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An Equilibrium Existence Theorem*

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1. INTRODUCTION

Bewley [4, Theorem 1] proved an infinite dimensional equilibrium existence theorem which is a significant extension of the classical finite dimensional theorem of Arrow and Debreu [1]. The assumptions on technology and preferences are natural and applicable in a wide variety of cases. The proof is based on a limit argument that makes direct use of the existence of equilibrium in the finite dimensional case.

This paper establishes the existence of equilibrium under assumptions which are essentially the same as those given by Bewley, with the additional assumption that the preference orderings of consumers are representable by real valued utility functions. This approach is related to the welfare approach of Negishi [9] and Arrow and Hahn [2, Chap. 5] in the finite dimensional case and simplifies the approach originally adopted by Bewley [5]. In addition to making clear the role played by each of the assumptions in establishing the existence of equilibrium, this approach has the merit of constructing directly a certain real valued function that is maximised at an equilibrium, a result that provides a powerful tool in the analysis of qualitative properties of an equilibrium.

A model of resource allocation in continuous time over an infinite horizon that may be viewed as an application of the model that follows is given in [8].

2. THE ECONOMY

In formulating the model of the economy I shall follow the notation of Bewley [4] and Debreu [6]. The commodity space $\mathscr{Y}$ is the space of essentially bounded vector valued functions defined on a σ-finite measure space $(I, \mathscr{I}, \lambda)$

$$\mathscr{Y} = \mathscr{L}_{\infty}^{k}(I, \mathscr{I}, \lambda) = \{\xi \in \mathscr{M}^{k} \mid \text{ess sup}\, \|\xi(s)\| < \infty\},$$

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where $\mathscr{M}^k$ denotes the space of R^k-valued ($k \geqslant 1$), $\mathscr{I}$-measurable functions defined on $(I, \mathscr{I})$. Prices will be elements of the space of bounded additive set functions, the norm dual of $\mathscr{T}$

$$\mathscr{T}^* = (\mathscr{L}^k_\infty(I, \mathscr{I}, \lambda))^*.$$

Theorem 3.1 establishes the existence of an equilibrium with prices in $\mathscr{T}^*$. The reader is referred to Bewley's paper [4, Theorems 2, 3] for additional assumptions ensuring that an equilibrium can be supported by prices drawn from

$$\mathscr{T}' = \mathscr{L}^k_1(I, \mathscr{I}, \lambda) = \left\{ \eta \in \mathscr{M}^k \,\middle|\, \int_I \|\eta(s)\| \, d\lambda(s) < \infty \right\},$$

the space of R^k-valued Lebesgue integrable functions on $(I, \mathscr{I})$.

Each of the n *consumers* in the economy is characterised by a *consumption* set $X_i \subset \mathscr{T}$, a *preference ordering* $\succsim_i$ on X_i, and an *endowment* of exogenously given resources $w_i \in \mathscr{T}$ and *ownership shares* $\theta_{ij} \geqslant 0$ in the profits of firms. All profits of firms are distributed to consumers so that $\sum_{i=1}^n \theta_{ij} = 1$, $j = 1,..., m$. Each of the m *producers* is characterised by a production set $Y_j \subset \mathscr{T}$. An *allocation* for the economy is a specification of the consumption $x_i \in \mathscr{T}$ of each consumer ($i = 1,..., n$) and the production $y_j \in \mathscr{T}$ of each producer ($j = 1,..., m$). An allocation will be written as

$$(x, y) = (x_1,..., x_n, y_1,..., y_m).$$

2.1. Definition. An allocation $(x, y) \in \mathscr{T}^{n+m}$ is feasible if

$$(x, y) \in \prod_{i=1}^n X_i \times \prod_{j=1}^m Y_j \qquad \text{and} \qquad \mathbf{x} = \mathbf{y} + \mathbf{w}, \tag{1}$$

where $\mathbf{x} = \sum_{i=1}^n x_i$, $\mathbf{y} = \sum_{j=1}^m y_j$, $\mathbf{w} = \sum_{i=1}^n w_i$. If we let $M_{\mathbf{w}} = \{(x, y) \in \mathscr{T}^{n+m} \mid \mathbf{x} = \mathbf{y} + \mathbf{w}\}$, then the *set of feasible allocations*, denoted by $\mathscr{F}$, may be written as

$$\mathscr{F} = \left(\prod_{i=1}^n X_i \times \prod_{j=1}^m Y_j \right) \cap M_{\mathbf{w}}. \tag{2}$$

2.2 Definition. $(x, y) \in \mathscr{F}$ is a *Pareto optimum* if there does not exist $(x', y') \in \mathscr{F}$ such that

$$x_i' \succsim_i x_i, \ i = 1,..., n, \qquad x_k' \succ_k x_k \text{ for some } k.$$

We let $\mathscr{F}^*$ denote the set of all Pareto optimal allocations.

2.3. DEFINITION. Let $(x, y) \in \mathscr{F}$ and $p \in \mathscr{Y}^*_+$, $p \neq 0$; then (x, y, p) is an *equilibrium with transfer payments* if

(i) $x_i \succsim_i x_i' \; \forall x_i' \in \{\xi \in X_i \mid p\xi \leqslant px_i\}$, $i = 1,..., n$,

(ii) $py_j \geqslant py_j' \; \forall y_j' \in Y_j$, $j = 1,..., m$.

The vector of *transfer payments* $\Delta_p = (\Delta_{1p},..., \Delta_{np})$ is defined by

$$\Delta_{ip}(x, y) = px_i - \left(pw_i + \sum_{j=1}^{m} \theta_{ij}\, py_j\right), \qquad i = 1,..., n. \tag{3}$$

If in addition $\Delta_p = 0$, then (x, y, p) is a *competitive equilibrium.*

The consumption set, preference ordering and endowment of each consumer are subject to the following *assumptions*:

A.1. *The consumption set X_i is a convex, $\sigma(\mathscr{Y}, \mathscr{Y}')$ closed subset of $\mathscr{Y}_+$.*

A.2. *Consumer choice among commodities in X_i is determined by a preference ordering $\succsim_i$ which is complete and transitive.*

A.3. (i) *The set $\{\xi' \in X_i \mid \xi' \succsim_i \xi\}$ is convex and $\sigma(\mathscr{Y}, \mathscr{Y}')$ closed, $\forall \xi \in X_i$.*

(ii) *The set $\{\xi' \in X_i \mid \xi' \succ_i \xi\}$ is open in the norm topology of $\mathscr{Y}$, $\forall \xi \in X_i$.*

A.4. *If $\xi \in X_i$, $z \in \mathscr{Y}$, $z \gg 0$, then $\xi + z \succ_i \xi$.*

A.5. *The initial endowment of each consumer is adequate in the sense that there exists $x_i^0 \in X_i$, $y_i \in A(Y)$, where $A(Y)$ denotes the asymptotic cone of $Y = \sum_{j=1}^m Y_j$, such that*

$$x_i^0 \ll y_i + w_i.$$

A.6. *The preference ordering $\succsim_i$ is representable by a utility function $u_i: X_i \to R$ so that $\xi' \succsim_i \xi$ if and only if $u_i(\xi') \geqslant u_i(\xi)$.*

2.4 *Remark.* $z \gg 0$ means that there exists $\varepsilon > 0$ such that $z(s) \geqslant \varepsilon$, a.e. The production set of each producer and the aggregate production set are subject to the following *assumptions*:

B.1. *The production set Y_j is a $\sigma(\mathscr{Y}, \mathscr{Y}')$ closed subset of $\mathscr{Y}$ and $0 \in Y_j$.*

B.2. *$Y_j \cap (\mathscr{Y}_+ - (\sum_{i=1, i \neq j}^m Y_i + w))$ is bounded in the norm topology.*

B.3. *$Y = \sum_{j=1}^m Y_j$ is convex.*

B.4. *There is free disposal $Y + w \supset (Y + w) - \mathscr{Y}_+$.*

3. Existence Theorem

3.1 Theorem. *Under Assumptions* A.1–A.6 *and* B.1–B.4 *there exists a competitive equilibrium* $(x, y, p) \in \mathscr{Y}^{n+m} \times \mathscr{Y}^{*}$.

Proof. Let $\mathscr{F}_0 = \{(x, y) \in \mathscr{F} \mid x_i \succsim_i x_i^0, i = 1,..., n\}$ and let $\mathscr{F}_0^* = \{(x, y) \in \mathscr{F}^* \mid (x, y) \in \mathscr{F}_0\}$. It is convenient to normalise the utility functions so that $u_i(x_i^0) = 0$, $i = 1,..., n$. Define $u: \mathscr{F} \to R^n$ by

$$u(x, y) = (u_1(x_1),..., u_n(x_n)), \qquad \forall (x, y) \in \mathscr{F}.$$

3.2 Lemma. $\mathscr{F}_0^* \neq \varnothing$ *and* $U_0^* = u(\mathscr{F}_0^*)$ *is homeomorphic to* $V_0^* = \{v \in R_+^n \mid \sum_{i=1}^n v_i = 1\}$.

Proof. By A.5 and A.4, $0 = u_i(x_i^0) < u_i(y_i + w_i) = \bar{u}_i$, $i = 1,..., n$. By B.4, $\xi_i(\lambda_i) = \lambda_i x_i^0 + (1 - \lambda_i)(y_i + w_i)$, $\lambda_i \in [0, 1]$, $i = 1,..., n$ is feasible. Since A.3 implies $u_i(\cdot)$ is continuous in the norm topology on X_i, $g_i(\lambda_i) = u_i(\xi_i(\lambda_i))$ is continuous on $[0, 1]$. Since $g_i(1) = 0$, $g_i(0) = \bar{u}_i > 0$, by the intermediate value theorem for any $\tilde{u}_i \in [0, \bar{u}_i]$ there exists $\tilde{\lambda}_i$ such that $g_i(\tilde{\lambda}_i) = \tilde{u}_i$. Thus

$$\mathscr{F}^{\tilde{u}} = \{(x, y) \in \mathscr{F} \mid u(x, y) = \alpha\tilde{u}, \alpha > 0\} \neq \varnothing, \qquad \forall \tilde{u} \in R_+^n. \tag{4}$$

Let $\sigma(\mathscr{Y}^{n+m}, \mathscr{Y}'^{n+m})$ denote the topology on $\mathscr{Y}^{n+m}$ which is the product of the $\sigma(\mathscr{Y}, \mathscr{Y}')$ topologies on $\mathscr{Y}$. By A.1, B.1, and (2), $\mathscr{F}$ is $\sigma(\mathscr{Y}^{n+m}, \mathscr{Y}'^{n+m})$ closed. By B.2 and (2), $\mathscr{F}$ is bounded in the product of the norm topologies. Thus, $\mathscr{F}$ is $\sigma(\mathscr{Y}^{n+m}, \mathscr{Y}'^{n+m})$ compact. For $u, u' \in R^n$ let $u \geqslant u'$ denote $u_i \geqslant u_i'$, $i = 1,..., n$. By A.3(i) and (4), for fixed $\tilde{u} \in R_+^n$,

$$\{z \in \mathscr{F} \mid u(z) \geqslant u(z')\}_{z' \in \mathscr{F}^{\tilde{u}}}$$

is a nonempty family of $\sigma(\mathscr{Y}^{n+m}, \mathscr{Y}'^{n+m})$ closed subsets of $\mathscr{F}$, which, since u induces an ordering which is complete and transitive on $\mathscr{F}^{\tilde{u}}$, has the finite intersection property. Since $\mathscr{F}$ is $\sigma(\mathscr{Y}^{n+m}, \mathscr{Y}'^{n+m})$ compact there exists $\bar{z} \in \bigcap_{z' \in \mathscr{F}^{\tilde{u}}} \{z \in \mathscr{F} \mid u(z) \geqslant u(z')\}$. Since utility is disposable by the argument given above, we may assume $u(\bar{z}) = \bar{\alpha}\tilde{u}$ for some $\bar{\alpha} > 0$, so that $\bar{z} \in \mathscr{F}^{\tilde{u}} \cap \mathscr{F}_0^*$. Let $(\tilde{u})$ denote the ray from the origin through $\tilde{u}$; then $(\tilde{u}) \cap U_0^* \neq \varnothing$. Thus the map $f: U_0^* \to V_0^*$ defined by

$$f(u) = \lambda(u)u, \qquad \lambda(u) = 1\Big/\sum_{i=1}^{n} u_i \tag{5}$$

is onto. By (5) and the definition of U_0^*, f is one-to-one. Since $\sum_{i=1}^n u_i \geqslant \min\{\bar{u}_1,..., \bar{u}_n\} > 0$, $\forall u \in U_0^*$, f is continuous. For $u, u' \in R^n$ let $u > u'$ denote $u \geqslant u_i'$, $i = 1,..., n$, $u_j > u_j'$ for some j. Let $\{u^k\}_{k=1}^{\infty}$ be a sequence in U_0^* such that $u^k \to u$ and let $u^* \in (u) \cap U_0^*$. Suppose $u^* \neq u$. If $u^* < u$, then

$u^k > u^*$ for some k contradicting $u^* \in U_0^*$. If $u < u^*$, then $u^k < u^*$ for some k contradicting $u^k \in U_0^*$. Thus U_0^* is closed. Suppose U_0^* is not bounded. Then there exists $\{u^m\}_{m=1}^{\infty} \subset U_0^* \subset R_+^n$ such that $\|u^m\| \to \infty$. But this contradicts $\mathscr{F}_0^* \neq \varnothing$. Thus U_0^* is compact and f is a homeomorphism. ▮

3.3 LEMMA. *If* $(x, y) \in \mathscr{F}_0^*$, *then there exists* $p \in \Pi = \{p \in \mathscr{T}_+^{\prime *} \mid \|p\|_* = 1\}$ *such that*

(i) $px_i \leqslant p\bar{x}_i$ *if* $u_i(\bar{x}_i) \geqslant u_i(x_i)$ $i = 1,..., n$,

(ii) $py_j \geqslant p\bar{y}_j$ *if* $\bar{y}_j \in Y_j$, $j = 1,..., m$.

Proof. Apply the standard separation theorem [7, Theorem 14.2, p. 118] to $Y + w$ and $\mathscr{P}$, where $Y = \sum_{j=1}^m Y_j$ and $\mathscr{P} = \sum_{i=1}^n \mathscr{P}_i$, $\mathscr{P}_i = \{\bar{x}_i \mid u_i(\bar{x}_i) \geqslant u_i(x_i)\}$ noting that $\mathscr{P}$ and Y are convex (A.3(i), B.3), the norm interior, $\operatorname{int}(Y + w) \neq \varnothing$ (B.4), and $\mathscr{P} \cap \operatorname{int}(Y + w) = \varnothing$, since $(x, y) \in \mathscr{F}_0^*$. Thus there exists $p' \in \mathscr{T}^{\prime *}$ satisfying (i) and (ii). Assumptions A.4 and A.6 imply $p' \geqslant 0$, $p' \neq 0$. Let $p = p'/\|p'\|_*$. ▮

Consider the correspondence $\pi: \mathscr{F}_0^* \to \Pi$ induced by (i) and (ii) above:

$$\pi(x, y) = \left\{ p \in \Pi \;\middle|\; \begin{array}{l} px_i \leqslant p\bar{x}_i \text{ if } u_i(\bar{x}_i) \geqslant u_i(x_i), i = 1,..., n \\ py_j \geqslant p\bar{y}_j \text{ if } \bar{y}_j \in Y_j, j = 1,..., m \end{array} \right\}.$$

Recalling $\Delta_p(x, y): \mathscr{F}_0^* \to R$ defined by (3), let $\Delta(x, y): \mathscr{F}_0^* \to R$ be defined by $\Delta(x, y) = \{\Delta_p(x, y) \mid p \in \pi(x, y)\}$. Define $\gamma: V_0^* \to \mathscr{F}_0^*$ by $\gamma(v) = \{(x, y) \in \mathscr{F}_0^* \mid f(u(x, y)) = v\}$ and let $\tau: V_0^* \to R$ be defined by $\tau(v) = \Delta(\gamma(v))$. Note that since $\mathscr{F}_0^*$ and Π are norm bounded subsets of $\mathscr{T}^{\prime n+m}$ and $\mathscr{T}^{\prime *}$, respectively, there exists $\alpha > 0$ such that if we define

$$T = \left\{ t \in R^n \;\middle|\; \sum_{i=1}^n t_i = 0, \; \sum_{i=1}^n |t_i| \leqslant \alpha \right\},$$

then $\tau(V_0^*) \subset T$. Let $\nu: V_0^* \times T \to V_0^*$ be defined by

$$\nu_i(v, t) = \frac{\max(0, v_i - t_i/\beta)}{\sum_{j=1}^n \max(0, v_j - t_j/\beta)}, \qquad \alpha < \beta < \infty, \; i = 1,..., n, \qquad (6)$$

with $\nu(v, t) = (\nu_1(v, t),..., \nu_n(v, t))$.

3.4 LEMMA. (i) $\nu(v, t): V_0^* \times T \to V_0^*$ *is a continuous function on* $V_0^* \times T$.

(ii) $\tau(v): V_0^* \to T$ *is an upper semicontinuous convex valued correspondence.*

Proof. Part (i) follows by noting that for all $t \in T$, $\sum_{i=1}^n (t_i/\beta) \leqslant \sum_{i=1}^n (|t_i|/\beta) \leqslant \alpha/\beta < 1$, so that

$$\sum_{i=1}^{n} \max\left(0, v_i - \frac{t_i}{\beta}\right) \geqslant \sum_{i=1}^{n} \left(v_i - \frac{t_i}{\beta}\right) = 1 - \sum_{i=1}^{n} \frac{t_i}{\beta} > 0 \tag{7}$$

for all $(v, t) \in V_0^* \times T$.

(ii) For fixed $v \in V_0^*$, let (x, y), $(x', y') \in \gamma(v)$; then $p \in \pi(x, y)$ implies $p \in \pi(x', y')$ and $\Delta_p(x, y) = \Delta_p(x', y')$. It suffices to note that $px_i < px'_i$ for some i and (or) $py_j > py'_j$ for some j contradicts $p(\mathbf{x} - \mathbf{y}) = p(\mathbf{x}' - \mathbf{y}') = p\mathbf{w}$ implied by (1). Thus $\pi(x, y) = \pi(x', y')$ and $\tau(v) = \Delta(x, y)$ for arbitrary fixed $(x, y) \in \gamma(v)$. Since $\pi(x, y)$ is convex, $\Delta(x, y)$, and thus $\tau(v)$, is convex.

Let $\{(v^s, \tau^s)\}_{s=1}^{\infty} \subset \mathscr{G}_\tau = \{(v, \tau) \mid \tau \in \tau(v), v \in V_0^*\}$ such that $(v^s, \tau^s) \to (v, \tau)$ as $s \to \infty$. Since V_0^* is closed, $v \in V_0^*$ and it remains to show that $\tau \in \tau(v)$. By definition $(v^s, \tau^s) \in \mathscr{G}_\tau$ implies that there exist $(x^s, y^s) \in \gamma(v^s)$ and $p^s \in \pi(x^s, y^s)$ such that

$$\tau_i^s = p^s x_i^s - p^s \left(w_i + \sum_{j=1}^{m} \theta_{ij} y_j^s\right), \qquad i = 1,..., n,$$

where $\tau^s = (\tau_1^s,..., \tau_n^s)$. Since $\{p^s\}_{s=1}^{\infty} \subset \Pi$ which is $\sigma(\mathscr{Y}^*, \mathscr{Y})$ compact, there exists a subsequence $\{p^k\}_{k=1}^{\infty} \subset \{p^s\}_{s=1}^{\infty}$ and $p \in \Pi$ such that

$$|(p - p^k)\xi| \to 0 \qquad \text{as} \quad k \to \infty, \qquad \forall \xi \in \mathscr{Y}. \tag{8}$$

It suffices to show that if $(x, y) \in \gamma(v)$ then $p \in \pi(x, y)$ and

$$\tau_i = px_i - p\left(w_i + \sum_{j=1}^{m} \theta_{ij} y_j\right), \qquad i = 1,..., n. \tag{9}$$

Since $f: U_0^* \to V_0^*$ is a homeomorphism, $v^k \to v$ is equivalent to $\mu^k = f^{-1}(v^k) \to f^{-1}(v) = \mu$. Let $\mu = (\mu_1,..., \mu_n)$ and let $\bar{x}_i \in \mathscr{Y}$ satisfy $u_i(\bar{x}_i) \geqslant \mu_i$. Since $u_i(\cdot)$ is continuous in the norm topology there exist $\varepsilon_k \in R$, $\varepsilon_k \to 0$ and $\xi_i^k \in \mathscr{Y}$ such that

$$u_i(\xi_i^k) \geqslant \mu_i^k, \qquad \|\xi_i^k - \bar{x}_i\|_\infty < \varepsilon_k, \tag{10}$$

so that

$$|p\bar{x}_i - p^k\xi_i^k| \leqslant |(p - p^k)\,\bar{x}_i| + |p^k(\bar{x}_i - \xi_i^k)| \to 0 \tag{11}$$

by (8), (10) and $\|p^k\|_* = 1$. Since $p^k \in \pi(x^k, y^k)$

$$p^k\xi_i^k \geqslant p^k x_i^k = \tau_i^k + p^k\left(w_i + \sum_{j=1}^{m} \theta_{ij} y_j^k\right)$$

$$\geqslant \tau_i^k + p^k\left(w_i + \sum_{j=1}^{m} \theta_{ij} y_j\right).$$

Taking limits of both sides and using (11)

$$p\bar{x}_i = \lim_{k\to\infty} p^k \xi_i^k \geqslant \lim_{k\to\infty} \left(\tau_i^k + p^k \left(w_i + \sum_{j=1}^{m} \theta_{ij} y_j\right)\right)$$
$$= \tau_i + p\left(w_i + \sum_{j=1}^{m} \theta_{ij} y_j\right). \tag{12}$$

Since $\sum_{i=1}^n \tau_i^k = 0$, $\tau^k \to \tau$ implies $\sum_{i=1}^n \tau_i = 0$ and since $p(\mathbf{x} - (\mathbf{y} + \mathbf{w})) = 0$, it follows that $\sum_{i=1}^n (px_i - \tau_i - p(w_i + \sum_{j=1}^m \theta_{ij} y_j)) = 0$. Thus since $u_i(x_i) = \mu_i$, (12) applied to x_i gives (9). Then (9) and (12) imply

$$px_i \leqslant p\bar{x}_i \text{ if } u_i(\bar{x}_i) \geqslant u_i(x_i), \qquad i = 1,\ldots, n,$$

$$\theta_{ij}\, py_j = px_i - \left(\tau_i + p\left(w_i + \sum_{r\neq j} \theta_{ir} y_r\right)\right) \geqslant \theta_{ij}\, py_j', \qquad \forall y_j' \in Y_j, j = 1,\ldots, m,$$

since $\theta_{ij} > 0$ for some i for each $j = 1,\ldots, m$, $p \in \pi(x, y)$. ∎

3.5 Lemma. *The correspondence* $\phi(v, t) = (\nu(v, t), \tau(v))$: $V_0^* \times T \to V_0^* \times T$ *has a fixed point* (v, t). *For any* $(x, y) \in \gamma(v)$ *there exists* $p \in \pi(x, y)$ *such that* (x, y, p) *is a competitive equilibrium.*

Proof. Since $V_0^* \times T$ is a nonempty compact convex subset of R^{2n} and since $\phi = \nu \times t$, as a product of upper semicontinuous mappings of $V_0^* \times T$ into V_0^* and T, respectively, is an upper semicontinuous mapping of $V_0^* \times T$ into $V_0^* \times T$ [3, p. 114, Theorem 4′] such that $\phi(v, t) \neq \varnothing$ and convex $\forall (v, t) \in V_0^* \times T$, by Kakutani's theorem [3, p. 174] ϕ has a fixed point (v, t) so that $v = \nu(v, t)$, $t \in \tau(v)$. Thus $\sigma v_i = \max(0, v_i - t_i/\beta)$ $i = 1,\ldots, n$ where $\sigma = \sum_{i=1}^n \max(0, v_i - t_i/\beta) > 0$ by (7). If $v_i(v, t) = 0$, then $t_i = 0$; if $v_i(v, t) > 0$, then $t_i = \beta(\sigma - 1)\, v_i$ so that t_i have the same sign for $i = 1,\ldots, n$. Since $t \in T$, $\sum_{i=1}^n t_i = 0$ so that $t_i = 0$, $i = 1,\ldots, n$. Consider any $(x, y) \in \gamma(v)$. Since $0 \in \tau(v) = \Delta(\gamma(v))$ there exists $p \in \pi(x, y)$ such that $\Delta_p(x, y) = 0$. Since $p \in A(Y)^* = \{p \in \mathscr{Y}^* \mid py \leqslant 0, \quad \forall y \in A(Y)\}$, by A.5 there exists $(x_1^0,\ldots, x_n^0) \in \prod_{i=1}^n X_i$ such that

$$px_i^0 < p(y_i + w_i) \leqslant pw_i \leqslant pw_i + \sum_{j=1}^{m} \theta_{ij}\, py_j = px_i, \qquad i = 1,\ldots, n, \tag{13}$$

since $\theta_{ij} \geqslant 0$ and $py_j \geqslant 0$ by B.1. Suppose for any $i = 1,\ldots, n$ there exists $\xi_i \in X_i$ such that $u_i(\xi_i) > u_i(x_i)$ and $p\xi_i = px_i$. Let $x_i(\lambda) = \lambda\xi_i + (1 - \lambda)\, x_i^0$. Since $u_i(\cdot)$ is lower semicontinuous in the norm topology (A.3(ii)) there exists $0 < \lambda < 1$ such that $u_i(x_i(\lambda)) > u_i(x_i)$. But (13) implies $px_i(\lambda) < px_i$, contradicting $p \in \pi(x, y)$. Thus (x, y, p) is a competitive equilibrium and the proof of the theorem is complete. ∎

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