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THE ORIGIN OF CYCLICAL MOTION IN DYNAMIC ECONOMIC MODELS*

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This paper examines the origin of cyclical motion in a class of deterministic infinite horizon problems that arise in dynamic economics. For this class of problems the optimal solution converges to a unique equilibrium point. Conditions are given under which the motion in a neighbourhood of the equilibrium point is cyclical. These conditions involve certain asymmetric stock–flow interaction terms that arise in the local equations of motion about the equilibrium point. The results are used to show how cyclical motion can arise in a rational expectations equilibrium for a competitive industry.

1. Introduction

This paper is part of a preliminary attempt to understand the forces that lead to cyclical motion in dynamic economic models that arise from the maximising behaviour of economic agents. While it is quite clear that uncertainty constitutes a major reason for the presence of cyclical behaviour in many economic problems,[1] in this paper I choose to focus attention on the sources of cyclical behaviour in the simplest deterministic framework. In order that the sources of cyclical behaviour should not appear ad hoc, in order that the model should appear at least in a preliminary way to be arbitrage proof, I concentrate attention on dynamical processes that arise from intertemporal maximising behaviour on the part of economic agents. To keep the analysis simple, I select a class of deterministic infinite horizon maximum problems for which we have the beginnings of a relatively complete theory and in which a unique equilibrium point or stationary state is the sole element of the ω-limit set of an optimal trajectory: this is the class of strictly concave, undiscounted, infinite horizon maximum problems where optimality is in the sense of the overtaking criterion.

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[1]See Magill (1977a) where spectral analysis is used to examine the cyclical properties of the long-run stationary stochastic process. See also Brock–Magill (1978).

The paper is arranged as follows. In section 2 the basic maximum problem is laid out, along with some of the basic definitions and properties needed in the subsequent analysis. Particular attention is paid to those properties that are required to make a local analysis in the neighbourhood of an equilibrium point meaningful. The results of Brock–Haurie (1976) are used to set the problem up in such a way that an optimal solution to the infinite horizon problem exists which has the additional property that it converges to a unique equilibrium point, whose existence is also assured. This justifies the subsequent local analysis of the nature of the motion in a neighbourhood of the equilibrium point, in that the system will always be observed in any such neighbourhood after a sufficient interval of time.

To characterise the nature of the motion in a neighbourhood of the equilibrium point the notion of a *symmetric* and an *asymmetric* equilibrium point is introduced in section 3. An equilibrium point is symmetric (asymmetric) when a certain $n \times n$ matrix of stock–flow interaction terms that appears in the linearised Euler–Lagrange equations is symmetric (asymmetric). It is the symmetry or asymmetry of this matrix that plays the crucial role in determining whether cyclical motion arises. I show that cyclical motion in a neighbourhood of the equilibrium point only arises when the equilibrium is asymmetric. This condition is necessary, but it is not in general sufficient. To obtain a precise characterisation of cyclical motion, the linearised Euler–Lagrange equations are first reduced by a non-singular transformation to a *normal form*. This form of the equations of motion is then used is section 4, after certain additional structural assumptions have been made, to obtain a precise characterisation of the conditions under which there is cyclical motion in a neighbourhood of the equilibrium point. It should be pointed out that these structural assumptions are rather harsh, so that an important part of the problem in the general case still remains to be solved.

In section 5 these results are used to throw light on the way in which technological forces can lead to cyclical motion in a rational expectations equilibrium for a competitive industry. In the Lucas (1967) and Mortensen (1973) framework of the adjustment cost theory of the firm these technological forces are found to arise from asymmetries in the effect of adjustments in one capital good on the productivity of another capital good. The analysis of this section may be viewed as part of a highly preliminary attempt to develop a theory of the business cycle based on the theory of resource allocation, that meets two important tenets proposed by Lucas (1976): first, that the sequence of prices and quantities be determined through a process of competitive equilibrium, and second, that the expectations of agents be rational, in the sense that the anticipated sequence of prices formed on the basis of their expectations, coincides with the actual sequence of prices generated on the markets by their maximising behaviour. It should perhaps be pointed out, however, that such a formulation is in turn preliminary to a theory in which the concept of equilibrium is itself reformulated so as to take account of the fact that in a world with imperfect information, in which the organisation of markets is costly, the complete clearing of markets at each instant of time is unlikely to prove worthwhile. This phenomenon is in turn intimately connected with the presence of cyclical behaviour in the process of intertemporal resource allocation.

When the analysis of this paper is applied to the class of discounted infinite horizon maximum problems, as in Magill (1978b), a complex interaction arises between the skew-symmetric forces and the stability or instability of equilibrium. In some instances the presence of the skew-symmetric forces induces a stabilising effect, in other instances a destabilising effect.

On a historical note I might add that in the classical investigations of Lotka (1924) and Volterra (1931) on the evolution of interacting predator–prey biological species, it is precisely the presence of a fundamental skew-symmetric matrix arising from the predator–prey interactions, that gives rise to the cyclical behaviour in the number of individuals of each species.[2]

2. The basic maximum problem

Let $k(t)$, $t \in I = [0, \infty)$ denote the state of the economic system at the instant t, where $k(t) \in K$ (the state space), a convex subset of R^n, $n \geqq 1$. $k(t)$ is typically a vector of capital stocks of n different commodities.

Definition. For fixed $k_0 \in K$, the class of absolutely continuous paths

$$k(t) = k_0 + \int_0^t \dot{k}(\tau)\,\mathrm{d}\tau : I \to K, \tag{1}$$

for which

$$\|k(t)\| \leqq \|k_0\| + \int_0^t \|\dot{k}(\tau)\|\,\mathrm{d}\tau < \infty \quad \text{for all} \quad t \in I, \tag{1'}$$

where $\| \; \|$ denotes the standard Euclidean norm, is called the class of *feasible paths* and is denoted by $\mathscr{P}$. It is convenient to let $\{k, \dot{k}\}$ denote the path (1).

Let $\Pi \subseteq R^s$, $s \geqq 1$, denote the parameter space. We consider a vector of exogenous parameters $\pi \in \Pi$ and a real valued instantaneous profit (utility)

[2]For an excellent exposition of the Lotka–Volterra theory in English the reader is referred to the book by D'Ancona (1954). See also Samuelson's (1971) discussion.

function

$$L(k, \dot{k}; \pi): K \times R^n \times \Pi \to R, \tag{2}$$

satisfying the following:

Assumption 1. (Concavity, differentiability). $L(\cdot;\pi)$ is a C^r strictly concave function on $K \times R^n$ for all $\pi \in \Pi$, where $r \geqq 2$.

The explicit dependence of $L(k, \dot{k}; \pi)$ on the parameter π is sometimes omitted to simplify the notation. The function L and the feasible paths $\mathscr{P}$ lead to the following:

Maximum Problem. Find a feasible path $\{k, \dot{k}\} \in \mathscr{P}$ such that

$$\varliminf_{T \to \infty} \int_0^T (L(k(t), \dot{k}(t)) - L(\bar{k}(t), \dot{\bar{k}}(t)))\,dt \geqq 0, \tag{$\mathscr{M}$}$$

for all $\{\bar{k}, \dot{\bar{k}}\} \in \mathscr{P}$. The path $\{k, \dot{k}\} \in \mathscr{P}$ is said to be *optimal.*

Definition. Let $\mathscr{P}^*$ denote the class of absolutely continuous price paths

$$p(t) = p_0 + \int_0^t \dot{p}(\tau)\,d\tau : I \to R^n, \tag{3}$$

for which

$$\|p(t)\| \leqq \|p_0\| + \int_0^t \|\dot{p}(\tau)\|\,d\tau < \infty \quad \text{for all} \quad t \in I.$$

We let $\{\dot{p}, p\}$ denote the path (3).

Definition. A feasible path $\{k, \dot{k}\} \in \mathscr{P}$ is *competitive* if there exists an absolutely continuous path of prices $\{\dot{p}, p\} \in \mathscr{P}^*$ such that

$$L(k(t), \dot{k}(t)) + p(t)\dot{k}(t) + \dot{p}(t)k(t) \geqq L(\bar{k}(t), \dot{\bar{k}}(t)) + p(t)\dot{\bar{k}}(t) + \dot{p}(t)\bar{k}(t), \tag{4}$$

for all $(\bar{k}(t), \dot{\bar{k}}(t)) \in K \times R^n$, for almost all $t \in I$.

Remark. $(1, p(t))$ denotes the vector of (imputed) output prices and $-\dot{p}(t)$ the vector of (imputed) rental costs at time $t \in I$ so that

$$L(k(t), \dot{k}(t)) + p(t)\dot{k}(t) + \dot{p}(t)k(t)$$

is the (imputed) profit which is maximised at almost every instant by a competitive path.

Lemma 1. (i) *If Assumption 1 holds then* $\{k,\dot{k}\}\in\mathscr{P}$ *is competitive if and only if*

$$(\dot{p},p)=-(L_k,L_{\dot{k}}) \quad \textit{for almost all} \quad t\in I, \tag{5}$$

which in turn is equivalent to the Euler–Lagrange equation

$$L_k-\frac{\mathrm{d}}{\mathrm{d}t}(L_{\dot{k}})=L_k-L_{\dot{k}\dot{k}}\ddot{k}-L_{\dot{k}k}\dot{k}=0. \tag{$\mathscr{L}$}$$

(ii) *A competitive path* $\{k,\dot{k}\}\in\mathscr{P}$ *which satisfies the transversality condition*

$$\overline{\lim_{t\to\infty}}\, p(t)k(t)<\alpha \quad \textit{for some constant } \alpha,$$

is optimal among paths $\{\bar{k},\dot{\bar{k}}\}\in\mathscr{P}$ *which satisfy*

$$\underline{\lim_{t\to\infty}}\, p(t)\bar{k}(t)>\beta \quad \textit{for some constant } \beta.$$

Proof. (i) Immediate from (4). (ii) See Magill (1977b, lemma 2, p. 177). ■

Definition. A path $\{k,\dot{k}\}\in\mathscr{P}$ which satisfies ($\mathscr{L}$) with $\ddot{k}(t)=\dot{k}(t)=0$, for all $t\in I$, is called an *equilibrium point* (*stationary state*);

$$\mathscr{E}=\{(k^*,\pi)\in K\times\Pi \,|\, L_k(k^*,0;\pi)=0\}$$

is called the *equilibrium set* for the maximum problem ($\mathscr{M}$).

Definition. The function $\phi(k;\pi):K\times\Pi\to R$ defined by the line-integral

$$\phi(k;\pi)={}_{\gamma}\int_0^k L_k(\bar{k},0;\pi)'\,\mathrm{d}\bar{k},$$

where γ denotes the line-segment joining 0 and k, is called the *steady state profit function*, in view of eq. (4).

Remark. The steady state profit function attains a maximum at an equilibrium point.

Definition. Let $(k^*,\pi)\in\mathscr{E}$. The local coordinates around the equilibrium point $k^*=k^*(\pi)$ are defined by

$$x=k-k^*.$$

Let $\mathscr{P}'$ denote the class of absolutely continuous paths $\{x,\dot{x}\}$ for which $(k,\dot{k})\in\mathscr{P}$. *The second variation problem* about k^*,

$$\inf_{\{x,\dot{x}\}\in\mathscr{P}'} -\tfrac{1}{2}\int_0^\infty L^0(x(t),\dot{x}(t))\,\mathrm{d}t, \qquad (\mathscr{M}')$$

where

$$L^0(x,\dot{x})=\begin{bmatrix}x\\ \dot{x}\end{bmatrix}'\begin{bmatrix}L^*_{kk} & L^*_{k\dot{k}}\\ L^*_{\dot{k}k} & L^*_{\dot{k}\dot{k}}\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\end{bmatrix},$$

and where the asterisk indicates that the Hessian matrices are evaluated at $(k^*,0;\pi)$, has associated with it the Euler–Lagrange equations

$$L^*_{\dot{k}\dot{k}}\ddot{x}+(L^*_{\dot{k}k}-L^*_{k\dot{k}})\dot{x}-L^*_{kk}x=0, \qquad (\mathscr{L}')$$

which are the linearised equations for $(\mathscr{L})$ about k^*.

In order to be sure that the linearised equations $(\mathscr{L}')$ reveal the local topological structure of the solution of $(\mathscr{M})$ in the neighbourhood of an equilibrium point we need to distinguish certain types of equilibria.

Definition. An equilibrium point $k^*=k^*(\pi)$ is called *regular* (*hyperbolic*) if $\lambda_i\neq 0(\mathrm{Re}(\lambda_i)\neq 0)$, $i=1,\dots,2n$, where $\lambda_i\in C$ is a root of the characteristic polynomial

$$D(\lambda_i)=|L^*_{\dot{k}\dot{k}}\lambda_i^2+(L^*_{\dot{k}k}-L^*_{k\dot{k}})\lambda_i-L^*_{kk}|=0. \qquad (6)$$

We let $\mathscr{E}^r(\mathscr{E}^h)$ denote the set of regular (hyperbolic) equilibria in $\mathscr{E}$.

Lemma 2. Let $k^*=k^*(\pi)$ where $(k^*,\pi)\in\mathscr{E}$, then $k^*\in\mathscr{E}^r$ if and only if

$$\Delta=|L_{kk}(k^*,0;\pi)|\neq 0. \qquad (7)$$

Proof. If $\lambda_1,\dots,\lambda_{2n}$ denote the roots of (6), then for some $\alpha\neq 0$

$$D(\lambda)=\alpha(\lambda_1-\lambda)\dots(\lambda_{2n}-\lambda),$$

so that $\Delta=(-1)^nD(0)=(-1)^n\alpha\lambda_1\dots\lambda_{2n}\neq 0$ if and only if $k^*\in\mathscr{E}^r$. ∎

Remark. An equilibrium point k^* is said to be isolated if there is a neighbourhood of k^* containing no other equilibrium points than k^*. In view of Lemma 2, it follows from the implicit function theorem, that regular equilibria are isolated. A regular equilibrium point $k^*(\pi)$ is a point of intersection of the n, $(n-1)$-dimensional hypersurfaces

$$L_{k_i}(k^*,0;\pi)=0, \qquad i=1,\ldots,n, \tag{8}$$

in R^n. Thus an equilibrium point $k^*(\pi)$ is regular if and only if the gradients

$$(L^*_{k_ik_1},\ldots,L^*_{k_ik_n}), \qquad i=1,\ldots,n,$$

exist and are linearly independent.

Remark. Hyperbolic equilibria are of importance in the analysis that follows since it is only for these equilibria that the linearised equations $(\mathscr{L}')$ reveal the topological structure of the trajectories that are solutions of $(\mathscr{L})$ in a neighbourhood of an equilibrium point. Since hyperbolic equilibria are the important subset of the set of regular equilibria for which the linear theory is applicable, it is desirable to have a sufficient condition ensuring the existence of such equilibria.

Assumption 2. (Productivity). There exist $\underline{k}_i$, $\bar{k}_i$, $i=1,\ldots,n$, such that

$$\underline{K}=\{k\in R^n|-\infty<\underline{k}_i<k_i<\bar{k}_i<\infty, \quad i=1,\ldots,n\}\subset K,$$

and for all $j=1,\ldots,n$,

$$L_{k_j}(k_1,\ldots,\underline{k}_j,\ldots,k_n,0,\ldots,0;\pi)>0,$$
$$L_{k_j}(k_1,\ldots,\bar{k}_j,\ldots,k_n,0,\ldots,0;\pi)<0,$$

for all $k_i\in(\underline{k}_i,\bar{k}_i), i\neq j$.

Lemma 3. If Assumptions 1, 2 are satisfied and if $\pi\in\Pi$ is a parameter value for which $L^0(x,\dot{x};\pi)$ is negative definite then there exists a hyperbolic equilibrium point $k^*\in\underline{K}$

Proof. It follows from the classical index theorem of Kronecker–Poincaré (1886, ch. XVIII) and Assumption 2 that there exists a regular equilibrium point $k^*\in\underline{K}$. Theorem 2 of Levhari–Liviatan (1972) and the fact that $L^0(x,\dot{x};\pi)$ is negative definite implies that no root of the characteristic polynomial (6) can be pure imaginary, so that $\mathrm{Re}(\lambda_i)\neq 0$, $i=1,\ldots,2n$, and k^* is a hyperbolic equilibrium point. ■

Remark. In view of Assumption 1 the equilibrium point k^* in Lemma 3 is unique.

Remark. Assumption 2 is a natural economic condition on the productivity of each of the capital goods. For each capital good j, the marginal profit (L_{k_j}) from an additional unit of j must be positive (negative) when the endowment of this capital good is sufficiently small (large), independent of the endowments of the other capital goods $(i \neq j)$.

The following proposition assures us that under certain additional conditions a solution of the maximum problem $(\mathcal{M})$ exists. This solution has, furthermore, the important property that it converges to the equilibrium point k^*.

Proposition 1. Let the assumptions of Lemma 3 be satisfied and let k^* denote the associated equilibrium point. If the feasible paths are restricted to the subset $\mathcal{F} \subset \mathcal{P}$ for which $(k(t), \dot{k}(t)) \in K \times Q$ for all $t \in I$, where $K \times Q$ is a compact convex subset of $R^n \times R^n$ and if there exists $(\bar{k}, \dot{\bar{k}}) \in \mathcal{F}$ such that for some $0 < T < \infty$, $\bar{k}(t) = k^*$ for all $t \geqq T$, then (i) *there exists an optimal path $(k, \dot{k}) \in \mathcal{F}$, and* (ii) *$k(t) \to k^*$ as $t \to \infty$.*

Proof. The result follows from Lemma 3 above and from Theorem 4.1 and Corollary 4.1 in Brock–Haurie (1976). ■

3. Symmetric and asymmetric equilibria and transformation to normal form

To obtain a more complete understanding of the behaviour of an optimal path in a neighbourhood of the equilibrium point k^* we need to make a distinction between two different types of equilibria that can arise.

Definition. Let $(k^*, \pi) \in \mathcal{E}$. k^* will be called a *symmetric* (*asymmetric*) *equilibrium point* if

$$L_{\dot{k}k}(k^*, 0; \pi) - L_{k\dot{k}}(k^*, 0; \pi) = 0 \; (\neq 0).$$

Lemma 4. If k^* is an asymmetric equilibrium point then $L^*_{\dot{k}k} - L^*_{k\dot{k}}$ is a skew-symmetric matrix.

Proof. $L^*_{\dot{k}k} = (L^*_{k\dot{k}})'$ implies $(L^*_{\dot{k}k} - L^*_{k\dot{k}})' = -(L^*_{\dot{k}k} - L^*_{k\dot{k}})$. ■

Definition. Let $k^* \in \mathcal{E}^h$. A solution of $(\mathcal{M})$ will be called *locally cyclical* (*locally monotone*) in a neighbourhood of k^* if the characteristic polynomial $D(\lambda)$ has at least one (no) pair of complex conjugate roots.

In the analysis that follows we restrict our attention to those parameter values $\pi \in \Pi$ for which the following assumption is satisfied:

Assumption 1'. (Definiteness). $(k^*, \pi) \in \mathscr{E}$ is such that $L^0(x, \dot{x}; \pi)$ is negative definite.

To simplify the notation we let

$$A = -L^*_{kk}, \quad B = -L^*_{\dot{k}\dot{k}}, \quad N = -L^*_{k\dot{k}}, \quad C = N - N', \tag{9}$$

so that the linearised Euler–Lagrange equation ($\mathscr{L}'$) reduces to

$$B\ddot{x} - C\dot{x} - Ax = 0, \tag{$\mathscr{L}'$}$$

where A and B are positive definite (symmetric) matrices in view of Assumption 1' and C is a skew-symmetric matrix by Lemma 4.

Definition. Let E and F be $n \times n$ matrices with real coefficients and let F be a positive definite (symmetric) matrix. We say that $\alpha_i \in C$ is an *eigenvalue of E in the metric of F* and $w^i \in C^n$, $w^i \neq 0$, is an associated *eigenvector* if

$$(E - \alpha_i F) w^i = 0. \tag{10}$$

Lemma 5. (i) *A has n real positive eigenvalues, $\alpha_1, \ldots, \alpha_n$, and n real associated eigenvectors, $w^1, \ldots, w^n$, in the metric of B.* (ii) *If the eigenvalues are placed in order of decreasing magnitude,*

$$\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq \alpha_n > 0,$$

then the following maximum property holds

$$\alpha_i = \max_{x \in \mathscr{B}_i} x'Ax = w^{i\prime} A w^i, \qquad i = 1, \ldots, n,$$

$$\mathscr{B}_i = \{x \in R^n | x'Bx = 1, \quad x'Bw^j = 0, \quad j = 1, \ldots, i-1\}.$$

Proof. (i) follows from Theorem 8 in Gantmacher (1960, p. 310) and the fact that A and B are positive definite (symmetric) matrices. (ii) This is the well-known Courant–Fischer result, see Gantmacher (1960, theorem 11, p. 319). ∎

Lemma 5 and the fact that $A = -\phi_{kk}(k^*; \pi)$ lead naturally to the following:

Definition. $\alpha_1, \ldots, \alpha_n$ will be called the *steady state profit rates* and $w^1, \ldots, w^n$ the *directions of maximum profit.*

With these definitions and properties in mind we obtain a complete characterisation of the behaviour of an optimal path in a neighbourhood of a symmetric equilibrium point.

Proposition 2. Let Assumption 1′ be satisfied. If k^* is a symmetric equilibrium point then (i) *a solution of* ($\mathcal{M}$) *is locally monotone in a neighbourhood of k^*,* (ii) *the eigenvalues $\pm\lambda_1,\ldots,\pm\lambda_n$ of the linearised Euler–Lagrange equation* ($\mathcal{L}'$) *are determined by the steady state profit rates,*

$$\lambda_i = \pm\sqrt{\alpha_i}, \qquad i=1,\ldots,n,$$

and the eigenvectors of ($\mathcal{L}'$) *coincide with the directions of maximum profit $w^1,\ldots,w^n$.*

Proof. (i) follows from (ii) and the fact that $\alpha_i>0, i=1,\ldots,n$. (ii) follows at once from ($\mathcal{L}'$) and the definition (10). ■

Remark. It is evident that only the negative eigenvalues $\lambda_i=-\sqrt{\alpha_i}$, $i=1,\ldots,n$, are used in characterising a solution of ($\mathcal{M}'$) and hence of ($\mathcal{M}$) in a neighbourhood of k^*.

Lemma 6. There exists a non-singular transformation to principal coordinates

$$x = Wy, \tag{11}$$

which reduces the linearised Euler–Lagrange equations ($\mathcal{L}'$) *to the normal form*

$$\ddot{y} - \Gamma\dot{y} - \mathcal{A}y = 0, \tag{$\mathcal{L}''$}$$

where

$$\Gamma = \begin{bmatrix} 0 & \gamma_{12} & \cdots & \gamma_{1n} \\ -\gamma_{12} & 0 & \cdots & \gamma_{2n} \\ \vdots & \vdots & & \vdots \\ -\gamma_{1n} & -\gamma_{2n} & \cdots & 0 \end{bmatrix}, \qquad \mathcal{A} = \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{bmatrix}.$$

Proof. By Theorem 9 in Gantmacher (1960, p. 314) the $n\times n$ matrix of eigenvectors $W=[w^1\ldots w^n]$ may be chosen in such a way that

$$W'BW = I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}, \qquad W'AW = \mathcal{A} = \begin{bmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{bmatrix}.$$

Under the transformation (11), ($\mathscr{L}'$) reduces to

$$W'BW\ddot{y}-W'CW\dot{y}-W'AWy=\ddot{y}-\Gamma\dot{y}-\mathscr{A}y=0,$$

where

$$W'CW=\Gamma. \quad \blacksquare \tag{12}$$

Remark. At a symmetric equilibrium point, the linearised Euler–Lagrange equations ($\mathscr{L}''$) in principal coordinates separate into n independent one-sector systems. At an asymmetric equilibrium point there is a skew-symmetric interaction between the n sectors (or capital goods). The term $\Gamma\dot{y}$ imposes velocity dependent rotational forces on the economic system which, under conditions to be examined in the next section lead to cyclical motion in a neighbourhood of the equilibrium point k^*.

It should be noted that the skew-symmetric terms $\Gamma\dot{y}$ have not been artificially introduced. These terms arise naturally from the structure of the maximum problem ($\mathscr{M}$) in the neighbourhood of an asymmetric equilibrium point and are present in maximum problems of a quite general form. It is these skew-symmetric forces that may be viewed as the cause of cyclical behaviour for the class of maximising problems considered in this paper.

4. Characterising locally cyclical motion

In this section I will show that if certain additional structural assumptions are made then we can obtain a precise statement of the conditions under which a solution of ($\mathscr{M}$) is locally cyclical in a neighbourhood of the equilibrium point k^*. I consider two cases. In the first case an assumption is made that reduces the number of interaction terms in the matrix Γ so that it reduces to block-diagonal form, while $\mathscr{A}$ is an arbitrary positive definite diagonal matrix. In the second case the eigenvalues of $\mathscr{A}$ are assumed to be identical, or by continuity, to differ by a very small amount, Γ being an arbitrary skew-symmetric matrix. It remains an interesting open problem to obtain a precise statement of the conditions under which the motion in a neighbourhood of k^* is locally cyclical in the general case where $\mathscr{A}$ is an arbitrary positive definite diagonal matrix and Γ is an arbitrary skew-symmetric matrix.

Proposition 3. Let Assumption 1′ be satisfied. If there exists a re-ordering of the principal coordinates $y_1,\ldots,y_n$, such that

$$\Gamma=\begin{bmatrix}\Gamma_1 & & 0\\ & \ddots & \\ 0 & & \Gamma_{(n/2)}\end{bmatrix}, \qquad \Gamma_i=\begin{bmatrix}0 & \gamma_i\\ -\gamma_i & 0\end{bmatrix}, \qquad i=1,\ldots,(n/2), \tag{13}$$

where $(n/2)=n/2$ when n is even, $(n/2)=(n+1)/2$ when n is odd and where $\Gamma_{(n+1)/2}=[0]$ when n is odd, then a solution of $(\mathcal{M})$ is locally cyclical (locally monotone) in a neighbourhood of k^* if and only if

$$|\gamma_i|>(<)|\sqrt{\alpha_i}-\sqrt{\alpha_{i'}}|, \qquad i=1,\ldots,(n/2), \tag{14}$$

where $(\alpha_i,\alpha_{i'})$ are the components of $\mathscr{A}$ associated with γ_i.

Proof. $(\mathscr{L}'')$ decomposes into $(n/2)$ pairs of second order differential equations when n is even $[(n/2)-1$ when n is odd$]$. For each such pair the characteristic polynomial is

$$D(\lambda_i)=\lambda_i^4+(\gamma_i^2-\alpha_i-\alpha_{i'})\lambda_i^2+\alpha_i\alpha_{i'}=0, \qquad i=1,\ldots,(n/2).$$

Let $\lambda_i=\mu_i+i\nu_i$ and let $\lambda_i^2=\xi_i=a_i+ib_i$, then

$$a_i=\tfrac{1}{2}(\alpha_i+\alpha_{i'}-\gamma_i^2), \qquad b_i=\tfrac{1}{2}\sqrt{(-H_i)J_i},$$

where

$$H_i=\gamma_i^2-(\sqrt{\alpha_i}+\sqrt{\alpha_{i'}})^2, \qquad J_i=\gamma_i^2-(\sqrt{\alpha_i}-\sqrt{\alpha_{i'}})^2,$$

so that $\lambda_i^2=\xi_i$ implies

$$\mu_i=\frac{1}{\sqrt{2}}\left(\frac{b_i}{\theta_i}\right), \qquad \nu_i=\frac{\theta_i}{\sqrt{2}}, \qquad \theta_i=\sqrt{-a_i+\sqrt{a_i^2+b_i^2}},$$

which in turn implies

$$\mu_i=\tfrac{1}{2}\sqrt{-H_i}, \qquad \nu_i=\tfrac{1}{2}\sqrt{J_i}. \tag{15}$$

Thus the roots of the characteristic polynomial are given by

$$\lambda_i=\tfrac{1}{2}(\pm\sqrt{-H_i}\pm\sqrt{-J_i}), \qquad i=1,\ldots,(n/2). \tag{16}$$

In view of Assumption 1′, $\sqrt{\alpha_i}>0$, $\sqrt{\alpha_{i'}}>0$, from which we readily deduce that the roots are complex if and only if $J_i>0$, $i=1,\ldots,(n/2)$. ■

Remark. Let $\mathcal{N}=W'NW=\{\nu_{ij}\}$ then Assumption 1′ implies

$$\begin{aligned}\sqrt{\alpha_i}+\sqrt{\alpha_{i'}}&>|\nu_{2i-1,2i}|+|\nu_{2i,2i-1}|\\ &\geqq|\nu_{2i-1,2i}-\nu_{2i,2i-1}|=|\gamma_i|, \qquad i=1,\ldots,(n/2),\end{aligned}$$

so that in view of (16), the characteristic polynomial has no pure imaginary roots, in accordance with the result of Levhari–Liviatan (1972, theorem 8, p. 91).

Under Assumption 1′ the trajectory which minimises ($\mathcal{M}'$) is asymptotically stable. Cyclical motion, when it arises, is therefore damped and the n roots in (16) which characterise the trajectory have negative real parts (as observed earlier). We may ask how a change in the magnitude $|\gamma_i|$ of any one of the components of the basic skew-symmetric matrix Γ affects (i) the real parts and hence the magnitude of the damping and (ii) the imaginary parts and hence the period of the cycles.

Corollary. If the components $(y_i(t), y_{i'}(t))$ *are cyclical, for any* $i=1,\dots,(n/2)$, *then an increase in the skew-symmetric component* $|\gamma_i|$ *leads to* (i) *a reduction in the exponential damping, and* (ii) *a reduction in the period of the cycles for the components* $(y_i(t), y_{i'}(t))$.

Proof. Set $\lambda_i=\mu_i+i\nu_i$ and use (15), then (i) is immediate and (ii) follows from the fact that the period of each cycle is $(2\pi/\nu_i)$. ■

Remark. A sufficient condition for Γ to have the block-diagonal form (13) is that the matrices in (9) have the block diagonal form

$$A=\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_{(n/2)} \end{bmatrix},\quad B=\begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_{(n/2)} \end{bmatrix},\quad N=\begin{bmatrix} N_1 & & 0 \\ & \ddots & \\ 0 & & N_{(n/2)} \end{bmatrix},$$

$$A_i=\begin{bmatrix} a_{i1} & a_{i2} \\ a_{i'1} & a_{i'2} \end{bmatrix},\quad B_i=\begin{bmatrix} b_{i1} & b_{i2} \\ b_{i'1} & b_{i'2} \end{bmatrix},\quad N_i=\begin{bmatrix} n_{i1} & n_{i2} \\ n_{i'1} & n_{i'2} \end{bmatrix},$$

for $i=1,\dots,(n/2)$, where the last elements $A_{(n/2)}$, $B_{(n/2)}$ and $N_{(n/2)}$ reduce to scalars when n is odd.

The cycling condition (14), stated in terms of the derived coefficients $(\alpha_i, \alpha_{i'}; \gamma_i)$ may be transformed into a condition on the coefficients of the original matrices $(A_i, B_i; N_i)$. Since the matrix of eigenvectors W is block-diagonal,

$$W=\begin{bmatrix} W_1 & & 0 \\ & \ddots & \\ 0 & & W_{(n/2)} \end{bmatrix},$$

this implies

$$|W_i'B_iW_i|=|W_i|^2|B_i|=1 \quad \text{so that} \quad W_i=1/\sqrt{|B_i|}.$$

Thus

$$\gamma_i=|W_i|g_i=g_i/\sqrt{|B_i|} \quad \text{where} \quad g_i=n_{i2}-n_{i'1},$$

so that

$$\alpha_i, \alpha_{i'}=(1/|B_i|)(\Delta_i/2\pm\sqrt{(\Delta_i/2)^2-|A_i|\,|B_i|}),$$

where

$$\Delta_i=a_{i1}b_{i'2}+a_{i'2}b_{i1}-2a_{i2}b_{i2}.$$

Thus the cycling condition (14) becomes

$$|g_i|>|\Delta_i-2\sqrt{|A_i|\,|B_i|}|. \tag{17}$$

When A and B are both diagonal this reduces to the simple condition

$$|g_i|>|\sqrt{a_{i1}b_{i'2}}-\sqrt{a_{i'2}b_{i'1}}|. \tag{17'}$$

Proposition 4. Let Assumption 1′ be satisfied. If $\alpha_1=\alpha_2=\ldots=\alpha_n=\alpha^*$, then a solution of $(\mathscr{M})$ is locally cyclical in a neighbourhood of k^* if and only if k^* is an asymmetric equilibrium point.

Proof. Since $\alpha_1=\alpha_2=\ldots=\alpha_n=\alpha^*$, $\mathscr{A}=\alpha^* I$ and the eigenvalue problem for $(\mathscr{L}'')$,

$$(\lambda^2 I-\ \Gamma\lambda-\mathscr{A})w=0, \tag{18}$$

reduces to an eigenvalue problem for Γ,

$$\left[\Gamma-\left(\frac{\lambda^2-\alpha^*}{\lambda}\right)I\right]w=0. \tag{19}$$

It is well-known that the eigenvalues of a skew-symmetric matrix are pure imaginary [Gantmacher (1960, p. 285)]. Let

$$\pm i\gamma_1,\ldots,\pm i\gamma_{(n/2)},$$

where $\gamma_{(n/2)}=0$ when n is odd, denote the eigenvalues of Γ and let $\lambda=\mu+i\nu$

denote an eigenvalue of (18). Then by (19), for some $j=1,\ldots,(n/2)$,

$$(\lambda^2-\alpha^*)/\lambda=i\gamma_j,$$

so that

$$\mu=\pm\sqrt{\alpha^*-(\gamma_j/2)^2}, \qquad \nu=(\gamma_j/2).$$

Thus the eigenvalues of (18) are given by

$$\lambda_j=\pm\sqrt{\alpha^*-(\gamma_j/2)^2}\pm i(\gamma_j/2), \qquad j=1,\ldots,(n/2),$$

from which the result follows. ■

5. Cyclical motion in rational expectations equilibrium

In several recent contributions Lucas (1975, 1976) has emphasised the importance of developing a theory of the business cycle in which prices and quantities are determined at each instant of time through competitive equilibrium and in which the expectations of agents are rational in the sense of Muth (1961). Lucas has also emphasised the role of uncertainty in generating the observed pattern of business cycles.

In this section I will use the results of the previous sections to examine a rational expectations equilibrium for a competitive industry with a fixed finite number of firms in which each firm behaves according to the standard Lucas (1967) and Mortensen (1973) adjustment cost theory of the firm. The analysis of rational expectations for the industry is made possible by the introduction of an extended integrand similar to that employed by Brock–Magill (1978) and Scheinkman (1976) and originally introduced by Lucas and Prescott (1971).

While the presence of uncertainty is of undisputed importance in generating the observed pattern of business cycles, it may well be of interest to seek causes of cycling which are independent of the presence of uncertainty but which are consistent with the postulates of competitive equilibrium and rational expectations. Thus while the analysis of this section in no way pretends to form a theory of the business cycle, it seeks to explore the ways in which technological forces arising from the recursive nature of the production process may act on representative firms within an industry so as to cause cycling in the process of competitive equilibrium over time.

Consider therefore an industry composed of M representative firms, each producing the same industry good with the aid of n capital goods. I assume that each firm forms identical expectations about the industry's product's price path which is a measurable function

$$r(t):[0,\infty)\to R^{+}. \tag{20}$$

The instantaneous flow of profit of the representative firm is given by $r(t)f(k(t),\dot{k}(t))$, where $f(k,\dot{k})\in C^2$, incorporates both adjustment costs and the cost of purchasing new capital equipment and is a strictly concave function in $(k,\dot{k})$, where $k=(k_1,\dots,k_n)$ denotes the vector of capital goods. To simplify the analysis and make possible the application of the results of the previous sections I assume that the representative firm faces a zero interest rate. Let $K=R^{n+}$ denote the non-negative orthant, then for a fixed initial capital endowment $k_0\in K$, the firm manager seeks an absolutely continuous capital expansion path

$$k(t)=k_0+\int_0^t \dot{k}(\tau)\,\mathrm{d}\tau : I\to K,$$

satisfying (1′), which maximises, in the sense of the overtaking criterion ($\mathscr{M}$), the future stream of profit

$$\varliminf_{T\to\infty}\int_0^T r(t)(f(k(t),\dot{k}(t))-f(\bar{k}(t),\dot{\bar{k}}(t)))\,\mathrm{d}t\geqq 0, \tag{R}$$

for all $\{\bar{k},\dot{\bar{k}}\}\in\mathscr{P}$.

The total market supply forthcoming at each instant on the product market

$$Q_S(t)=Mf(k(t),\dot{k}(t)),\qquad t\in[0,\infty),$$

has a complex functional dependence on the price path (20), since it arises as a by-product of the solution of the basic maximum problem (R) by each firm. On the demand side of the market I make the simplifying assumption that the total market demand depends only on the current market price

$$Q_D(t)=\psi^{-1}(r(t)),\qquad t\in[0,\infty),$$

where

$$\psi\in C^1,\quad \psi(Q)>0,\quad \psi'(Q)<0,\quad Q\geqq 0.$$

Definition. A *rational expectations equilibrium* for the product market of the industry is a measurable price path (20) such that

$$Q_D(t)=Q_S(t)\quad\text{for almost all}\quad t\in[0,\infty). \tag{E}$$

Definition. The function $\Psi(Mf(k,\dot{k}))$ where

$$\Psi(Q)=\int_0^Q \psi(y)\,\mathrm{d}y, \qquad Q\geqq 0,$$

denotes the integral of the demand function, so that

$$\Psi\in C^2, \quad \Psi'(Q)=\psi(Q), \quad \Psi''(Q)=\psi'(Q)<0, \quad Q\geqq 0,$$

is called the *extended integrand.* I call the extended *integrand problem,* the problem of finding an absolutely continuous path $\{k,\dot{k}\}\in\mathscr{P}$ which maximises this integrand, in the sense of the overtaking criterion $(\mathscr{M})$,

$$\varliminf_{T\to\infty}\int_0^T [\Psi(Mf(k(t),\dot{k}(t)))-\Psi(Mf(\bar{k}(t),\dot{\bar{k}}(t)))]\,\mathrm{d}t\geqq 0, \tag{$\mathscr{I}$}$$

for all $\{\bar{k},\dot{\bar{k}}\}\in\mathscr{P}$.

Our analysis of the cyclical properties of rational expectations equilibrium will be based on the following proposition which is a straightforward adaptation of the result of Brock–Magill (1978, theorem 5) and Scheinkman (1976, section 4) to the undiscounted case. This proposition transforms the analysis of rational expectations equilibrium from a direct analysis of the representative firms problem (R) combined with the market equilibrium condition (E) to an indirect analysis of the extended integrand problem $(\mathscr{I})$.

Proposition 5. If $k(t)$ is the solution of the Euler–Lagrange equation for $(\mathscr{I})$,

$$\Psi'(Mf(k,\dot{k}))f_k(k,\dot{k})-\frac{\mathrm{d}}{\mathrm{d}t}(\Psi'(Mf(k,\dot{k}))f_{\dot{k}}(k,\dot{k}))=0, \tag{21}$$

which satisfies the initial condition $k(0)=k_0$ and the transversality condition,

$$\varlimsup_{t\to\infty}[-\Psi'(Mf(k(t),\dot{k}(t)))f_{\dot{k}}(k(t),\dot{k}(t))]k(t)<\alpha, \tag{22}$$

and if for any feasible path $\{\bar{k},\dot{\bar{k}}\}\in\mathscr{P}$ with $\bar{k}(0)=k_0$,

$$\varliminf_{t\to\infty}[-\Psi'(Mf(k(t),\dot{k}(t)))f_{\dot{k}}(k(t),\dot{k}(t))]\bar{k}(t)>\beta, \tag{23}$$

for constants α,β, then the price path,

$$r(t)=\Psi'(Mf(k(t),\dot{k}(t))), \qquad t\in I, \tag{24}$$

is a rational expectations equilibrium for the product market of the industry.

Proof. (21) and (24) imply that the Euler–Lagrange equation ($\mathscr{L}$) for the problem (R) is satisfied,

$$r(t)f_k(k,\dot{k})-\frac{\mathrm{d}}{\mathrm{d}t}(r(t)f_{\dot{k}}(k,\dot{k}))=0.$$

(22)–(24) imply that the conditions of Lemma 1(ii) are satisfied for the problem (R). Since $\Psi'=\psi$, (24) implies that (E) holds. Thus each firm maximises (R), in the overtaking sense, and the market equilibrium condition (E) is satisfied. ■

Does a solution of the extended integrand problem ($\mathscr{I}$) exist which is a solution of the Euler–Lagrange equation (21)? Proposition 1(i) asserts the existence of a solution to the extended integrand problem, under suitable additional assumptions, but it does not assure us that this solution will satisfy the Euler–Lagrange equation (21). Some additional analysis is needed at this point which is examined in greater detail in the paper of Magill (1978a). For the present purposes we will suppose that suitable additional conditions can be imposed so that the path whose existence is asserted in Proposition 1(i) does indeed satisfy (21). Then the condition, $(k(t),\dot{k}(t))\in K\times Q$ for all $t\in I$, in Proposition 1, implies that (22) and (23) are satisfied. It follows from Proposition 5 that an industry rational expectations equilibrium exists. By Proposition 1(ii), $k(t)\to k^*$ as $t\to\infty$, so that the equilibrium path converges to a stationary equilibrium.

To analyse the behaviour of the equilibrium path in a neighbourhood of the steady state k^*, we evaluate the matrices in (9) for the extended integrand

$$A=-\Psi'^*f^*_{kk},\quad B=-(\Psi'^*f^*_{\dot{k}\dot{k}}+M\Psi''^*f^*_{\dot{k}}f^{*\prime}_{\dot{k}}),\quad N=-\Psi'^*f^*_{k\dot{k}},$$

so that

$$C=\Psi'^*(f^*_{\dot{k}k}-f^*_{k\dot{k}}),$$

where C is the basic skew-symmetric matrix to which the origin of cyclical motion in a neighbourhood of k^* may be traced. The cyclical forces thus have their origin in asymmetries in the effect of adjustments in one capital good on the marginal productivity of another capital good. When these asymmetries are absent as in the original model of Lucas (1967), where the adjustment costs are separable, $f(k,\dot{k})=u(k)+v(\dot{k})$ then the motion in a neighbourhood of k^* is locally monotone. The simplest precise condition for cycling is given by (17′). If we let $n=2$ and assume in addition that the production function $f(k,\dot{k})$ satisfies $f^*_{k_1k_2}=f^*_{\dot{k}_1\dot{k}_2}=0$ and $f^*_{\dot{k}}=0$ (or alternatively that $\Psi''^*=0$), then (17)′ gives

$$|f^*_{\dot{k}_2k_1}-f^*_{\dot{k}_1k_2}|>|\sqrt{f^*_{k_1k_1}f^*_{\dot{k}_2\dot{k}_2}}-\sqrt{f^*_{k_2k_2}f^*_{\dot{k}_1\dot{k}_1}}|.$$

Proposition 4 provides the rather special condition where any asymmetry in the effect of adjustments in one capital good on the productivity of another capital good immediately leads to cyclical motion.

Lucas (1975) has emphasised the difficulty of generating an equilibrium process in which there is cyclical motion and in which 'persistent, recurrent, unexploited profit opportunities' are absent. In the present context, if the industry good is storable there is an incentive for speculators to carry the commodity over from periods of relative abundance, when the price is low, to periods of relative scarcity, when the price is high, the extent of such arbitrage activity depending on the cost of storage. This arbitrage activity reduces the extent of cycling in both the price and the quantity traded. However if the commodity is perishable or if the cost of storage is sufficiently high the cycles will tend to persist.[3] These issues lead us to the theory of inventories, which constitute the classic method of inducing smoothing in the process of production over time and which must clearly play a central role in any more general theory of the business cycle.

[3]For an analysis of the way transactions costs influence inventory behaviour when the inventory consists of a portfolio of assets held by an investor, see Magill-Constantinides (1976). For a further discussion of the relation between inventories and investment the reader is referred to the original discussion of Eisner and Strotz (1963).

References

Brock, William A. and A. Haurie, 1976, On existence of overtaking optimal trajectories over an infinite time horizon, Mathematics of Operations Research 1, 337–346.

Brock, William A. and Michael J.P. Magill, 1978, Dynamics under uncertainty, Discussion Paper no. 324 (Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, IL), forthcoming in Econometrica.

Brock, William A. and José Schienkman, 1976, Global asymptotic stability of optimal control systems with applications to the theory of economic growth, Journal of Economic Theory 12, 164–190.

Cass, David and Karl Shell, 1976, The structure and stability of competitive dynamical systems, Journal of Economic Theory 12, 31–70.

D'Ancona, Umberto, 1954, The struggle for existence (E.J. Brill, Leiden).

Eisner, Robert and Robert Strotz, 1963, Determinants of business investment, in: Commission on Money and Credit, Impacts of monetary policy (Prentice-Hall, Englewood Cliffs, NJ).

Gantmacher, F.R., 1960, The theory of matrices, Vol. 1 (Chelsea, New York).

Levhari, David and Nissan Liviatan, 1972, On stability in the saddle-point sense, Journal of Economic Theory 4, 88-93.

Lotka, Alfred J., 1924, Elements of physical biology (Williams and Wilkins, Baltimore, MD).

Lucas, Robert E., 1967, Optimal investment policy and the flexible. accelerator, International Economic Review 8, 78–85.

Lucas, Robert E., 1975, An equilibrium model of the business cycle, Journal of Political Economy 83, 1113–1144.

Lucas, Robert E., 1976, Understanding business cycles, Paper prepared for the Kiel Conference on Growth without Inflation, June 22–23.

Lucas, Robert E. and Edward C. Prescott, 1971, Investment under uncertainty, Econometrica 39, 659-682.

McKenzie, Lionel W., 1976, Turnpike theory, Econometrica 44, 841–865.
Magill, Michael J.P., 1977a, A local analysis of n-sector capital accumulation under uncertainty, Journal of Economic Theory 15, 211–219.
Magill, Michael, J.P., 1977b, Some new results on the local stability of the process of capital accumulation, Journal of Economic Theory 15, 174–210.
Magill, Michael J.P., 1978a, Intertemporal equilibrium and the theory of investment, Discussion Paper (Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, IL).
Magill, Michael J.P., 1978b, The stability of equilibrium, Discussion Paper no. 306 (Center for Mathematical Studies in Economics and Management Science, Northwestern University, Evanston, IL), forthcoming in the International Economic Review.
Magill, Michael J.P. and George M. Constantinides, 1976, Portfolio selection with transactions costs, Journal of Economic Theory 13, 245–263.
Magill, Michael J.P. and José A. Scheinkman, 1977, Stability of regular equilibria and the correspondence principle for symmetric variational problems, Report no. 7736 (Center for Mathematical Studies in Business and Economics, University of Chicago, Chicago, IL), forthcoming in the International Economic Review.
Muth, John F., 1961, Rational expectations and the theory of price movements, Econometrica 29, 315–335.
Mortensen, Dale T., 1973, Generalized costs of adjustment and dynamic factor demand theory, Econometrica 41, 657–665.
Poincaré, Henri, 1886, Sur les courbes définies par les équations différentielles, Journal de Mathématiques (Ser. 4) 2, 151–217. Reprinted in: 1951, Oeuvres de Henri Poincaré, Vol. I (Gauthier-Villars, Paris) 167–222.
Ryder, Harl E. and Geoffrey M. Heal, 1973, Optimum growth with intertemporally dependent preferences, Review of Economic Studies 40, 1–31.
Samuelson, Paul, A., 1971, Generalized predator–prey oscillations in ecological and economic equilibrium, Proceedings of the National Academy of Sciences 68. Reprinted in: 1972, The collected scientific papers of Paul A. Samuelson, Vol. III (M.I.T. Press, Cambridge, MA) 487–490.
Scheinkman, José A., 1976, Stability of separable Hamiltonians and investment theory, Report no. 7648 (Center for Mathematical Studies in Business and Economics, University of Chicago, Chicago, IL), forthcoming in the Review of Economic Studies.
Volterra, Vito, 1931, Leçons sur la théorie mathématique de la lutte pour la vie (Gauthier-Villars, Paris).