $f(\bar{p},\bar{Q}) = 0$, $\psi(\bar{p},\bar{Q}) = 0$. But then $\|\bar{p} - f(\bar{p},\bar{Q})\| \geqq \delta$ and since $\theta(\delta) = 1$, $f(\bar{p},\bar{Q}) = \phi_0(\bar{p},\bar{Q}) \in \partial B(0,1)$ which is a contradiction.

Step 2. For each integer $v \geqq 2$ define $\phi_v(p,Q) = (1-1/v)\phi(p,Q)$. Then the hypothesis of Theorem B** implies that ϕ_v satisfies (20) with $\varepsilon = 1/v$. Thus for each $v \geqq 2$ we can apply step 1 to (ϕ_v,ψ) to obtain a sequence $\{p_v,Q_v\} \subset B(0,1) \times O^{n,n-k}$ satisfying $\phi_v(p_v,Q_v) = p_v, \psi(p_v,Q_v) = 0$. By the compactness of $B(0,1) \times O^{n,n-k}$ we can select a convergent subsequence $\{p_m,Q_m\}$ where $(p_m,Q_m) \to (\bar{p},\bar{Q})$. By the continuity of (ϕ,ψ), $\phi(\bar{p},\bar{Q}) = \bar{p}$, $\psi(\bar{p},\bar{Q}) = 0$.

Step 3. Since C is a compact convex subset and $\operatorname{int} C \neq \varnothing$ there exists a homeomorphism $\alpha: H^m \to R^m$ such that $\alpha(C) = B(0,1)$. Step 2 can now be applied to the pair of functions

$$(\tilde{\phi}(p,Q),\tilde{\psi}(p,Q)) = (\alpha(\phi(\alpha^{-1}(p),Q)),\psi(\alpha^{-1}(p),Q))$$

yielding the solution $(\tilde{p},\tilde{Q}) \in B(0,1) \times O^{n,n-k}$. Then $(\bar{p},\bar{Q}) = (\alpha^{-1}(\tilde{p}),\tilde{Q})$ is the desired solution for (ϕ,ψ). □

These first three steps have reduced the proof of Theorem B** to the proof of the following theorem.

Theorem E. *Let* (f,ψ) *be continuous functions* $f: D^m \times O^{n,n-k} \to R^m$, $\psi: D^m \times O^{n,n-k} \to (R^{n-k})^k$ *such that* $f(p,Q)=p$, $\forall p \in \partial D^m$, $\forall Q \in O^{n,n-k}$, $f(p,gQ)=f(p,Q)$, $\psi(p,gQ)=g\psi(p,Q)$, $\forall g \in T_{n-k}$, $\forall (p,Q) \in D^m \times O^{n,n-k}$, *then there exists* $(\bar{p},\bar{Q}) \in D^m \times O^{n,n-k}$ *such that* $f(\bar{p},\bar{Q})=0$, $\psi(\bar{p},\bar{Q})=0$.

Remark 9. The basic idea of the proof that follows is identical to that used in proving Theorem B*. The proof 'appears' more complicated because of the need to cope with the Brouwer component f, which leads us to introduce the *relative cohomology* H^* (X,A) of pairs of spaces (X,A) where A is a subset of X.

Proof. Step 1. The problem can be reduced to the action of a T_{n-k} map between a pair of spaces as follows. Define the action of T_{n-k} on $D^m \times O^{n,n-k}$ by $(p,Q) \in D^m \times O^{n,n-k}$ implies $g(p,Q)=(p,gQ)$ and define the action of T_{n-k} on $R^m \times (R^{n-k})^k$ by $(\eta,\xi) \in R^m \times (R^{n-k})^k$ implies $g(\eta,\xi)=(\eta,g\xi)$ where g is the diagonal action on $(R^{n-k})^k$. Then the hypothesis of the behaviour of (f,ψ) under the action of T_{n-k} becomes

$$(f(g(p,Q)),\psi(g(p,Q)))=(f(p,gQ),\psi(p,gQ))=(f(p,Q),g\psi(p,Q))$$

$$=g(f(p,Q),\psi(p,Q))$$

so that $h=(f,\psi)$ is a T_{n-k} map. Recall: (i) (X,A) and (Y,B) are *pairs of spaces* if $A \subset X$ and $B \subset Y$, (ii) h is a map between pairs of spaces $h:(X,A) \to (Y,B)$ if $h: X \to Y$ such that $h(A) \subset B$. If we let

$$\begin{aligned} &(X,A)=(D^m \times O^{n,n-k}, \partial D^m \times O^{n,n-k}), \\ &(Y,B)=(R^m \times R^\mu, \partial D^m \times R^\mu), \quad \mu=(n-k)k, \end{aligned} \tag{22}$$

then the boundary behaviour of f, namely $f|_{\partial D^m \times O^{n,n-k}}=\pi_{\partial D^m}$, implies that h: $(X,A) \to (Y,B)$ is a T_{n-k} map.

Step 2. Suppose therefore that $h^{-1}(0,0)=\emptyset$, then there is a T_{n-k} map h': $(X,A) \to (Y',B)$ where $Y'=Y \backslash 0$. As in the proof of Theorem B* let $(ET_{n-k}, BT_{n-k}, p_{n-k})$ denote the universal bundle for T_{n-k} with $ET_{n-k}=O^{\infty,n-k}$, $BT_{n-k}=F^{\infty,n-k}$ and define

$$H^*_{T_{n-k}}(X,A;Z_2)=H^*(ET_{n-k} \times_{T_{n-k}} (X,A);Z_2). \tag{23}$$

Using the cup product this Z_2-module can be extended[11] to an $\mathscr{H}^*$-module where $\mathscr{H}^* = H^*(F^{\infty,n-k}; Z_2)$. Applying the cohomology in (23), h' induces a homomorphism of $\mathscr{H}^*$-modules

$$H^*_{T_{n-k}}(Y', B) \xrightarrow{h'^*} H^*_{T_{n-k}}(X, A). \tag{24}$$

Step 3. Consider the embedding $(Y'', B'') \xrightarrow{i} (Y', B)$ where $(Y'', B'') = (S^{m+\mu-1}, S^{m-1})$. Since Y'' is a deformation retract of Y' and B'' is a deformation retract of B [Spanier (1966, p. 30)] and since homotopic spaces have isomorphic cohomology modules, $H^*_{T_{n-k}}(Y') = H^*_{T_{n-k}}(Y'')$, $H^*_{T_{n-k}}(B) = H^*_{T_{n-k}}(B'')$. Consider the long exact cohomology sequences [Spanier (1966, p. 240)] of the pairs (Y', B) and (Y'', B'')

$$\begin{array}{ccccccccc} \cdots \to & H^i_{T_{n-k}}(Y') & \to & H^i_{T_{n-k}}(B) & \xrightarrow{\partial^*} & H^{i+1}_{T_{n-k}}(Y', B) & \to \cdots \\ & \downarrow \cong & & \downarrow \cong & & \downarrow & \\ \cdots \to & H^i_{T_{n-k}}(Y'') & \to & H^i_{T_{n-k}}(B'') & \xrightarrow{\partial^*} & H^{i+1}_{T_{n-k}}(Y'', B'') & \to \cdots \end{array}$$

Applying Steenrod's five lemma [Spanier (1966, p. 185)] implies $H^{i+1}_{T_{n-k}}(Y', B) \cong H^{i+1}_{T_{n-k}}(Y'', B'')$ so that

$$H^*_{T_{n-k}}(Y', B) \cong H^*_{T_{n-k}}(Y'', B''). \tag{25}$$

Step 4. The following two lemmas generalise Lemmas a^* and b^* in the proof of Theorem B*; the proofs are given below.

*Lemma a**.* $H^*_{T_{n-k}}(Y'', B'') \cong H^m(D^m, \partial D^m) \otimes Z_2[t_1, \ldots, t_{n-k}]/\mathscr{M}$ *is an* $\mathscr{H}^*$-*isomorphism where* $\mathscr{M}$ *is the ideal generated by* $(t_1, \ldots, t_{n-k})^k$.

*Lemma b**.* $H^*_{T_{n-k}}(X, A) \cong H^m(D^m, \partial D^m) \otimes Z_2[t_1, \ldots, t_{n-k}]/\mathscr{N}$ *is an* $\mathscr{H}^*$-*isomorphism where* $\mathscr{N}$ *is the ideal generated by the polynomials* $p_1, \ldots, p_{n-k}$ *in Lemma b*.*

Eq. (25), Lemmas a^* and b^* imply that (24) becomes

$$H^m(D^m, \partial D^m) \otimes Z_2[t_1, \ldots, t_{n-k}]/\mathscr{M} \xrightarrow{h'^*} H^m(D^m, \partial D^m) \otimes Z_2[t_1, \ldots, t_{n-k}]/\mathscr{N}.$$

[11] Pick $\xi \in X$ and let $c(x) = \xi \; \forall x \in X$ so that $c: X \to \xi$. Then $c^*: H^*_{T_{n-k}}(\xi, \varnothing) \to H^*_{T_{n-k}}(X, A)$ with $H^*_{T_{n-k}}(\xi, \varnothing) = \mathscr{H}^*$. For $x \in H^*_{T_{n-k}}(X, A)$, $\alpha \in \mathscr{H}^*$ define $\alpha \cdot x = c^*(\alpha) \cup x$, then $H^*_{T_{n-k}}(X, A)$ becomes an $\mathscr{H}^*$-module.

Writing $(X,A)=(D^m,\partial D^m)\times O^{n,n-k}$ and recalling the assumptions made on h'^* maps $H^m(D^m,\partial D^m)$ to itself identically. Since h'^* is an $\mathscr{H}^*$-homomorphism, $\mathscr{M}\subset\mathscr{N}$ which is impossible. □

*Proof of Lemma a**.* Note that

$$H^*_{T_{n-k}}(Y'',B'')\cong H^*_{T_{n-k}}(S^{m+\mu-1},S^{m+\mu-1}\backslash S^{\mu-1})$$

since $S^{m+\mu-1}\cong S^{\mu-1}*S^{m-1}$ and $S^{m-1}\subset S^{\mu+m-1}\backslash S^{\mu-1}$ is a deformation retract. But $H^*_{T_{n-k}}(S^{m+\mu-1},S^{m+\mu-1}\backslash S^{\mu-1})\cong H^*_{T_{n-k}}((D^m,\partial D^m)\times S^{m-1})$ by Excision. By the Künneth formula [Spanier (1966, p. 249)] the latter term is isomorphic to $H^m(D^m,\partial D^m)\otimes H^*_{T_{n-k}}(S^{\mu-1})$. Applying Lemma a* gives the result. □

*Proof of Lemma b**.* Since T_{n-k} acts trivially on D^m we have the homeomorphism $ET_{n-k}\times_{T_{n-k}}((D^m,\partial D^m)\times O^{n,n-k})\cong(D^m,\partial D^m)\times(ET_{n-k}\times_{T_{n-k}}O^{n,n-k})$. Since T_{n-k} acts freely on $O^{n,n-k}$, the projection $ET_{n-k}\times O^{n,n-k}\to O^{n,n-k}$ induces a homotopy equivalence $ET_{n-k}\times_{T_{n-k}}O^{n,n-k}\cong O^{n,n-k}/T_{n-k}$. Identifying the latter space with $F^{n,n-k}$ and applying Lemma b* completes the proof. □

Appendix

The following lemma was used in constructing the price adjustment functions Φ and ϕ in section 3. It suffices to establish the result in the Stiefel manifold case.

Lemma 4. There exists a continuous function $\delta:\Delta^{r-1}_{+}\times O^{n,n-k}\to[0,1]$ such that the function $\phi:\Delta^{r-1}_{+}\times O^{n,n-k}\to E^{r-1}$ defined by

$$\phi(p,Q)=\delta(p,Q)(p+p\,\square\,z(p,Q))+(1-\delta(p,Q))u \tag{26}$$

where $u=(1/r,\dots,1/r)$ satisfies

(i) $\phi(p,Q)=p\Leftrightarrow z(p,Q)=0$,

(ii) $\phi(\partial\Delta^{r-1}_{+},Q)\subset\Delta^{r-1}_{+}\ \forall Q\in O^{n,n-k}$,

(iii) $\phi(p,gQ)=\phi(p,Q)\ \forall g\in O_{n-k},\forall(p,Q)\in\Delta^{r-1}_{+}\times O^{n,n-k}$.

Proof. The basic construction follows Dierker (1974, p. 79). We need to check that the Q-dependence of z does not create a problem. Let

$$v_j=\left\{(p,Q)\in\Delta^{r-1}_{++}\times O^{n,n-k}\,|\,z_j(p,Q)>0,\ p_j<\frac{1}{r}\right\},\qquad j=1,\dots,r.$$

Since $z_j(p,gQ)=z_j(p,Q)\ \forall g\in O_{n-k},\forall(p,q)\in\Delta^{r-1}_{+}\times O^{n,n-k}$, each set v_j is

O_{n-k}-invariant $((p,gQ)\in v_j \Leftrightarrow (p,Q)\in v_j,\ \forall g\in O_{n-k})$. We claim that candidate equilibrium (p,Q) pairs will lie in the set

$$K=(\Delta_{++}^{r-1}\times O^{n,n-k})\Big\backslash \bigcup_{j=1}^{n} v_j.$$

By a standard result in demand theory, under Assumption U, for any sequence $p^m \to \bar{p}\in\partial\Delta_+^{r-1}$, $\|F^1(p^m;p^m w^1)\|\to\infty$. Since $f^i\geqq 0$, $i=1,\ldots,m$, $\|z(p^m, Q)\|\to\infty$. Since $z(\cdot)$ is continuous and $O^{n,n-k}$ is compact, there is no sequence $(p^m,Q^m)\in K$, $(p^m,Q^m)\to(\bar{p},\bar{Q})$ with $\bar{p}\in\partial\Delta_+^{r-1}$. Thus K is compact as a subset of $\Delta_+^{r-1}\times O^{n,n-k}$ and $K\cap(\partial\Delta_+^{r-1}\times O^{n,n-k})=\varnothing$.

Let $\alpha:\Delta_+^{r-1}\times O^{n,n-k}\to[0,1]$ be a continuous function such that $\alpha^{-1}(1)\supset K$ and $\alpha^{-1}(0)$ contains an O_{n-k}-invariant neighborhood of $\partial\Delta_+^{r-1}\times O^{n,n-k}$. To obtain an O_{n-k}-invariant map define $\delta:\Delta_+^{r-1}\times O^{n,n-k}\to[0,1]$,

$$\delta(p,Q)=\frac{1}{\mu(O_{n-k})}\int_{O_{n-k}}\alpha(p,gQ)\,\mathrm{d}\mu(g),$$

where μ is the left-invariant Haar measure on the compact group O_{n-k}. Then $\delta^{-1}(1)\supset K$, since K is O_{n-k}-invariant and $\delta^{-1}(0)$ is a neighborhood of $\partial\Delta_{++}^{r-1}\times O^{n,n-k}$, since $\alpha^{-1}(0)$ contains an O_{n-k}-invariant neighborhood of $\Delta_{++}^{r-1}\times O^{n,n-k}$ and

$$\delta(p,gQ)=\delta(p,Q),\quad \forall g\in O_{n-k},\quad \forall(p,Q)\in\Delta_{++}^{r-1}\times O^{n,n-k}.$$

Let $\phi(p,Q)$ be defined by (26), then $\sum_{j=1}^{r}\phi_j(p,Q)=1$ since $\sum_{j=1}^{r}p_j z_j(p,Q)=0$. It is clear that ϕ satisfies (ii) and (iii). It remains to establish (i). $(\Leftarrow)$ Suppose $z(p,Q)=0$, then $(p,Q)\in K$ so that $\delta=1$ and $\phi_j(p,Q)=p_i+p_j z_j(p,Q)=p_j$, $j=1,\ldots,r$. $(\Rightarrow)$ Suppose $\phi(p,Q)=p$. There are three cases (a): $(p,Q)\in K$, (b): $(p,Q)\in v_j$, for some j, (c): $(p,Q)\in\partial\Delta_{++}^{r-1}\times O^{n,n-k}$. In case (a) $\delta=1$ so that $\phi_j(p,Q)=p_j+p_j z_j(p,Q)$ and $p_j>0\Rightarrow z_j(p,Q)=0$, $j=1,\ldots,r$. In case (b) $(p,Q)\in v_j\Rightarrow p_j=\phi_j(p,Q)>p_j$ which is impossible and in case (c) $\phi(p,Q)=(1/r,\ldots,1/r)\notin\partial\Delta_{++}^{r-1}$, so only case (a) can arise. □

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