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DYNAMICS UNDER UNCERTAINTY[1]

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1. INTRODUCTION

THIS PAPER IS a preliminary investigation of dynamics under uncertainty. We attempt to develop *a general approach to the continuous time stochastic processes that arise in dynamic economics from the maximizing behavior of agents*. The analysis builds on recent results of Bismut [**2**, **3**] concerning the characterization of the extrema of stochastic variational problems over a finite horizon and on our own investigations [**6**, **7**, **20**, **21**] of the stability properties of the *equations of dynamic economics*.[2]

We consider a class of discounted infinite horizon maximum problems. While it is convenient to pose the basic economic problem as a *stochastic control problem*, to obtain the full benefit of Bismut's elegant characterization of a maximizing process it is convenient to transform this problem into an equivalent *stochastic variational problem* along the lines indicated by Rockafellar [**27**] in the deterministic case and generalized by Bismut [**2**] to the stochastic case. Within this framework we show that the idea of a *competitive path* introduced in the continuous time deterministic case in [**21**] generalizes in a natural way in the case of uncertainty to a *competitive process*. We show, under a *concavity* assumption on the basic integrand of the problem, that a competitive process which satisfies a *transversality condition* is optimal under a discounted *catching up* criterion (Section 2).

In Section 3 we examine the sample path properties of a competitive process. If for almost every realization of a competitive process the associated dual price process generates a path of subgradients for the value function, we call the process McKenzie competitive, since it was McKenzie [**22**] who first recognized the importance of this property in the deterministic case. We show that two McKenzie competitive processes starting from distinct nonrandom initial conditions converge almost surely if the processes are *bounded almost surely* and if a certain *curvature condition* is satisfied by the *Hamiltonian* of the system. The earlier convergence result extensively studied in the deterministic case thus continues to hold in the stochastic case. The problem of finding sufficient conditions for the existence of a McKenzie competitive process remains an open problem.

Section 4 examines the long-run behavior of the *probability measure* associated with a competitive process. We give conditions under which a McKenzie competitive process is a *Markov process* with an *invariant probability measure* and show that under the curvature conditions of Section 3 the competitive process converges to a unique *stationary stochastic process*.

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[2] We should also refer to the related work of Cass-Shell [**8**], McKenzie [**22**], and Rockafellar [**28**].

Section 5 is a preliminary analysis of *intertemporal market equilibrium.* We consider the simplest case of *rational expectations equilibrium* for a *competitive industry* with a fixed number of firms producing a single output where the firms face an exogenously given *demand function* for their final product. We show that a solution of an associated consumers' surplus problem, which we call the *extended integrand problem,* generates a rational expectations equilibrium. We use the results of Section 4 to give conditions under which the resulting Markov process converges to a unique *stationary stochastic equilibrium process.* This generalizes the earlier work of Lucas-Prescott [**17**] and the subsequent results of Brock [**5**] and Scheinkman [**29**].

A number of questions raised in this section are examined in greater detail in [**19**] and [**4**]. In [**19**] Magill provides a more detailed analysis of the *shadow prices* and *risk costs* on which firms base their investment decisions. The security market, by indirectly informing producers of these variables in the process of valuing the securities of the firms, is shown to play an important role in determining an *optimal allocation of investment among the firms.* An alternative approach to the problem of intertemporal equilibrium is developed in the paper of Brock [**4**] where the capital theoretic framework of this paper is related more directly to the well-known *financial* theory of the capital market.

2. COMPETITIVE PROCESSES AND THE TRANSVERSALITY CONDITION

Let $(\Omega, \mathcal{F}, P)$ denote a *complete probability space,* $\mathcal{F}$ a σ-field on Ω, and P a probability measure on $\mathcal{F}$. Let $I=[0,\infty)$ denote the nonnegative *time interval* and $(I, \mathcal{M}, \mu)$ the *complete measure space* of Lebesgue measurable sets $\mathcal{M}$, with Lebesgue measure μ. Let $(\Omega\times I, \mathcal{H}, P\times\mu)$ denote the associated *complete product measure space* with complete measure $P\times\mu$ and σ-field $\mathcal{H}\supset\mathcal{F}\times\mathcal{M}$. Let $(R^n, \mathcal{M}^n)$, with $n\geq 1$, denote the measurable space formed from the n-dimensional real Euclidean space R^n with σ-field of Lebesgue measurable sets $\mathcal{M}^n$. Let

$$k(\omega, t)\colon (\Omega\times I, \mathcal{H})\to(R^n, \mathcal{M}^n)$$

be an *$\mathcal{H}$-measurable function* (*random process*) induced by the following problem.

STOCHASTIC CONTROL PROBLEM: Find an $\mathcal{H}$-measurable control $v(\omega, t)\in U\subseteq R^s$, $s\geq 1$, such that

$$\text{(1)}\qquad \sup_{v\in U}\int_\Omega\int_I e^{-\delta t}u(\omega, t, k(\omega, t), v(\omega, t))\,dt\,dP(\omega), \qquad \delta>0,$$

$$\text{(2)}\qquad k(\omega, t)=k_0+\int_0^t f(\omega, \tau, k(\omega, \tau), v(\omega, \tau))\,d\tau$$
$$+\int_0^t \sigma(\omega, \tau, k(\omega, \tau), v(\omega, \tau))\,dz(\omega, \tau)$$

where $u \in R^1, f = (f^1, \ldots, f^n) \in R^n$,

$$\sigma = \begin{bmatrix} \sigma^{11} \ldots \sigma^{1m} \\ \vdots \quad\quad \vdots \\ \sigma^{n1} \ldots \sigma^{nm} \end{bmatrix} \in R^{nm},$$

and where $k_0 \in K \subseteq R^n$ is a nonrandom initial condition. $u(\cdot, k, v)$, $f(\cdot, k, v)$, $\sigma(\cdot, k, v)$ are *$\mathscr{H}$-measurable random processes* for all (k, v) in $K \times U \subseteq R^n \times R^s$ and $u(\omega, \cdot)$, $f(\omega, \cdot)$, $\sigma(\omega, \cdot)$ are *continuous* on $I \times K \times U$ for almost all ω, while $z(\omega, t) \in R^m$, $m \geq 1$, is a *Brownian motion process.* Let

$$\mathscr{F}_t = \mathscr{I}(z(\omega, \tau), \tau \in [0, t])$$

denote the smallest complete σ-field on Ω relative to which the random variables $\{z(\omega, \tau), \tau \in [0, t]\}$ are measurable. We require that $k(\omega, t)$ be $\mathscr{F}_t$-measurable for all $t \in I$, so that $f(\cdot)$ and $\sigma(\cdot)$ are *nonanticipating* with respect to the family of σ-fields $\{\mathscr{F}_t, t \in I\}$. To ensure the existence of a unique random process $k(\omega, t)$ as a solution of (2) we make the following assumptions.

ASSUMPTION 1 (Lipshitz and Growth Conditions): *There exist positive constants α, β, such that*

(i) $$\|f(\omega, t, k, v) - f(\omega, t, \bar{k}, v)\| + \|\sigma(\omega, t, k, v) - \sigma(\omega, t, \bar{k}, v)\| \leq \alpha \|k - \bar{k}\|$$

for all (k, v), $(\bar{k}, v) \in K \times U$, for almost all $(\omega, t) \in \Omega \times I$.

(ii) $$\|f(\omega, t, k, v)\|^2 + \|\sigma(\omega, t, k, v)\|^2 \leq \beta(1 + \|k\|^2)$$

for all $(k, v) \in K \times U$, for almost all $(\omega, t) \in \Omega \times I$.

We will exhibit a sufficient condition for a random process to be a solution of the problem (1), (2) in terms of a certain price support property, the nature of which is most clearly revealed by restricting the stochastic control problem (1), (2) in the manner of Rockafellar-Bismut [**27**, p. 188; **2**, p. 393] as follows. Consider the new integrand

$$L(\omega, t, k, \dot{k}, \sigma) = \begin{cases} \sup_{v \in U} u(\omega, t, k, v) \mid f(\omega, t, k, v) = \dot{k}, \quad \sigma(\omega, t, k, v) = \sigma \\ -\infty, \quad \text{if there is no } v \in U \text{ such that } f(\omega, t, k, v) = \dot{k}, \\ \qquad \sigma(\omega, t, k, v) = \sigma. \end{cases}$$

REMARK: $L(\omega, t, \cdot)$ is *upper semicontinuous* for all $(\omega, t) \in \Omega \times I$ and $L(\omega, t, k(\omega, t), \dot{k}(\omega, t), \sigma(\omega, t))$ is *$\mathscr{H}$-measurable* whenever $k(\omega, t)$, $\dot{k}(\omega, t)$, and $\sigma(\omega, t)$ are $\mathscr{H}$-measurable.

We impose indirect concavity and boundedness conditions on the functions $u(\omega, t, \cdot)$, $f(\omega, t, \cdot)$, and $\sigma(\omega, t, \cdot)$ and a convexity condition on the domain $K \times U$ by the following assumption.

ASSUMPTION 2 (Concavity-Boundedness): $L(\omega, t, \cdot)$ is *concave in* $(k, \dot{k}, \sigma)$ for all $(k, \dot{k}, \sigma) \in R^n \times R^n \times R^{nm}$, for all $(\omega, t) \in \Omega \times I$, and there exists $\gamma \in R$, $|\gamma| < \infty$, such that $L(\cdot) < \gamma$ for all $(\omega, t, k, \dot{k}, \sigma) \in \Omega \times I \times R^n \times R^n \times R^{nm}$.

EXAMPLE: Consider the following matrices, whose coefficients are *$\mathscr{H}$-measurable nonanticipating random processes.* Let $\{A(\omega, t), F(\omega, t), H_i(\omega, t)\}$ be $n \times n$, $\{N(\omega, t), G(\omega, t), D_i(\omega, t)\}$ be $n \times s$, $\{B(\omega, t)\}$ be $s \times s$, $\sigma_0^i(\omega, t)$ be $n \times 1$, for $i = 1, \ldots, m$. We require that

$$\begin{bmatrix} A & N \\ N' & B \end{bmatrix}$$

be *positive definite* for all $(\omega, t) \in \Omega \times I$, $K \times U = R^n \times R^s$, and let

$$u(\omega, t, k, v) = -\tfrac{1}{2}\begin{bmatrix} k \\ v \end{bmatrix}' \begin{bmatrix} A(\omega, t) & N(\omega, t) \\ N'(\omega, t) & B(\omega, t) \end{bmatrix} \begin{bmatrix} k \\ v \end{bmatrix},$$

$$f(\omega, t, k, v) = F(\omega, t)k + G(\omega, t)v,$$

$$\sigma(\omega, t, k, v)\, dz = \sum_{i=1}^{m} \sigma^i\, dz_i = \sum_{i=1}^{m} (H_i(\omega, t)k + D_i(\omega, t)v + \sigma_0^i(\omega, t))\, dz_i.$$

It is immediate that Assumption 2 is satisfied.

DEFINITION: Let $(k, \dot{k}, \sigma) = (k(\omega, t), \dot{k}(\omega, t), \sigma(\omega, t))$ denote the $\mathscr{H}$-measurable random process defined by the equation

$$k(\omega, t) = k_0 + \int_0^t \dot{k}(\omega, \tau)\, d\tau + \int_0^t \sigma(\omega, \tau)\, dz(\omega, \tau), \tag{3}$$

where $k_0 \in K \subseteq R^n$ is a nonrandom initial condition, and where there exists an $\mathscr{H}$-measurable control $v(\omega, t) \in U$ such that

$$\dot{k}(\omega, \tau) = f(\omega, \tau, k(\omega, \tau), v(\omega, \tau)), \qquad \sigma(\omega, \tau) = \sigma(\omega, \tau, k(\omega, \tau), v(\omega, \tau))$$

for almost all $(\omega, \tau) \in \Omega \times I$. In view of Assumption 1

$$\int_\Omega \left(\int_0^t \|\dot{k}(\omega, \tau)\|^2\, d\tau + \int_0^t \|\sigma(\omega, \tau)\|^2\, d\tau \right) dP(\omega) < \infty, \qquad \text{for all } t \in I. \tag{3'}$$

We let $\mathscr{P}$ denote the class of random processes satisfying (3) and (3)', where $\dot{k}(\omega, \tau)$, $\sigma(\omega, \tau)$ are $\mathscr{H}$-measurable and *nonanticipating* with respect to the family of σ-fields $\{\mathscr{F}_t, t \in I\}$. The control problem (1), (2) then reduces to the following problem.

STOCHASTIC VARIATIONAL PROBLEM: Let L satisfy Assumption 2, let $L(\omega, t, \cdot)$ be *upper semicontinuous* for all $(\omega, t) \in \Omega \times I$, and let $L(\cdot, x, v, s)$ be *$\mathscr{H}$-measurable* for all $(x, v, s) \in R^n \times R^n \times R^{nm}$. Find an $\mathscr{H}$-measurable random process $(k, \dot{k}, \sigma) \in \mathcal{P}$ such that

$$\sup_{(k,\dot{k},\sigma)\in\mathcal{P}} \int_\Omega \int_I e^{-\delta t} L(\omega, t, k(\omega, t), \dot{k}(\omega, t), \sigma(\omega, t))\, dt\, dP(\omega). \tag{4}$$

In order to give (4) a broad interpretation we introduce the following definition.

DEFINITION: Let $\mathcal{K} \subset \mathcal{P}$ denote a class of $\mathcal{H}$-measurable random processes $(k, \dot{k}, \sigma)$. A random process $(\bar{k}, \dot{\bar{k}}, \bar{\sigma}) \in \mathcal{K}$ is *optimal* (*in* $\mathcal{K}$) if

$$\varliminf_{T\to\infty} \int_\Omega \int_0^T e^{-\delta\tau}(L(\omega, \tau, \bar{k}, \dot{\bar{k}}, \bar{\sigma}) - L(\omega, \tau, k, \dot{k}, \sigma))\, d\tau\, dP(\omega) \geq 0 \tag{5}$$

for all random processes $(k, \dot{k}, \sigma) \in \mathcal{K}$.

DEFINITION: Let

$$p(\omega, t): (\Omega \times I, \mathcal{H}) \to (R^n, \mathcal{M}^n)$$

denote an *$\mathcal{H}$-measurable random price process* dual to $k(\omega, t)$. We let $(\dot{p} - \delta p, p, \pi) = (\dot{p}(\omega, t) - \delta p(\omega, t), p(\omega, t), \pi(\omega, t))$ denote the $\mathcal{H}$-measurable random process defined by the equation

$$p(\omega, t) = p_0 + \int_0^t \dot{p}(\omega, \tau)\, d\tau + \int_0^t \pi(\omega, \tau)\, dz(\omega, t) \tag{6}$$

where $p_0 \in R^n$ is nonrandom and where $\dot{p}(\omega, \tau)$ and $\pi(\omega, \tau)$ are $\mathcal{H}$-measurable random processes, *nonanticipating* with respect to the family of σ-fields $\{\mathcal{F}_t, t \in I\}$, with values in $(R^n, \mathcal{M}^n)$ and $(R^{nm}, \mathcal{M}^{nm})$, respectively, and which satisfy

$$\int_\Omega \left(\int_0^t \|\dot{p}(\omega, \tau)\|^2\, d\tau + \int_0^t \|\pi(\omega, \tau)\|^2\, d\tau \right) dP(\omega) < \infty, \qquad \text{for all } t \in I. \tag{6'}$$

Let $\mathcal{P}^*$ denote the class of random processes defined in this way.

The following concept is fundamental to all the analysis that follows.

DEFINITION: A random process $(\bar{k}, \dot{\bar{k}}, \bar{\sigma}) \in \mathcal{P}$ is *competitive* if there exists a dual random price process $(\dot{\bar{p}} - \delta\bar{p}, \bar{p}, \bar{\pi}) \in \mathcal{P}^*$ such that

$$\begin{aligned} &(\dot{\bar{p}} - \delta\bar{p})'\bar{k} + \bar{p}'\dot{\bar{k}} + \operatorname{tr}(\bar{\pi}\bar{\sigma}') + L(\omega, t, \bar{k}, \dot{\bar{k}}, \bar{\sigma}) \\ &\quad \geq (\dot{\bar{p}} - \delta\bar{p})'k + \bar{p}'\dot{k} + \operatorname{tr}(\bar{\pi}\sigma') + L(\omega, t, k, \dot{k}, \sigma) \end{aligned} \tag{7}$$

for all $(k, \dot{k}, \sigma) \in R^n \times R^n \times R^{nm}$, for almost all $(\omega, t) \in \Omega \times I$.

REMARK (Economic Interpretation): *A competitive random process is a random process* $(\bar{k}, \dot{\bar{k}}, \bar{\sigma}) \in \mathcal{P}$ *that has associated with it a dual random price process* $(\dot{\bar{p}} - \delta\bar{p}, \bar{p}, \bar{\pi}) \in \mathcal{P}^*$ *under which it maximizes profit almost surely, at almost every instant.* For $-(\dot{\bar{p}} - \delta\bar{p})$ denotes the vector of *unit rental costs*, $-\bar{\pi}$ denotes the matrix of *unit risk costs* induced by the disturbance matrix $\bar{\sigma}$, while $(1, \bar{p})$ is the vector of *unit output prices*, so that

$$L + \bar{p}'\dot{\bar{k}} + (\dot{\bar{p}} - \delta\bar{p})'\bar{k} + \operatorname{tr}(\bar{\pi}\bar{\sigma}')$$

is the (imputed) *profit* which is maximized almost surely, at almost every instant, by a competitive random process.

REMARK (Geometric Interpretation): *The random process* $(\dot{\bar{p}} - \delta\bar{p}, \bar{p}, \bar{\pi}) \in \mathscr{P}^*$ *generates supporting hyperplanes to the epigraph of* $-L(\omega, t, k, \dot{k}, \sigma)$ *at the point* $(\bar{k}, \dot{\bar{k}}, \bar{\sigma})$ *for almost all* $(\omega, t) \in \Omega \times I$. The hyperplanes parallel to a given supporting hyperplane indicate hyperplanes of constant profit, so that the supporting hyperplanes are precisely the hyperplanes of maximum profit at each instant.

REMARK: *Under Assumption* 2 *a random process* $(k, \dot{k}, \sigma) \in \mathscr{P}$ *is competitive if and only if*

(8) $\quad (\dot{p}(\omega, t) - \delta p(\omega, t), p(\omega, t), \pi(\omega, t)) \in -\partial L(\omega, t, k(\omega, t), \dot{k}(\omega, t), \sigma(\omega, t))$

for almost all $(\omega, t) \in \Omega \times I$, *where* ∂L *denotes the subdifferential*[3] *of* $L(\omega, t, \cdot)$. Equation (8) is a generalization of the standard *Euler-Lagrange equation.*

DEFINITION: The Fenchel conjugate of $-L(\omega, t, k, \dot{k}, \sigma)$ with respect to $(\dot{k}, \sigma)$ will be called the *generalized Hamiltonian*

$$\mathscr{G}(\omega, t, k, p, \pi) = \sup_{(\dot{k}, \sigma) \in R^n \times R^{nm}} \{p'\dot{k} + \operatorname{tr}(\pi\sigma') + L(\omega, t, k, \dot{k}, \sigma)\}. \tag{9}$$

REMARK: $\mathscr{G}(\omega, t, k, p, \pi)$ is *concave in* k and *convex in* (p, π) for all $(\omega, t) \in \Omega \times I$ and is defined for all $(k, p, \pi) \in R^n \times R^n \times R^{nm}$.

The following characterization of a competitive path will be used in the Corollary of Theorem 2.

LEMMA 1: *Under Assumption* 2, *if* $\mathscr{G}(\omega, t, \cdot)$ *is differentiable, a random process* $(k, \dot{k}, \sigma) \in \mathscr{P}$ *is competitive if and only if*

$$\begin{aligned} k(\omega, t) &= k_0 + \int_0^t \mathscr{G}_p(\omega, \tau)\, d\tau + \int_0^t \mathscr{G}_\pi(\omega, \tau)\, dz(\omega, \tau), \\ p(\omega, t) &= p_0 + \int_0^t [-\mathscr{G}_k(\omega, \tau) + \delta p(\omega, \tau)]\, d\tau + \int_0^t \pi(\omega, \tau)\, dz(\omega, \tau). \end{aligned} \tag{$\mathscr{G}$}$$

The equations ($\mathscr{G}$), which will be called the *stochastic Hamiltonian equations*, are a generalization of the standard *Hamiltonian canonical equations* for a discounted stochastic variational problem.

[3] See Rockafellar [**27**, p. 207] and Bismut [**2**, p. 398].

DEFINITION: Assume that for all $(x, v, s) \in R^n \times R^n \times R^{nm}$

$$L(\omega, t, x, v, s) = L(x, v, s) \qquad \text{for all} \quad (\omega, t) \in \Omega \times I$$

so that L is nonrandom and time-independent. When (4) is finite we define the *current value function,* $W(k)$: $R^n \to R$

$$W(k(t)) = \sup_{(k,\dot{k},\sigma)\in\mathcal{P}} E_t \int_t^\infty e^{-\delta(\tau-t)} L(k(\omega, \tau), \dot{k}(\omega, \tau), \sigma(\omega, \tau))\, d\tau, \tag{10}$$

where E_t denotes the conditional expectation given k at time t, and where $k(t)$ replaces k_0 as the initial condition in (3).

REMARK: Under Assumption 2, $W(k)$ is a *concave* function for all $k \in K$.

In establishing convergence properties, the following class of competitive processes is of especial importance.

DEFINITION: A random process $(\bar{k}, \dot{\bar{k}}, \bar{\sigma}) \in \mathcal{P}$ is *McKenzie competitive* if it is competitive and if the dual random price process $\bar{p}(\omega, t)$ supports the value function

$$W(\bar{k}(\omega, t)) - \bar{p}(\omega, t)'\bar{k}(\omega, t) \geq W(k) - \bar{p}(\omega, t)'k \tag{11}$$

for all $k \in R^n$, for almost all $(\omega, t) \in \Omega \times I$.

REMARK: If $(k, \dot{k}, \sigma) \in \mathcal{P}$ is McKenzie competitive then p_0 in (6) is determined by the condition $p_0 \in \partial W(k_0)$.

It is convenient for our purposes to recall *Ito's Lemma* [**15**, Theorem 6, p. 59] in the following form. This result will be used repeatedly in the analysis that follows.

LEMMA 2 (ITO): *Let $\dot{y}(\omega, t), \rho(\omega, t)$ denote $\mathcal{H}$-measurable random processes*

$$\dot{y}(\omega, t)\colon (\Omega \times I, \mathcal{H}) \to (R^r, \mathcal{M}^r), \qquad \rho(\omega, t)\colon (\Omega \times I, \mathcal{H}) \to (R^{rm}, \mathcal{M}^{rm})$$

which are nonanticipating with respect to the m-dimensional Brownian motion process z, and satisfy, for almost all $\omega \in \Omega$,

$$\int_0^t \|\dot{y}(\omega, \tau)\|\, d\tau < \infty, \qquad \int_0^t \|\rho(\omega, \tau)\|^2\, d\tau < \infty, \qquad t \in I,$$

and let $y(\omega, t)$: $(\Omega \times I, \mathcal{H}) \to (R^r, \mathcal{M}^r)$ be defined by

$$y(\omega, t) = y_0 + \int_0^t \dot{y}(\omega, \tau)\, d\tau + \int_0^t \rho(\omega, \tau)\, dz(\omega, \tau), \qquad t \in I.$$

If $V(t, y)$: $R^{r+1} \to R$ is C^1 in t and C^2 in y, then

$$V(t, y(\omega, t)) = V(0, y_0) + \int_0^t [V_t(\tau, y(\omega, \tau)) + \mathscr{D}V(\tau, y(\omega, \tau))] \, d\tau + \int_0^t V_y(\tau, y(\omega, \tau))'\rho(\omega, \tau) \, dz(\omega, \tau), \qquad t \in I,$$

where $\mathscr{D}V(t, y(\omega, t))$ is the differential generator of the process $V(t, y(\omega, t))$:

(12) $$\mathscr{D}V(t, y(\omega, t)) = V_y(t, y)'\dot{y}(\omega, t) + \tfrac{1}{2} \operatorname{tr} (V_{yy}(t, y)\rho(\omega, t)\rho(\omega, t)').$$

THEOREM 1 (Transversality Condition): *A competitive random process $(\bar{k}, \dot{\bar{k}}, \bar{\sigma}) \in \mathscr{K}$ with dual process $(\dot{\bar{p}} - \delta\bar{p}, \bar{p}, \bar{\pi}) \in \mathscr{P}^*$, which satisfies the transversality condition*

$$\overline{\lim_{T\to\infty}} \, E_0 \, e^{-\delta T}\bar{p}(\omega, T)'\bar{k}(\omega, T) \leq 0$$

is optimal in the class $\mathscr{K}$ of random processes for which

(13) $$\underline{\lim_{T\to\infty}} \, E_0 \, e^{-\delta T}\bar{p}(\omega, T)'k(\omega, T) \geq 0.$$

PROOF: The proof is a generalization of the proof of Lemma 2 in Magill [**21**]. Let

(14) $$\bar{\mathscr{L}}(\omega, t, k, \dot{k}, \sigma) = L(\omega, t, \bar{k}, \dot{\bar{k}}, \bar{\sigma}) - L(\omega, t, k, \dot{k}, \sigma) + (\dot{\bar{p}} - \delta\bar{p})'(\bar{k} - k) + \bar{p}'(\dot{\bar{k}} - \dot{k}) + \operatorname{tr} (\bar{\pi}(\bar{\sigma} - \sigma)')$$

denote the *flow value loss* function for the random process $(k, \dot{k}, \sigma)$ induced by the random price process $(\dot{\bar{p}} - \delta\bar{p}, \bar{p}, \bar{\pi})$ of the competitive process $(\bar{k}, \dot{\bar{k}}, \bar{\sigma})$. Multiplying by $e^{-\delta t}$, integrating and rearranging terms, gives

(15) $$E_0 \int_0^T e^{-\delta\tau}(L(\omega, \tau, \bar{k}, \dot{\bar{k}}, \bar{\sigma}) - L(\omega, \tau, k, \dot{k}, \sigma)) \, d\tau = E_0 \int_0^T e^{-\delta\tau}\bar{\mathscr{L}}(\omega, \tau, k, \dot{k}, \sigma) \, d\tau + E_0 \int_0^T e^{-\delta\tau}[(\dot{\bar{p}} - \delta\bar{p})'(k - \bar{k}) + \bar{p}'(\dot{k} - \dot{\bar{k}}) + \operatorname{tr} (\bar{\pi}(\sigma - \bar{\sigma})')] \, d\tau.$$

Lemma 2 applied to the function

$$V(t, k - \bar{k}, \bar{p}) = e^{-\delta t}\bar{p}'(k - \bar{k}),$$

with $(k-\bar{k}, \bar{p})$ satisfying equations (3) and (6), leads to

$$e^{-\delta T}\bar{p}(\omega, T)'(k(\omega, T)-\bar{k}(\omega, T))-\bar{p}(\omega, 0)'(k(\omega, 0)-\bar{k}(\omega, 0))$$
$$=\int_0^T e^{-\delta\tau}[(\dot{\bar{p}}-\delta\bar{p})'(k-\bar{k})+\bar{p}'(\dot{k}-\dot{\bar{k}})+\operatorname{tr}(\bar{\pi}(\sigma-\bar{\sigma})')]\,d\tau$$
$$+\int_0^T e^{-\delta\tau}(\bar{p}'(\sigma-\bar{\sigma})+(k-\bar{k})'\bar{\pi})\,dz(\omega, \tau).$$

In view of (3)′, (6)′, and Proposition I-1 in [**2**], we have

$$E_0\int_0^T e^{-\delta\tau}(\bar{p}'(\sigma-\bar{\sigma})+(k-\bar{k})'\bar{\pi})\,dz(\omega, \tau)=0$$

so that

$$E_0\int_0^T e^{-\delta\tau}[(\dot{\bar{p}}-\delta\bar{p})'(k-\bar{k})+\bar{p}'(\dot{k}-\dot{\bar{k}})+\operatorname{tr}(\bar{\pi}(\sigma-\bar{\sigma})')]\,d\tau \tag{16}$$
$$=E_0\, e^{-\delta T}\bar{p}(\omega, T)'(k(\omega, T)-\bar{k}(\omega, T))-E_0\bar{p}(\omega, 0)'(k(\omega, 0)-\bar{k}(\omega, 0)).$$

Equations (15), (16), and $k_0=\bar{k}_0$ imply

$$E_0\int_0^T e^{-\delta\tau}(L(\omega, \tau, \bar{k}, \dot{\bar{k}}, \bar{\sigma})-L(\omega, \tau, k, \dot{k}, \sigma))\,d\tau \tag{17}$$
$$=E_0\int_0^T e^{-\delta\tau}\bar{\mathscr{L}}(\omega, \tau, k, \dot{k}, \sigma)\,d\tau$$
$$+E_0[e^{-\delta T}\bar{p}(\omega, T)'(k(\omega, T)-\bar{k}(\omega, T))].$$

Since the competitivity of the process $(\bar{k}, \dot{\bar{k}}, \bar{\sigma})\in\mathscr{K}$ implies $\bar{\mathscr{L}}(\omega, t, k, \dot{k}, \sigma)\geqslant 0$, (17) gives at once

$$\varliminf_{T\to\infty} E_0\int_0^T e^{-\delta\tau}(L(\omega, \tau, \bar{k}, \dot{\bar{k}}, \bar{\sigma})-L(\omega, \tau, k, \dot{k}, \sigma))\,d\tau$$
$$\geqslant \varliminf_{T\to\infty} E_0\, e^{-\delta T}\bar{p}(\omega, T)'k(\omega, T)-\varlimsup_{T\to\infty} E_0 e^{-\delta T}\bar{p}(\omega, T)'\bar{k}(\omega, T)\geqslant 0$$

so that $(\bar{k}, \dot{\bar{k}}, \bar{\sigma})$ is optimal in the class of random processes for which (13) is satisfied. *Q.E.D.*

3. CONVERGENCE OF McKENZIE COMPETITIVE PROCESSES

The sample paths of a McKenzie competitive process starting from nonrandom initial conditions have a remarkable *convergence property*. Consider a point $k_0\in K$ and a McKenzie competitive process emanating from this point. Under assumptions, which include a strict concavity assumption on the basic integrand L, a McKenzie competitive process emanating from *any other point* $k_0\in K$ converges almost surely to the first process. This result, which has its origin in the *dual* relationship between the *prices* and *quantities* of a McKenzie competitive process, may be stated as follows.

THEOREM 2 (Almost Sure Convergence): *Let Assumption* 2 *be satisfied and let the function L be time-independent and nonrandom as in* (10). *If two McKenzie competitive random processes*

$$(k, \dot{k}, \sigma) \in \mathcal{P}, \qquad (\bar{k}, \dot{\bar{k}}, \bar{\sigma}) \in \mathcal{P}, \tag{18}$$

with associated dual price processes

$$(\dot{p} - \delta p, p, \pi) \in \mathcal{P}^*, \qquad (\dot{\bar{p}} - \delta \bar{p}, \bar{p}, \bar{\pi}) \in \mathcal{P}^*, \tag{18'}$$

starting from the nonrandom initial conditions

$$(k_0, p_0), \qquad (\bar{k}_0, \bar{p}_0),$$

satisfy the following conditions: (i) *there exists a compact convex subset* $M \subset R^n \times R^n$ *such that for all* $t \in I$

$$(k(\omega, t), p(\omega, t)) = (k(\omega, t; k_0), p(\omega, t; p_0)) \in M,$$

$$(\bar{k}(\omega, t), \bar{p}(\omega, t)) = (\bar{k}(\omega, t; \bar{k}_0), \bar{p}(\omega, t; \bar{p}_0)) \in M,$$

for almost all $\omega \in \Omega$; (ii) *there exists* $\mu > 0$ *such that the function*

$$V(k - \bar{k}, p - \bar{p}) = -(p - \bar{p})'(k - \bar{k}) \tag{19}$$

satisfies

$$\mathcal{D}V(k - \bar{k}, p - \bar{p}) \leq -\mu \|k - \bar{k}, p - \bar{p}\|^2 \tag{20}$$

for all

$$(k - \bar{k}, p - \bar{p}) \in Y = \{(k - \bar{k}, p - \bar{p}) \,|\, (k, p), (\bar{k}, \bar{p}) \in M\};$$

(iii) *the value function is* (*a*) *strictly concave*, (*b*) *differentiable*, (*c*) *strictly concave and differentiable, for all* $k \in \mathring{K}$ *where* $K = \{k \,|\, (k, p) \in M\}$; *then* (*i*), (*ii*), *and* (*iii*) (*a*) *imply*

$$k(\omega, t) - \bar{k}(\omega, t) \to 0 \qquad \text{a.s. as } t \to \infty;$$

(*i*), (*ii*), *and* (*iii*) (*b*) *imply*

$$p(\omega, t) - \bar{p}(\omega, t) \to 0 \qquad \text{a.s. as } t \to \infty;$$

(*i*), (*ii*), *and* (*iii*) (*c*) *imply*

$$(k(\omega, t) - \bar{k}(\omega, t), p(\omega, t) - \bar{p}(\omega, t)) \to 0 \qquad \text{a.s. as } t \to \infty.$$

PROOF:[4] The first step is to show that V is a *nonnegative supermartingale*.[5] To

[4] We are grateful to F. R. Chang for helping to modify the proof into its present form.

[5] For a thorough analysis of this remarkable class of stochastic processes the reader is referred to Doob [**9**, Ch. VII] and Meyer [**23**, Chs. V, VI].

this end consider two points $k, \bar{k} \in K$. Using the subgradient inequality (11) we define the *value losses* at k and $\bar{k}$ by

$$\text{(21)}\qquad \begin{aligned} \bar{\Delta}(k;\bar{p}) &= W(\bar{k}) - W(k) + \bar{p}'(k-\bar{k}), \qquad \bar{p} \in \partial W(\bar{k}),\\ \Delta(\bar{k};p) &= W(k) - W(\bar{k}) + p'(\bar{k}-k), \qquad p \in \partial W(k). \end{aligned}$$

The concavity of $W(k)$ at $\bar{k}$ and k implies $\bar{\Delta} \geqslant 0$, $\Delta \geqslant 0$ so that

$$\text{(21')}\qquad \bar{\Delta} + \Delta = -(p-\bar{p})'(k-\bar{k}) = V \geqslant 0.$$

To simplify notation let $y = (y_1, y_2) = (k-\bar{k}, p-\bar{p})$ and let $y_0 = (k_0 - \bar{k}_0, p_0 - \bar{p}_0)$. Since the processes (18), (18′) are nonanticipating with respect to the Brownian motion process z, and since they satisfy (3′) and (6′), we may apply Lemma 2 to the function $V(y) = -y_1'y_2$ to obtain

$$\text{(22)}\qquad V(y(\omega,t)) = V(y_0) + \int_0^t \mathscr{D}V(y(\omega,\tau))\,d\tau + \int_0^t h(\omega,t)\,dz(\omega,\tau) \qquad \text{where}$$

$$h = -[(k-\bar{k})'(\pi - \bar{\pi}) + (p-\bar{p})'(\sigma - \bar{\sigma})].$$

In view of (3′), (6′) and Proposition I-1 in [**2**],

$$\int_\Omega \int_0^t \|h(\omega,\tau)\|^2\,d\tau\,dP(\omega) < \infty \qquad \text{for all } t \in I,$$

so that

$$\int_0^t h(\omega,\tau)\,dz(\omega,\tau), \qquad t \in I,$$

is a *martingale* with respect to the family of σ-fields $\{\mathscr{F}_t, t \in I\}$. But then

$$E(h(\omega,t)\,|\,\mathscr{F}_\tau) = h(\omega,\tau) = 0, \qquad \tau \leqslant t, \qquad \tau, t \in I,$$

for almost all $\omega \in \Omega$, so that (22) implies

$$E(V(y(\omega,t))\,|\,\mathscr{F}_\tau) = V(y(\omega,\tau)) + E\left(\int_\tau^t \mathscr{D}V(y(\omega,\theta))\,d\theta\,|\,\mathscr{F}_\tau\right)$$

and by (20)

$$E(V(y(\omega,t)) - V(y(\omega,\tau))|\mathscr{F}_\tau) = E\left(\int_\tau^t \mathscr{D}V(y(\omega,\theta))\,d\theta|\mathscr{F}_\tau\right) \leqslant 0.$$

Thus

$$E(V(y(\omega,t))\,|\,\mathscr{F}_\tau) \leqslant V(y(\omega,\tau)), \qquad \tau \leqslant t, \qquad \tau, t \in I,$$

for almost all $\omega \in \Omega$, so that $V(y(\omega,t))$ is a *supermartingale* relative to the family of σ-fields $\{\mathscr{F}_t, t \in I\}$. Since $V(y(\omega,\cdot))$ is continuous for almost all $\omega \in \Omega$ by the almost sure continuity of $y(\omega,\cdot)$, the conditions of Doob's *supermartingale convergence theorem* [**23**, p. 96] are satisfied. Thus there exists a random variable

$\nu_0(\omega) \geqslant 0$ a.s. such that

(23) $$V(y(\omega, t; y_0)) \to \nu_0(\omega) \qquad \text{a.s. as } t \to \infty.$$

The next step is to show that $\nu_0(\omega) = 0$ for almost all $\omega \in \Omega$. Suppose therefore that $\nu_0(\omega) > 0$ for all $\omega \in A \subset \Omega$ with $P(A) > 0$. Then

$$0 < \bar{\nu}_0 = \int_\Omega \nu_0(\omega)\, dP(\omega) = \int_A \nu_0(\omega)\, dP(\omega).$$

The inequality $0 \leqslant (\|y_1\| - \|y_2\|)^2 = \|y_1\|^2 + \|y_2\|^2 - 2\|y_1\|\,\|y_2\|$ implies

(24) $$\tfrac{1}{2}\|y\|^2 \geqslant \|y_1\|\,\|y_2\| \geqslant |y_1' y_2|.$$

Equations (20), (22), and (24) imply

$$\begin{aligned}\beta(T) = EV(y(\omega, T; y_0)) - V(y_0) &= E\int_0^T \mathscr{D}V(y(\omega, \tau; y_0))\, d\tau \\ &\leqslant E\int_0^T -\mu\|y(\omega, \tau; y_0)\|^2\, d\tau \leqslant E\int_0^T -2\mu V(y(\omega, \tau; y_0))\, d\tau.\end{aligned}$$

In view of (21′), (24), and the fact that Y is a bounded set, there exists $0 < \alpha < \infty$ such that

(25) $$0 \leqslant V(y(\omega, t; y_0)) \leqslant \tfrac{1}{2}\|y(\omega, t; y_0)\|^2 < \alpha \qquad \text{a.s. for all } t \in I.$$

Equation (25) and the fact that $V(y(\omega, t; y_0))$ is a measurable function of (ω, t) for all $(\omega, t) \in \Omega \times [0, T]$, allow us to apply Fubini's theorem [**14**, p. 147] for all $T < \infty$:

$$\begin{aligned}\beta(T) &\leqslant -2\mu \int_\Omega \int_0^T V(y(\omega, \tau; y_0))\, d\tau\, dP(\omega) \\ &= -2\mu \int_0^T \int_\Omega V(y(\omega, \tau; y_0))\, dP(\omega)\, d\tau.\end{aligned}$$

If we let

$$\bar{V}_0(t) = \int_\Omega V(y(\omega, t; y_0))\, dP(\omega),$$

then

$$\beta(T) \leqslant -2\mu \int_0^T \bar{V}_0(\tau)\, d\tau.$$

Equations (23), (25), and the bounded convergence theorem [**14**, p. 110] imply

$$\bar{V}_0(t) \to \bar{\nu}_0 \qquad \text{as } t \to \infty.$$

Hence for any $0 < \varepsilon < \bar{\nu}_0$ there exists T_ε such that $t > T_\varepsilon$ implies

$$\bar{V}_0(t) > \bar{\nu}_0 - \varepsilon > 0$$

so that

$$\beta(T) \leqslant -2\mu \int_{T_\varepsilon}^{T} (\bar{\nu}_0 - \varepsilon)\, d\tau.$$

Thus $\beta(T)$ can be made arbitrarily negative by suitable choice of $T < \infty$, thus contradicting $V \geqslant 0$. Thus $\nu_0(\omega) = 0$ for almost all $\omega \in \Omega$, as was to be shown.

To complete the proof we note that $\bar{\Delta} \geqslant 0$, $\Delta \geqslant 0$ implies that $V = 0$ if and only if $\bar{\Delta} = \Delta = 0$, which implies that W is an affine function on the line segment connecting k and $\bar{k}$. Thus if W is strictly concave $k - \bar{k} = 0$. If W is differentiable in k, the affine function between k and $\bar{k}$ defines a hyperplane which coincides with the tangent hyperplanes at k and $\bar{k}$ so that $p - \bar{p} = W_k(k) - W_k(\bar{k}) = 0$. If W is differentiable and strictly concave, $V = 0$ implies $k - \bar{k} = p - \bar{p} = 0$ and the proof is complete. *Q.E.D.*

REMARK: It is natural to conjecture that the results of Theorem 2 may be extended to the case where L is *time-dependent* provided $W(k, t)$ satisfies the conditions in (iii) *uniformly* for all $t \in I$.

REMARK: Consider the *flow value loss* $\bar{\mathscr{L}}$ in (14) and recall the definition of the *value loss* $\bar{\Delta}$ in (21). If we define $\mathscr{L} = \mathscr{L}(\omega, t, \bar{k}, \dot{\bar{k}}, \bar{\sigma})$ by transposing the barred and unbarred terms in (14), as in the definition of Δ in (21), then

$$\bar{\mathscr{L}} + \mathscr{L} = \mathscr{D}V - \delta V.$$

If flow value loss arguments are made on the surface $L(k, \dot{k}, \sigma)$ in the same way that value loss arguments are made on the surface $W(k)$ in the proof of Theorem 2, it may be possible to show that

$$(26) \qquad (\dot{k}(\omega, t), \sigma(\omega, t)) - (\dot{\bar{k}}(\omega, t), \bar{\sigma}(\omega, t)) \to 0 \qquad \text{a.s. as } t \to \infty$$

by imposing boundedness conditions on each of the terms in (26) and giving sufficient conditions for $\mathscr{D}V \to 0$ a.s. as $t \to \infty$.

NOTATION: It is useful to view the matrix of risk induced prices π as being composed of m n-dimensional column vectors

$$\pi = [\pi^1 \ldots \pi^m] = \begin{bmatrix} \pi^{11} & \ldots & \pi^{1m} \\ \vdots & & \vdots \\ \pi^{n1} & \ldots & \pi^{nm} \end{bmatrix}.$$

The generalized Hamiltonian may then be viewed as a function of $m+2$ n-dimensional column vectors

$$\mathscr{G}(k, p, \pi) = \mathscr{G}(k, p, \pi^1, \ldots, \pi^m).$$

COROLLARY (Hamiltonian Curvature Condition): *The following conditions are sufficient for* (20). *For all* $(k, p, \pi) \in M \times R^{nm}$

(i) $$K^{\delta} = \begin{bmatrix} -\mathcal{G}_{kk} & \frac{\delta}{2} I \\ \frac{\delta}{2} I & \mathcal{G}_{pp} \end{bmatrix}$$ *is uniformly positive definite*;

(ii) $\mathcal{G}_{\pi^i \pi^i}$, $i = 1, \ldots, m$, *are non-negative definite.*

PROOF: In view of Lemma 1, the competitive process $(k, \dot{k}, \sigma) \in \mathcal{P}$, $(\dot{p} - \delta p, p, \pi) \in \mathcal{P}^*$ is a solution of the stochastic Hamiltonian equation ($\mathcal{G}$). Similarly $(\bar{k}, \dot{\bar{k}}, \bar{\sigma}) \in \mathcal{P}$, $(\dot{\bar{p}} - \delta \bar{p}, \bar{p}, \bar{\pi}) \in \mathcal{P}^*$ is a solution of a Hamiltonian system ($\bar{\mathcal{G}}$). If we apply Lemma 2 to the function (19), then (20) becomes

$$\text{(27)} \qquad \mathcal{D}V(y) = -\Big((k-\bar{k})'(\bar{\mathcal{G}}_k - \mathcal{G}_k) + (p-\bar{p})'(\mathcal{G}_p - \bar{\mathcal{G}}_p) + \sum_{i=1}^{m} (\pi^i - \bar{\pi}^i)'(\mathcal{G}_{\pi^i} - \bar{\mathcal{G}}_{\pi^i}) + \delta(p-\bar{p})'(k-\bar{k})\Big)$$

where $\bar{\mathcal{G}}_k = \mathcal{G}_k(\bar{k}, \bar{p}, \bar{\pi})$, and similarly for $\bar{\mathcal{G}}_p, \bar{\mathcal{G}}_{\pi^i}, i = 1, \ldots, m$. Let $A, B, C_1, \ldots, C_m$ be $n \times n$ matrices, A and B positive definite and $C_1, \ldots, C_m$ nonnegative definite. Consider the following curvature condition on the generalized Hamiltonian $\mathcal{G}(k, p, \pi)$:

$$\mathcal{G}(\bar{k}, p, \pi) \leq \mathcal{G}(k, p, \pi) + \mathcal{G}_k'(\bar{k} - k) - \tfrac{1}{2}(\bar{k} - k)'A(\bar{k} - k),$$

$$\text{(28)} \qquad \mathcal{G}(k, \bar{p}, \bar{\pi}) \geq \mathcal{G}(k, p, \pi) + \mathcal{G}_p'(\bar{p} - p) + \sum_{i=1}^{m} \mathcal{G}_{\pi^i}(\bar{\pi}^i - \pi^i) + \tfrac{1}{2}(\bar{p} - p)'B(\bar{p} - p) + \tfrac{1}{2}\sum_{i=1}^{m} (\bar{\pi}^i - \pi^i)'C_i(\bar{\pi}^i - \pi^i).$$

If we make the same evaluation at $(\bar{k}, \bar{p}, \bar{\pi})$ in place of (k, p, π), we obtain

$$\mathcal{G}(k, \bar{p}, \bar{\pi}) \leq \mathcal{G}(\bar{k}, \bar{p}, \bar{\pi}) + \bar{\mathcal{G}}_k'(k - \bar{k}) - \tfrac{1}{2}(k - \bar{k})'A(k - \bar{k}),$$

$$\text{(29)} \qquad \mathcal{G}(\bar{k}, p, \pi) \geq \mathcal{G}(\bar{k}, \bar{p}, \bar{\pi}) + \bar{\mathcal{G}}_p'(p - \bar{p}) + \sum_{i=1}^{m} \bar{\mathcal{G}}_{\pi^i}(\pi^i - \bar{\pi}^i) + \tfrac{1}{2}(p - \bar{p})'B(p - \bar{p}) + \tfrac{1}{2}\sum_{i=1}^{m} (\pi^i - \bar{\pi}^i)'C_i(\pi^i - \bar{\pi}^i).$$

Multiplying (28) and (29) by -1 and adding all four inequalities gives

$$0 \leq (k - \bar{k})'(\bar{\mathcal{G}}_k - \mathcal{G}_k) + (p - \bar{p})'(\mathcal{G}_p - \bar{\mathcal{G}}_p) + \sum_{i=1}^{m} (\pi^i - \bar{\pi}^i)'(\mathcal{G}_{\pi^i} - \bar{\mathcal{G}}_{\pi^i}) - (k - \bar{k})'A(k - \bar{k}) - (p - \bar{p})'B(p - \bar{p}) - \sum_{i=1}^{m} (\pi^i - \bar{\pi}^i)'C_i(\pi^i - \bar{\pi}^i),$$

but then by (27),

$$\mathscr{D}V(y) \leqslant -\Big((k-\bar{k})'A(k-\bar{k}) + (p-\bar{p})'B(p-\bar{p}) + \sum_{i=1}^{m} (\pi^i - \bar{\pi}^i)'C_i(\pi^i - \bar{\pi}^i) + \delta(p-\bar{p})'(k-\bar{k})\Big).$$

We now choose the matrices $A = -\mathscr{G}_{kk}$, $B = \mathscr{G}_{pp}$, $C_i = \mathscr{G}_{\pi^i\pi^i}$, $i = 1, \ldots, m$, so that

$$\mathscr{D}V(y) \leqslant -\Big((k-\bar{k})'(-\mathscr{G}_{kk})(k-\bar{k}) + (p-\bar{p})'\mathscr{G}_{pp}(p-\bar{p}) + \sum_{i=1}^{m} (\pi^i - \bar{\pi}^i)'\mathscr{G}_{\pi^i\pi^i}(\pi^i - \bar{\pi}^i) + \delta(p-\bar{p})'(k-\bar{k})\Big)$$

$$\leqslant -\begin{bmatrix} k-\bar{k} \\ p-\bar{p} \end{bmatrix}' \begin{bmatrix} -\mathscr{G}_{kk} & \frac{\delta}{2}I \\ \frac{\delta}{2}I & \mathscr{G}_{pp} \end{bmatrix} \begin{bmatrix} k-\bar{k} \\ p-\bar{p} \end{bmatrix} \leqslant -\mu\|y\|^2$$

since $\Sigma_{i=1}^{m} (\pi^i - \bar{\pi}^i)'\mathscr{G}_{\pi^i\pi^i}(\pi^i - \bar{\pi}^i) \geqslant 0$, $\mu > 0$ being the minimum eigenvalue of K^δ.

Q.E.D.

DEFINITION: The function $\hat{\mathscr{G}}(k, p, \pi; v) = u(k, v) + p'f(k, v) + \Sigma_{i=1}^{m} \pi^{i\prime}\sigma^i(k, v)$ will be called the *generalized pre-Hamiltonian.*

REMARK: $\mathscr{G}(k, p, \pi) = \sup_{v \in U} \hat{\mathscr{G}}(k, p, \pi; v)$.

If we assume that the maximum lies in the interior of U, then the condition $\hat{\mathscr{G}}_v = 0$ leads to the optimal control

$$v^* = v^*(k, p, \pi)$$

so that the generalized Hamiltonian is given by

$$\mathscr{G}(k, p, \pi) = \hat{\mathscr{G}}(k, p, \pi; v^*(k, p, \pi)).$$

Using the fact that $\hat{\mathscr{G}}_v = 0$ we obtain

$$\mathscr{G}_k = \hat{\mathscr{G}}_k + \hat{\mathscr{G}}_v \cdot v_k^* = \hat{\mathscr{G}}_k,$$

$$\mathscr{G}_p = \hat{\mathscr{G}}_p + \hat{\mathscr{G}}_v \cdot v_p^* = \hat{\mathscr{G}}_p = f(k, v^*(k, p, \pi)),$$

$$\mathscr{G}_{\pi^i} = \hat{\mathscr{G}}_{\pi^i} + \hat{\mathscr{G}}_v \cdot v_{\pi^i}^* = \hat{\mathscr{G}}_{\pi^i} = \sigma^i(k, v^*(k, p, \pi)) \qquad (i = 1, \ldots, m),$$

so that the $m+2$ matrices required for the *Hamiltonian curvature condition* are readily evaluated without explicitly calculating the generalized Hamiltonian,

$$\mathscr{G}_{kk} = \hat{\mathscr{G}}_{kk} + \hat{\mathscr{G}}_{kv} \cdot v_k^*, \quad \mathscr{G}_{pp} = f_v \cdot v_p^*, \quad \mathscr{G}_{\pi_i\pi_i} = \sigma_v^i \cdot v_{\pi^i}^* \quad (i = 1, \ldots, m). \tag{30}$$

EXAMPLE:

$$\hat{\mathscr{G}}(k, p, \pi; v) = -\tfrac{1}{2}\begin{bmatrix} k \\ v \end{bmatrix}'\begin{bmatrix} A & N \\ N' & B \end{bmatrix}\begin{bmatrix} k \\ v \end{bmatrix} + p'(Fk + Gv)$$

$$+ \sum_{i=1}^{m} \pi^{i'}(H_i k + D_i v + \sigma_0^i);$$

$$\hat{\mathscr{G}}_v = p'G - k'N - v'B + \sum_{i=1}^{m} \pi^{i'} D_i = 0.$$

Since B is positive definite, B^{-1} exists and

$$v^* = v^*(k, p, \pi) = B^{-1}\left(G'p - N'k + \sum_{i=1}^{m} D_i' \pi^i\right);$$

$$\hat{\mathscr{G}}_{kk} = -A, \qquad \hat{\mathscr{G}}_{kv} = -N, \qquad v_k^* = -B^{-1}N'$$

$$\text{so that} \quad \mathscr{G}_{kk} = -(A - NB^{-1}N');$$

$$f_v = G, \qquad v_p^* = B^{-1}G' \qquad \text{so that} \quad \mathscr{G}_{pp} = GB^{-1}G';$$

$$\sigma_v^i = D_i, \qquad v_{\pi^i}^* = B^{-1}D_i' \qquad \text{so that} \quad \mathscr{G}_{\pi^i\pi^i} = D_i B^{-1} D_i' \quad (i = 1, \ldots,).$$

Thus the Corollary requires that

$$K^\delta = \begin{bmatrix} (A - NB^{-1}N') & \frac{\delta}{2} I \\ \frac{\delta}{2} I & GB^{-1}G' \end{bmatrix}$$

be positive definite, since B^{-1} *positive definite implies* $\mathscr{G}_{\pi^i\pi^i} = D_i B^{-1} D_i'$ *is non-negative definite,* $i = 1, \ldots, m$.

The reader is referred to Magill [**21**, Section 4] for a detailed *geometric* and *economic* interpretation of the K^δ curvature condition.

4. CONVERGENCE TO INVARIANT PROBABILITY MEASURE

In this section we consider the long-run behavior of the *probability measure* of the process arising from the maximizing behavior of agents. To this end we consider problem (1), (2) with u, f, and σ nonrandom and constant over time.

Let $W(k(t))$ denote the *current value function*

$$W(k(t)) = \sup_{v \in U} E_t \int_t^\infty e^{-\delta(\tau - t)} u(k(\omega, \tau), v(\omega, \tau))\, d\tau \qquad \text{where} \tag{31}$$

$$k(\omega, t) = k_0 + \int_0^t f(k(\omega, \tau), v(\omega, \tau))\, d\tau + \int_0^t \sigma(k(\omega, \tau), v(\omega, \tau))\, dz(\omega, \tau). \tag{32}$$

Let $W(k)$ be a C^2 solution of the *generalized Hamilton-Jacobi equation* [**12**, p. 159]

$$u(k, v^*)+\mathscr{D}^{v^*}W(k)-\delta W(k)=0 \qquad \text{where}$$

$$u(k, v^*)+\mathscr{D}^{v^*}W(k)-\delta W(k)=\sup_{v\in U}\{u(k, v)+\mathscr{D}^{v}W(k)-\delta W(k)\},$$

$$\mathscr{D}^{v}W(k)= W_k(k)'f(k, v)+\tfrac{1}{2}\operatorname{tr}(W_{kk}(k)\sigma(k, v)\sigma(k, v)'),$$

and where we assume that v^* is a unique interior maximum such that

$$v^*(k)=v^*(k, W_k(k), W_{kk}(k)).$$

DEFINITION: Let v^* denote an optimal control which solves (1), (2). If v^* depends only on the current state, $v^*=v^*(k)$, then it will be called an *optimal policy function.*

LEMMA 3: *If an optimal policy function $v^*(k)$ exists such that*

$$f^*(k)=f(k, v^*(k)), \qquad \sigma^*(k)=\sigma(k, v^*(k))$$

satisfy a Lipshitz condition for some $\alpha\in R$, for all $k, \bar{k}\in K$,

$$\|f^*(k)-f^*(\bar{k})\|+\|\sigma^*(k)-\sigma^*(\bar{k})\|\leq \alpha\|k-\bar{k}\|, \tag{33}$$

then the optimal process

$$k^*(\omega, t)=k_0+\int_0^t f^*(k(\omega, \tau))\,d\tau+\int_0^t \sigma^*(k(\omega, \tau))\,dz(\omega, \tau) \tag{34}$$

is a continuous[6] *Markov process.*

PROOF: (Dynkin [**11**, Theorem 11.4, p. 349].)

ASSUMPTION 3 (Compactness): K is compact and $k^*(\omega, t)\in K$ for all $(\omega, t)\in \Omega\times I$.

We recall some basic facts associated with the analysis of the homogeneous Markov process (34). The reader is referred to Dynkin [**11**] for a complete analysis.

Consider the measurable space $(K, \mathscr{B})$ with Borel σ-field $\mathscr{B}$, the σ-field generated by the open sets of K. Let $\mathscr{C}(K, \mathscr{B})$ denote the space of *continuous functions* on K and let $\mathscr{V}(K, \mathscr{B})$ denote the space of finite *countably additive set functions* defined on $\mathscr{B}$. The Markov process (34) induces a *transition function* on $(K, \mathscr{B})$

$$P_t(k, A), \qquad k\in K, \qquad A\in\mathscr{B}, \qquad t\in I,$$

[6] We say that the process $k^*(\omega, t)$ is *continuous* if $k^*(\omega, \cdot)$ is continuous for all $t\in I$, for all $\omega\in\Omega$.

where $P_t(\cdot, A)$ is a *$\mathscr{B}$-measurable function* for each $A \in \mathscr{B}$ and $P_t(k, \cdot)$ is a *countably additive set function* for each $k \in K$, satisfying

(35) $$P_t(k, A) \geq 0 \quad \text{for all } A \in \mathscr{B}, \quad P_t(k, K) = 1, \quad t \in I.$$

The transition function leads in turn to a *family of linear mappings* on the spaces $\mathscr{C}(K, \mathscr{B})$ and $\mathscr{V}(K, \mathscr{B})$ defined by

$$T_t g(k) = \int_K g(y) P_t(k, dy), \quad g \in \mathscr{C}, \quad k \in K, \quad t \in I,$$

$$T_t^* \phi(A) = \int_K P_t(y, A) \phi(dy), \quad \phi \in \mathscr{V}, \quad A \in \mathscr{B}, \quad t \in I.$$

The property of conditional expectations combined with the Markov principle and the time homogeneity of the process leads to the *semigroup property*

(36) $$T_t^* T_s^* = T_{s+t}^*, \quad s, t \in I.$$

The spaces $\mathscr{C}(K, \mathscr{B})$ and $\mathscr{V}(K, \mathscr{B})$ are related by the scalar product

$$(g, \phi) = \int_K g(k) \phi(dk), \quad g \in \mathscr{C}, \quad \phi \in \mathscr{V}$$

in terms of which the mappings T_t and T_t^*, $t \in I$, satisfy the basic relation

(37) $$(T_t g, \phi) = (g, T_t^* \phi), \quad g \in \mathscr{C}, \quad \phi \in \mathscr{V}, \quad t \in I.$$

DEFINITION: $\phi \in \mathscr{V}(K, \mathscr{B})$ is called a *probability measure* if $\phi(A) \geq 0$ for all $A \in \mathscr{B}$ and $\phi(K) = 1$. The set of probability measures in $\mathscr{V}(K, \mathscr{B})$ is denoted by $\Pi(K, \mathscr{B})$. We say that $\phi \in \Pi(K, \mathscr{B})$ is an *invariant probability measure* if

$$T_t^* \phi = \phi \quad \text{for all } t \in I.$$

ϕ is thus a *fixed point* for the family of mappings $\{T_t^*, t \in I\}$. Since K is compact by the Riesz representation theorem[7] [**10**, p. 265], $\mathscr{C}(K, \mathscr{B})^* = \mathscr{V}(K, \mathscr{B})$. If we endow $\mathscr{V}(K, \mathscr{B})$ with the *weak* topology*, then we say that the net $\phi_\tau \in \mathscr{V}$ *converges weakly* to $\phi \in \mathscr{V}$ (in short $\phi_\tau \xrightarrow{w} \phi$) if

$$(g, \phi_\tau - \phi) \to 0 \quad \text{as } \tau \to \infty \quad \text{for all } g \in \mathscr{C}(K, \mathscr{B}).$$

LEMMA 4: *If an optimal policy function exists that satisfies* (33) *and if Assumption* 3 *is satisfied then* (i) T_t^* *is continuous in the weak* topology of* $\mathscr{V}(K, \mathscr{B})$ *and*

$$T_t^* : \Pi(K, \mathscr{B}) \to \Pi(K, \mathscr{B}) \quad \textit{for all } t \in I;$$

(ii) $\Pi(K, \mathscr{B})$ *is a convex weak* compact subset of* $\mathscr{V}(K, \mathscr{B})$.

[7] Since $K \subset R^n$ is a metric space, every finite countably additive set function is *regular.* See Parthasarathy [**26**, Theorem 1.2, p. 27].

PROOF: (i) We start by showing that (33) implies

(38) $$T_t\colon \mathscr{C}(K,\mathscr{B})\to\mathscr{C}(K,\mathscr{B}),\qquad t\in I.$$

To this end let $k(\omega, t; k_0)$ and $k(\omega, t; k_n)$ denote solutions of the Ito equation (34) where k_0 and k_n are nonrandom initial conditions such that $\|k_n-k_0\|\to 0$ as $n\to\infty$. In view of the Lipshitz condition (33), by a result of Dynkin [**11**, p. 344, equation 11.36] for any $\varepsilon>0$ there exists a nondecreasing function $\gamma(t)$, $t\in I$, such that

$$P(\omega\,|\,\|k(\omega,t;k_n)-k(\omega,t;k_0)\|>\varepsilon)\leq\frac{\gamma(t)}{\varepsilon^2}\|k_n-k_0\|^2.$$

Thus for any $g\in\mathscr{C}(K,\mathscr{B})$, $t\in I$,

$$\int_\Omega [g(k(\omega,t;k_n))-g(k(\omega,t;k_0))]\,dP(\omega)$$
$$=Eg(k(\omega,t;k_n))-Eg(k(\omega,t;k_0))$$
$$=T_tg(k_n)-T_tg(k_0)\to 0\qquad\text{as } n\to\infty,\qquad\text{since}$$
$$P(\omega|\,|g(k(\omega,t;k_n))-g(k(\omega,t;k_0))|>\varepsilon)\to 0\qquad\text{as } n\to\infty.$$

Since $k_0\in K$ is arbitrary, $T_tg(k)$ is continuous in k for all $k\in K$ and (38) follows. Now suppose

$$(g,\phi_\tau-\phi)\to 0\qquad\text{as } \tau\to\infty\qquad\text{for all } g\in\mathscr{C}(K,\mathscr{B});$$

then by (37)

$$(g, T_t^*(\phi_\tau-\phi))=(T_tg,\phi_\tau-\phi)\to 0\qquad\text{as } \tau\to\infty$$

for all $g\in\mathscr{C}(K,\mathscr{B})$, since $T_tg\in\mathscr{C}(K,\mathscr{B})$ by (38). To complete the proof of (i) we note that (35) implies that for any $\phi\in\Pi(K,\mathscr{B})$, $t\in I$,

$$\phi_t(A)=T_t^*\phi(A)=\int_K P_t(k,A)\phi(dk)\geq 0,\qquad A\in\mathscr{B},$$

$$\phi_t(K)=T_t^*\phi(K)=\int_K P_t(k,K)\phi(dk)=\phi(K)=1,$$

so that $T_t^*\phi\in\Pi(K,\mathscr{B})$.

(ii) This follows from Assumption 3 and [**26**, Theorem 6.4, p. 45]. *Q.E.D.*

THEOREM 3 (Invariant Probability Measure): *If an optimal policy function exists that satisfies* (33) *and if Assumption* 3 *is satisfied, then the Markov process* (34) *has at least one invariant probability measure.*

PROOF: Since $\mathscr{V}(K,\mathscr{B})$ is a linear topological space in its weak* topology and since the semigroup property (36) implies that $\{T_t^*, t\in I\}$ is a *commuting* family of mappings, the result follows from Lemma 4 and the Markov-Kakutani fixed point theorem [**10**, Theorem 6, p. 456]. *Q.E.D.*

REMARK: Let Π_{T^*} denote the *set of all invariant probability measures* of the process (34) under the condition (33). Theorem 3 asserts $\Pi_{T^*} \neq \emptyset$.

THEOREM 4 (Convergence to Invariant Probability Measure): *If an optimal policy function exists that satisfies* (33), *if conditions* (*i*), (*ii*) *and* (*iii*) (*c*) *of Theorem 2 are satisfied, and if* $\phi_t \in \Pi(K, \mathscr{B})$ *denotes the probability measure at time* $t \in I$ *for the process* (34), *starting from a nonrandom initial condition* $k_0 \in K$, *then* ϕ_t *converges weakly to a unique invariant probability measure* ϕ *as* $t \to \infty$, *for all nonrandom initial conditions* $k_0 \in K$.

PROOF: By Theorem 3 $\phi \in \Pi_{T^*}$ exists. Let $F(x) = \phi(k \mid k \leq x)$ denote its *distribution function.* By a well-known result of real variable theory since $K \subset R^n$, ϕ may be characterized by F [**16**, pp. 95–98]. Choose $k_0 \in K$. Let $k(\omega, t; k_0)$ denote the solution of (34) and let ϕ_t and $F_t(x)$ denote the associated probability measure and distribution function at time $t \in I$. We recall the following definition.

DEFINITION: Let $F_t(x)$, $t \in I$, and $F(x)$ denote distribution functions defined on $K \subset R^n$. We say that F_t *converges weakly* to F (in short $F_t \xrightarrow{\mathscr{D}} F$) if

$$F_t(x) - F(x) \to 0 \qquad \text{as } t \to \infty \qquad \text{for all } x \in C(F)$$

where $C(F)$ denotes the *set of continuity points of F.*

The following result is well-known [**1**, p. 18]. $F_t \xrightarrow{\mathscr{D}} F$ *if and only if* $\phi_t \xrightarrow{w} \phi$.

Now consider any $\varepsilon > 0$. Let η be a random variable with distribution function $F(x)$ and let $k(\omega, t; \eta)$ denote the solution of (34) with initial condition η. Then by the formula for total probability

$$\Pr\{(\omega, \eta) \mid \|k(\omega, t; k_0) - k(\omega, t; \eta)\| > \varepsilon\}$$
$$= \int_K P(\omega \mid \|k(\omega, t; k_0) - k(\omega, t \mid \eta = y)\| \geq \varepsilon)\, dF(y);$$

furthermore, by the bounded convergence theorem [**14**, p. 110],

$$\lim_{t\to\infty} \int_K P(\omega \mid \|k(\omega, t; k_0) - k(\omega, t \mid \eta = y)\| \geq \varepsilon)\, dF(y) \tag{39}$$
$$= \int_K \lim_{t\to\infty} P(\omega \mid \|k(\omega, t; k_0) - k(\omega, t \mid \eta = y)\| \geq \varepsilon)\, dF(y).$$

Using the result on almost sure convergence in Theorem 2 and the fact that almost sure convergence implies convergence in probability [**16**, p. 151], we obtain

$$P(\omega \mid \|k(\omega, t; k_0) - k(\omega, t \mid \eta = y)\| \geq \varepsilon) \to 0 \qquad \text{as } t \to \infty$$

for all $y \in K$, so that the limit in (39) is zero. But then

$$\|k(\omega, t; k_0) - k(\omega, t; \eta)\| \to 0 \qquad \text{in probability as } t \to \infty.$$

Since convergence in probability implies convergence in distribution [**1**, Theorem 4.3, p. 26] we obtain

$$(40) \qquad F_t \xrightarrow{\mathcal{D}} F \qquad \text{so that} \qquad \phi_t \xrightarrow{w} \phi.$$

Consider $\bar{k}_0 \in K$, $\bar{k}_0 \neq k_0$. Let $\bar{k}(\omega, t; \bar{k}_0)$ denote the solution of (34) and let $\bar{\phi}_t$ and $\bar{F}_t(x)$ denote the associated probability measure and distribution function at time $t \in I$. By an argument similar to the one above we may select $\bar{\phi} \in \Pi_{T^*}$, with distribution function $\bar{F}(x)$, and show that

$$(41) \qquad \bar{F}_t \xrightarrow{\mathcal{D}} \bar{F} \qquad \text{so that} \qquad \bar{\phi}_t \xrightarrow{w} \bar{\phi}.$$

We now make use of the following lemma.

LEMMA 5: *If $k(\omega, t)$, $\bar{k}(\omega, t)$ denote random processes with distribution functions $F_t(x)$, $\bar{F}_t(x)$, if there exists $F(x)$ such that $F_t \xrightarrow{\mathcal{D}} F$, and if $\|k(\omega, t) - \bar{k}(\omega, t)\| \to 0$ in probability as $t \to \infty$, then $\bar{F}_t \xrightarrow{\mathcal{D}} F$.*

PROOF: (See Billingsley [**1**, Theorem 4.1, p. 25].)
Since it follows from Theorem 2 that

$$\|k(\omega, t; k_0) - \bar{k}(\omega, t; \bar{k}_0)\| \to 0 \qquad \text{in probability as } t \to \infty,$$

(40), (41) and Lemma 5 imply $\bar{\phi} = \phi$. Since in the weak* topology ϕ_t $(\bar{\phi}_t)$ can converge to at most one limit, the proof is complete. *Q.E.D.*

5. RATIONAL EXPECTATIONS EQUILIBRIUM

In this section we will show how the concept of a competitive process in conjunction with the stochastic Hamiltonian equations ($\mathcal{G}$), provide a useful framework for the analysis of rational expectations equilibrium. We examine in particular a rational expectations equilibrium for a competitive industry in which a fixed finite number of firms behave according to a stochastic adjustment cost theory, by creating an *extended integrand problem* analogous to that of Lucas-Prescott [**18**]. By applying Theorem 4 to the extended integrand problem, we show that the capital accumulation process of the firms in the industry converges to a stationary stochastic equilibrium process.

Consider therefore an industry composed of $N \geq 1$ firms, each producing the same industry good with the aid of $n \geq 1$ capital goods. All firms have identical expectations regarding the industry product's price process, which is an $\mathcal{H}$-measurable, nonanticipating process:

$$(42) \qquad r(\omega, t): (\Omega \times I, \mathcal{H}) \to (R^+, \mathcal{M}).$$

The instantaneous flow of profit of the ith firm is the difference of its *revenue* $r(\omega, t)f^i(k^i(\omega, t))$ and its *costs* $C^i(v^i(\omega, t))$ where $k^i = (k^{i1}, \ldots, k^{in})$ and $v^i = (v^{i1}, \ldots, v^{in})$ denote the capital stocks and investment rates of the ith firm, and where $f^i(k^i)$ and $-C^i(v^i)$ are the standard strictly concave production and adjustment cost functions. If $\delta > 0$ denotes the nonrandom interest rate, then each firm seeks to maximize its *expected discounted profit* by selecting an $\mathscr{H}$-measurable, nonanticipating investment process

$$v^i(\omega, t): (\Omega \times I, \mathscr{H}) \to (R^n, \mathscr{M}^n) \qquad (i = 1, \ldots, N)$$

such that

$$\sup_{v^i(\omega,t)} E_0 \int_0^\infty e^{-\delta\tau}[r(\omega, \tau)f^i(k^i(\omega, \tau)) - C^i(v^i(\omega, \tau))]\, d\tau,$$

$$(43) \qquad k^i(\omega, t) = k_0^i + \int_0^t v^i(\omega, \tau)\, d\tau + \int_0^t \sigma^i(k^i(\omega, \tau))\, dz^i(\omega, \tau),$$

$$(44) \qquad \sigma^i(k^i)\, dz^i = \sum_{j=1}^m (H^{ij}k^i + \sigma_0^{ij})\, dz^{ij},$$

where H^{ij}, σ_0^{ij} are $n \times n$ and $n \times 1$ matrices with constant coefficients and $z^i(\omega, \tau)$ is an m-dimensional Brownian motion process. This model is a simple stochastic version of the basic Lucas-Mortensen [**17**, **24**] adjustment cost model, with the standard additional neoclassical assumption that the investment and output processes of the ith firm have no direct external effects on the investment and output processes of the kth firm, for $i \neq k$.

On the product market, the total *market supply* which is given by

$$Q_S(\omega, t) = \sum_{i=1}^N f^i(k^i(\omega, t))$$

depends in a complex way through the maximizing behavior of firms on the price process (42). On the demand side of the product market we make the simplifying assumption that the total *market demand* depends only on the current market price

$$Q_D(\omega, t) = \psi^{-1}(r(\omega, t)), \qquad r \geq 0, \qquad \text{where}$$

$$\psi(Q) > 0, \qquad \psi'(Q) < 0, \qquad Q \geq 0.$$

DEFINITION: A *rational expectations equilibrium* for the *product market* of the industry is an $\mathscr{H}$-measurable, nonanticipating random process (42) such that

$$(45) \qquad Q_D(\omega, t) = Q_S(\omega, t) \qquad \text{for almost all} \quad (\omega, t) \in \Omega \times I.$$

REMARK: The firms' expectations are *rational* in that the *anticipated price process* coincides almost surely with the *actual price process* generated on the market by their maximizing behavior [**25**].

Consider the *integral of the demand function*

$$\Psi(Q)=\int_0^Q \psi(y)\,dy, \qquad Q\geqslant 0, \qquad \text{so that}$$

$$\Psi'(Q)=\psi(Q), \qquad \Psi''(Q)=\psi'(Q)<0, \qquad Q\geqslant 0.$$

DEFINITION: We call the problem of finding N $\mathscr{H}$-measurable, nonanticipating investment processes

$$(v^1(\omega,t),\dots,v^N(\omega,t))\colon(\Omega\times I,\mathscr{H})\to(R^{nN},\mathscr{M}^{nN})$$

such that

$$(\mathscr{E}) \qquad \sup_{(v^1(\omega,t),\dots,v^N(\omega,t))} E_0\int_0^\infty e^{-\delta\tau}\left[\Psi\left(\sum_{i=1}^N f^i(k^i(\omega,\tau))\right)-\sum_{i=1}^N C^i(v^i(\omega,\tau))\right]d\tau,$$

where $(k^1(\omega,t),\dots,k^N(\omega,t))$ satisfy (43), (44), almost surely, the *extended integrand problem.*

THEOREM 5 (Rational Expectations Equilibrium): *If the generalized Hamiltonian of the extended integrand problem* $(\mathscr{E})$ *is differentiable, if*

$$(\bar{k}(\omega,t),\bar{p}(\omega,t))=(\bar{k}^1(\omega,t),\dots,\bar{k}^N(\omega,t),\bar{p}^1(\omega,t),\dots,\bar{p}^N(\omega,t))$$

is a competitive process for $(\mathscr{E})$ *which satisfies the transversality condition*

$$(46) \qquad \overline{\lim_{T\to\infty}}\, E_0\, e^{-\delta T}\bar{p}(\omega,T)'\bar{k}(\omega,T)\leqslant 0,$$

and if for any alternative random process $k(\omega,t)$ *with* $k_0=\bar{k}_0$

$$(47) \qquad \underline{\lim}_{T\to\infty}\, E_0\, e^{-\delta T}\bar{p}(\omega,T)'k(\omega,T)\geqslant 0,$$

then the $\mathscr{H}$*-measurable, nonanticipating random process*

$$(48) \qquad \bar{r}(\omega,t)=\psi\left(\sum_{i=1}^N f^i(\bar{k}^i(\omega,t))\right)$$

is a rational expectations equilibrium for the product market of the industry.

PROOF: Since the generalized Hamiltonian for the extended integrand problem is differentiable, $(\bar{k}(\omega,t),\bar{p}(\omega,t))$ is competitive if and only if, writing $(\mathscr{G})$ in shorthand form,

$$(49) \qquad \begin{aligned} dk^i&=h^i(p^i)\,dt+\sigma^i(k^i)\,dz^i,\\ dp^i&=\left[\delta p^i-\Psi'\left(\sum_{i=1}^N f^i(k^i)\right)f^i_{k^i}-\sum_{j=1}^m \pi^{ij}\sigma^{ij}_{k^i}\right]dt+\pi^i\,dz^i,\end{aligned} \qquad (i=1,\dots,N)$$

where $h^i = (C^i_{v^i})^{-1}$. Equations (48) and (49) imply

$$dk^i = h^i(p^i) + \sigma^i(k^i)\,dz^i,$$
$$dp^i = \left[\delta p^i - r(\omega, t) f^i_{k^i} - \sum_{j=1}^{m} \pi^{ij}\sigma^{ij}_{k^i}\right] dt + \pi^i\,dz^i. \qquad (i = 1, \ldots, N) \tag{50}$$

Equations (46), (47), and (50) are sufficient conditions for each firm to maximize expected discounted profit, by Theorem 1. Equation (48) implies that (45) is satisfied and the proof is complete. *Q.E.D.*

REMARK: Theorem 5 reduces the analysis of a rational expectations equilibrium to the much simpler analysis of the extended integrand problem ($\mathscr{E}$).

Let $\mathscr{G}(k, p, \pi)$ denote the generalized Hamiltonian for ($\mathscr{E}$); then the K^δ condition reduces to

$$\min_{x \neq 0} \frac{x'(-\mathscr{G}_{kk})x}{x'(\mathscr{G}_{pp})^{-1}x} > \left(\frac{\delta}{2}\right)^2 \quad \text{for all } (k, p) \in R^n \times R^n \quad \text{where} \tag{51}$$

$$\mathscr{G}_{kk} = \begin{bmatrix} (\Psi'' f^1_{k^1}(f^1_{k^1})' + \Psi' f^1_{k^1k^1}) \cdots & \Psi'' f^1_{k^1}(f^N_{k^N})' \\ \vdots & \vdots \\ \Psi'' f^N_{k^N}(f^1_{k^1})' & \cdots (\Psi'' f^N_{k^N}(f^N_{k^N})' + \Psi' f^N_{k^Nk^N}) \end{bmatrix},$$

$$(\mathscr{G}_{pp})^{-1} = \begin{bmatrix} C^1_{v^1v^1} & & 0 \\ & \cdot & \\ 0 & & C^N_{v^Nv^N} \end{bmatrix} = C_{vv}.$$

Inequality (51) thus imposes a curvature condition on the production and demand functions relative to the adjustment cost function. The condition is particularly simple in the case of an infinitely elastic demand curve ($\Psi'' = 0$, $\Psi' = r$) with quadratic production and adjustment cost functions. In this case the smallest eigenvalue of the matrix

$$\Psi' f_{kk} = \Psi' \begin{bmatrix} f^1_{k^1k^1} & & 0 \\ & \ddots & \\ 0 & & f^N_{k^Nk^N} \end{bmatrix}$$

in the metric induced by the matrix C_{vv} must exceed $(\delta/2)^2$, which is a condition on the productivity of capital for each producer relative to the cost of adjustment similar to that examined by Magill [**21**].

Let $W(k) = W(k^1, \ldots, k^N)$ denote the current value function (31) associated with the extended integrand problem ($\mathscr{E}$); then the optimal policy function is

$$v^*(k) = (h^1(W_{k^1}(k)), \ldots, h^N(W_{k^N}(k))) = f^*(k). \tag{52}$$

Since $\sigma^*(k)$ is given by (44), if (52) satisfies a Lipshitz condition, if (51) and Assumption 3 are satisfied, then by Theorem 4 there exists a *stationary distribution function* $F(x) = F(x^1, \ldots, x^N)$, for the capital stocks of the N firms, such that

$F_t \xrightarrow{\mathscr{D}} F$, where $F_t(x) = F_t(x^1, \ldots, x^N)$ denotes the *distribution function at time* $t \in I$ for the process (34) generated by (44) and (52). The rational expectations equilibrium for the competitive industry thus converges to a stationary stochastic equilibrium process.

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