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SOME RESULTS ON COMPARATIVE STATICS UNDER UNCERTAINTY*

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1. INTRODUCTION

Since Arrow [1965] first posed and studied the two-asset portfolio problem much interest has centered on the comparative statics analysis of decision problems under uncertainty. An interesting class of such problems arises when an agent's choice among uncertain prospects can be represented by a von Neumann-Morgenstern utility function. Thus, if an agent's random wealth is determined by a decision variable (x), an exogenous parameter (α) and an exogenous random variable (r) and if the decision variable is chosen so as to maximise the expected utility of wealth, then one can seek to characterise the qualitative change in the optimal decision when either the parameter α or the random variable r is changed.

In this paper, we present a complete solution of the comparative statics problem for the basic two asset portfolio problem: we give necessary and sufficient conditions for qualitative results both in the case of parameter changes and in the case of stochastically dominant shifts in the random variable. We also present results for the variant of the Ramsey saving problem first studied by Mirman [1971] and a model of the production decision of an individually owned firm which faces a random price for its output, first studied by Sandmo [1971]. In each case, the conditions required reduce to restrictions on the behavior of one of the three local measures of risk aversion associated with a von Neumann-Morgenstern utility function — absolute, relative or partial relative risk aversion (Arrow [1965], Pratt [1964], Menezes & Hanson [1970]).

Rothschild and Stiglitz [1971] have studied the comparative statics properties of this class of problems when the random variable r is subjected to a mean preserving spread. In this paper, we analyse the effect on the optimal decision of replacing the random variable r by a random variable $\hat{r}$ which dominates it with probability one. For the class of problems that we consider, the comparative statics properties arising from this type of change in r are equivalent to those that arise from a stochastically dominant shift in r (see footnote 3). This latter type of shift has been extensively studied (Hadar & Russel [1971], Hanoch & Levy [1969]).

Section 2 gives a brief statement of the approach used to study the comparative statics problems. Section 3 studies the two asset portfolio problem: this extends

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the earlier results of Arrow [1965], Fishburn & Porter [1976] and Merton [1982]. We give necessary and sufficient conditions for comparative statics properties for the optimal holdings of both assets for changes in the price of the risky asset (Propositions 1, 2), the initial wealth (Propositions 3, 4) and stochastically dominant shifts in the return on the risky asset (Proposition 5). This provides a complete characterisation of comparative statics for the two asset case.[2]

In Section 4, we show how the results on the two-asset portfolio problem can be used to analyse the theory of liquidity preference. Recall that two basic steps are involved in the Hicks-Tobin [1935, 1958] reformulation of the Keynesian theory of liquidity preference. The first is to show that the demand for money is positive even though its expected return is less than that on alternative risky assets; the second is to show that the demand for money is inversely related to the rate of interest. Tobin showed the first but not the second property. For the Tobin model, Arrow [1965] showed that decreasing absolute risk aversion gives the second property. The Tobin-Arrow model is based however on the restrictive assumption that the price of the risky asset (consol) is unaffected by changes in the current interest rate. In section 4, we drop this restrictive assumption and give a precise condition for the demand for money to be inversely related to the rate of interest. This is the condition that the agent's partial relative risk aversion be bounded above by 1. This condition also enables us to show that the demand for money increases when the agent's expectations regarding the next period interest rate shift up ward.

Section 5 studies a two-period version of the Ramsey saving problem under uncertainty: how much of its output should a nation save when the return on saving is uncertain? We study the behaviour of optimal saving (investment) when there is an improvement in technology modelled as a stochastically dominant shift in the random variable summarising the state of technology in the second period. We show that an improvement in technology affects the incentive to save in two opposing ways: by increasing output in the second period it lowers the marginal utility of consumption (the magnitude of this effect depending on the elasticity of the marginal utility of consumption) and hence the incentive to save; on the other hand it raises the marginal product of capital (the magnitude of this effect depending on the elasticity of substitution between capital and the technology input) which increases the incentive to invest. We show that an improvement in technology leads to an increase (decrease) in optimal saving if the product of these two elasticities is less (greater) than one. This provides a

[2] It is perhaps of some interest to note that while we studied the portfolio problem with two or more risky assets we were unable to extend these comparative statics results except for the class of utility functions for which the separation property holds (see Merton [1982, p. 635]). This suggests that Hart's [1975] important negative result that comparative statics properties hold with respect to changes in initial wealth in the case of two or more risky assets if and only if the utility function has the separation property, applies to more general comparative statics changes. Future research will thus need to focus on appropriate restrictions on the returns on the risky assets to obtain comparative statics results in the multi-asset case.

rather concise answer to the question of how technological improvements can be expected to affect aggregate saving under uncertainty.

In Section 6, we analyse the production decision of a firm which faces uncertainty about the price of its output. We assume that the firm is manager owned — its ownership is not spread amongst a collection of distinct share-holders. The firm's manager in deciding on his production decision is solving a part of his own portfolio problem and seeks to maximise the expected utility of the earnings from this part of his activity. This is a natural formulation of the problem faced for example by a farm manager who makes a production decision in the spring but faces an uncertain price for his output in the fall (harvest time). We ask the following question: if the random price shifts to a new price which is greater in every state of nature how does the optimal production decision respond? Two forces are at work — the incentive to increase output because of the upward shift in price (the return effect) and the incentive to reduce output in the face of increased variability in profit (the risk effect). We show that if the agent has less than unit relative risk aversion then the return effect dominates and output is increased.

2. THE COMPARATIVE STATICS PROBLEM

Let r denote an exogenously given random variable, α an exogenously given parameter and let x denote the basic scalar decision variable to be chosen by an agent. Suppose the choice of the decision x leads to a random variable w which represents the agents wealth, consumption or profit through the relation

$$w = h(x, \alpha, r). \tag{1}$$

If the agent has a von Neumann-Morgenstern utility function u then the decision variable x is chosen so as to maximise the expected utility

$$U(x, \alpha) = Eu(h(x, \alpha, r)). \tag{2}$$

We assume that both h and u are twice differentiable and satisfy $u'>0$, $u''<0$, $h_x \neq 0$, $h_{xx} \leqq 0$. The problem of maximising (2) has a solution under a variety of conditions on u, h, α and r: we do not enter into this here.[3] In the sequel when we refer to "all parameter values α" or "all random variables r", we mean all those for which the problem of maximising (2) has a solution. Since $U(\cdot, \alpha)$ is strictly concave the optimal solution is unique. The comparative statics problem consists in the study of the qualitative change in the optimal solution x^* when the parameter α or the random variable r is changed.

When considering changes in the random variable r we want to express the idea that r is replaced by a random variable $\hat{r}$ which is "greater than" r. This is ex-

[3] For a study of the existence problem for the subclass of portfolio problems in which h is linear see Bertsekas [1974] and Leland [1972].

pressed in a natural way in the following definition. The random variable $\hat{r}$ is said to dominate r with probability one if $Pr(\hat{r} \geqq r) = 1$. If in addition $Pr(\hat{r} > r) > 0$ then $\hat{r}$ is said to strictly dominate r with probability one. A weaker notion of dominance that is more frequently used is the following. $\hat{r}$ is said to stochastically dominate r if $Pr(\hat{r} > \xi) \geqq Pr(r > \xi)$ for all $\xi \in R$. If in addition strict inequality holds for some $\xi \in R$ then $\hat{r}$ is said to strictly stochastically dominate r.

It can be shown[4] that the comparative statics properties are the same for these two types of dominance for the class of decision problems that involve maximising (2). Thus, for brevity we will say that $\hat{r}$ (strictly) dominates r and write $\hat{r} \succsim r$ ($\hat{r} \succ r$) if either of these two types of (strict) dominance holds between $\hat{r}$ and r. However, in proving each of the propositions in which dominance appears (Propositions 5–7) we make use of the first definition (dominance with probability one) since this substantially simplifies the proofs.

A word is in order regarding the basic procedure that we use in analysing comparative statics problems. When the comparative statics change involves an infinitesimal change in the parameter α then the standard comparative statics procedure can be employed. Assuming that the optimal solution x^* is defined via the function in (2) by the first order condition $U_x(x^*, \alpha) = 0$, it follows from the implicit function theorem that $\frac{dx^*}{d\alpha} = -\frac{U_{x\alpha}(x^*, \alpha^*)}{U_{xx}(x^*, \alpha)}$. Since $u' > 0$, $u'' < 0$ and $h_x \neq 0$, $h_{xx} \leqq 0$ imply $U_{xx} = E(u''h_x^2 + u'h_{xx}) < 0$, the sign of $\frac{dx^*}{d\alpha}$ is the same as the sign of $U_{x\alpha}(x^*, \alpha)$. This is the first step involved in demonstrating each of the Propositions 1–4.

When considering a dominant shift in the random variable r we use the following trick which maps the problem back into the above procedure. A dominant shift in the random variable from r to $\hat{r}$ involves a discrete change in the random variable. Define the intermediate random variable $r_t = r + t(\hat{r} - r)$, $t \in [0, 1]$ and view the function U in (2) as a function of (x, t): this transformation converts the discrete change into a one-parameter family of continuous changes, $t \in [0, 1]$. For each $t \in [0, 1]$ the first order condition gives $U_x(x_t^*, t) = 0$. To show that $x^*(\hat{r}) \geqq x^*(r)$ it suffices to show that $x_\tau \geqq x_t$ for all $\tau \geqq t$ or $\frac{dx_t^*}{dt} \geqq 0$, $t \in [0, 1]$. Setting $t = \alpha$ in the method of the preceeding paragraph this follows if and only if $U_{xt}(x_t^*, t) \geqq 0$, $t \in [0, 1]$. This is the first step involved in demonstrating each of the Propositions 5–7.

3. PORTFOLIO THEORY

We will now present a complete characterization of comparative statics for the

[4] This follows at once from the following result which is proved, among others, by Hansen, Holt & Peled [1978, p. 317].

PROPOSITION. *$\hat{r}$ stochastically dominates r if and only if there exist random variables $(\hat{r}', r')$, where $\hat{r}'$ has the same probability distribution as $\hat{r}$ and r' has the same probability distribution as r, and $\hat{r}'$ dominates r' with probability one.*

basic two asset portfolio problem. In Section 4, we will show how these results can be used to study the Keynesian theory of liquidity preference. We fórmulate the problem as follows. Let there be two assets, one riskless and earning a zero rate of return, the other risky and earning a nonnegative (gross) random return r, the nonnegativity reflecting the limited liability of the risky asset. Let the price for each unit of the first asset be 1 and that for the second asset P. The investor has an initial wealth w_0 and subject to the budget condition

$$x + py = w_0 \tag{3}$$

chooses the amounts (x, y) of each asset to purchase so as to maximize the expected utility of wealth

$$Eu(x+yr). \tag{4}$$

Note that (3) and (4) imply that the risky asset has a net rate of return $r-p$; in addition, since short sales are permitted, even if initial wealth w_0 is positive, terminal wealth $w=x+yr$ can be negative.

To give a complete characterization of comparative statics for this problem we will need to make use of three local measures of risk averions derived from the utility function u, namely

$$A(w) = -\frac{u''(w)}{u'(w)}, \quad R(w) = -\frac{u''(w)}{u'(w)}\,w, \quad P(\gamma, w_0) = -\frac{u''(w_0+\gamma)}{u'(w_0+\gamma)}\,\gamma$$

where A denotes absolute risk aversion, R relative risk aversion and P partial relative risk aversion. The interpretation of A and R are familiar (see Arrow [1965]). An interpretation of P can be obtained from the standard local expression for A as follows. Define the risk premium $\rho=\rho(z, w_0)$ for an agent with an initial wealth w_0 faced with a random prospect z with expected value $\bar{z}=Ez$ by $u(w_0+\bar{z}-\rho)=Eu(w_0+z)$. Then by a standard argument when the variability of the prospect is small, we find that $\rho=\frac{A(w_0+\bar{z})}{2}\sigma_z^2$. Thus, for a normalized risk ($\sigma_z=1$) A is equal to (twice) the risk premium and the function $A(\cdot)$ measures the response of the risk premium to an increase in wealth w_0. If we define ρ_p by $u(w_0+\bar{z}(1-\rho_p))=Eu(w_0+z)$ then from the equation for A it follows that $\rho_p=\frac{P(\bar{z}, w_0)}{2}\left(\frac{\sigma_z}{\bar{z}}\right)^2$. Thus, for a normalized prospect ($\sigma_z/\bar{z}=1$) P represents (twice) the proportion of the actuarial value of the prospect that the agent is prepared to pay as a risk premium and the function $P(\cdot, w_0)$ measures the proportional response of the risk premium to an increase in the scale of the risk.[5]

The problem defined by (3) and (4) can be solved as an unconstrained problem in either x or y by using equation (3). Solving for y leads to the problem

$$\max_y Eu(w_0+y(r-p)). \tag{5}$$

[5] For a further discussion of the properties of these functions see Menezes & Hanson [1970, p. 484] and Zeckhauser & Keeler [1970].

A natural question to ask is the following. Under what conditions is the demand for the risky asset y^* a decreasing function of its price p?

PROPOSITION 1. *For a fixed initial wealth* w_0, $\frac{dy^*}{dp} \leqq 0$ *for all* (r, p) *if and only if the absolute risk aversion function satisfies*

$$(6) \qquad A(b) - \frac{1}{b-w_0} \leqq A(a) - \frac{1}{a-w_0} \quad \text{for all } (a, b) \text{ with } a < w_0 < b.$$

PROOF. The function in (1) is given by $w = h(y, p, r) = w_0 + y(r-p)$ and y^* is a maximum if and only if $U_y(y^*, p) = Eu'(w_0 + y^*(r-p))(r-p) = 0$ which is equivalent (for $y^* \neq 0$) to

$$(7) \qquad Eu'(w)(w-w_0) = 0.$$

By the procedure explained in Section 2, $\frac{dy^*}{dp} \leqq 0$ if and only if $U_{yp}(y^*, p) \leqq 0$. The proof thus reduces to showing that

$$(8) \qquad U_{yp}(y^*, p) = -Eu'(w)(1 - A(w)(w-w_0)) \leqq 0$$

for all random variables $w = w_0 + y^*(r-p)$ such that (7) holds if and only if (6) is satisfied. But (6) is equivalent to

$$\sup_{b > w_0} \left(A(b) - \frac{1}{b-w_0} \right) \leqq \inf_{a < w_0} \left(A(a) - \frac{1}{a-w_0} \right)$$

which implies that there exists λ such that

$$A(b) - \frac{1}{b-w_0} \leqq \lambda \leqq A(a) - \frac{1}{a-w_0} \qquad \text{for all} \quad a < w_0 < b$$

so that $1 - A(w)(w-w_0) \geqq -\lambda(w-w_0)$ for all $w \in R$. Thus,

$$Eu'(w)(1 - A(w)(w-w_0)) \geqq -\lambda Eu'(w)(w-w_0) = 0.$$

To show necessity fix any a and b with $a < w_0 < b$ and let w take the values a and b with probabilities $1-\pi$ and π where

$$(9) \qquad \pi = -\frac{u'(a)\,(a-w_0)}{u'(b)(b-w_0) - u'(a)(a-w_0)}.$$

Note that with this choice of π, (7) holds and (8) reduces to

$$(1-\pi)u'(a)(1 - A(a)(a-w_0)) + \pi u'(b)(1 - A(b)(b-w_0)) \geqq 0$$

which in view of (9) becomes

$$(b-w_0)(1 - A(a)(a-w_0)) - (a-w_0)(1 - A(b)(b-w_0)) \geqq 0.$$

Since $a < w_0 < b$ are arbitrary the result follows. Q.E.D.

Since the net return on the risky asset in (5) is given by $r-p$, it follows from Proposition 1 that if (6) is satisfied then a uniform downward shift in the random return r leads to a decrease in y^*. Proposition 1 also includes as a special case two sufficient conditions that are known in the literature (see Merton [1982, p. 613]), namely that absolute risk aversion decreases or the partial relative risk aversion is bounded above by one. These are summarised in the following corollary.

COROLLARY 1. *If* (i) $A(b) \leqq A(a)$ *for all* $b > a$ *or* (ii) $P(\gamma, w_0) \leqq 1$, *for all* $\gamma > 0$, *then* $\frac{dy^*}{dp} \leqq 0$.

Proposition 1 can be interpreted intuitively as follows. When p increases there is a wealth effect and a substitution effect and the latter is always negative. When wealth falls, if absolute risk aversion decreases, then y^* can increase. (6) places a bound on the increase in the absolute risk aversion arising from an increase in wealth which prevent the wealth effect from dominating the substitution effect. Under what conditions is the demand for the risky asset y^* a decreasing function of its price p for all initial wealth levels w_0?

COROLLARY 2. $\frac{dy^*}{dp} \leqq 0$ *for all* w_0 *and for all* (r, p) *if and only if*

$$(A(b) - A(a))(b - a) \leqq 4 \qquad \text{for all} \quad b > a. \tag{10}$$

PROOF. The problem $\min_{w_0} \left(\frac{1}{b - w_0} + \frac{1}{w_0 - a} \right)$ has the solution $w_0 = \frac{a+b}{2}$. The result follows by applying Proposition 1.

Note that $\frac{dy^*}{dp} < 0$ replaces $\frac{dy^*}{dp} \leqq 0$ in Proposition 1 and the corollaries when strict equality holds in (6), (i), (ii) or (10).

Under what conditions does the demand for the riskless asset increase when the price of the risky asset p increases? To answer this let us solve (3) and (4) in terms of x. This leads to the problem

$$\max_x Eu\left(x + \left(\frac{w_0 - x}{p} \right) r \right). \tag{11}$$

Let x^* denote the optimal solution to (11). It follows from the budget constraint (3) that if $y^*(p) > 0$ then $\frac{dx^*}{dp} \geqq 0$ if and only if $\frac{p\,dy^*}{y^* dp} \leqq -1$. The condition for the demand for the riskless asset to increase is thus equivalent to an elasticity condition on the demand for the risky asset. We would thus naturally expect that a more restrictive condition than (6) will be needed to show that $\frac{dx^*}{dp} \geqq 0$. The next result gives the precise condition.

PROPOSITION 2. *For a fixed initial wealth* w_0, $\frac{dx^*}{dp} \geqq 0$ *for all* (r, p) *if and only if the partial relative risk aversion function satisfies*

$$P(\gamma, w_0) \leqq 1 \quad \text{for all} \quad \gamma > 0. \tag{12}$$

PROOF. Let $U(x, p)$ denote the integral in (11), then x^* is a maximum if and only if $U_x(x^*, p)=0$ which is equivalent to (7). Thus $\frac{dx^*}{dp} \geqq 0$ if and only if $U_{xp}(x^*, p) \geqq 0$ so that the proof reduces to showing that

$$U_{xp}(x^*, p) = -Eu'(w)(P(w-w_0, w_0)-1)\frac{r}{p^2} \geqq 0 \tag{13}$$

for all random variables $w = x^* + \left(\frac{w_0 - x^*}{p}\right) r$ such that (7) holds if and only if (12) is satisfied. Sufficiency is immediate. To establish necessity fix $a < w_0 < b$ and let w take the values a and b with probabilities $1-\pi$ and π, with π given by (9) so that (7) holds. Let $r_1 = \frac{(b-a)p}{w_0 - a}$, $r_2 = 0$, and let $p > 0$, $x^* = a$ then (13) reduces to $u'(b)(P(b-w_0, w_0)-1)\frac{r_1}{p^2}\pi \leqq 0$ from which (12) follows. Q. E. D.

We now consider the effect of changes in the wealth w_0 on the agent's demand for the two assets. The qualitative change can be studied either for all initial wealth levels as in the two corollaries that follow and as in the paper of Arrow [1965] or more generally at a fixed initial wealth level (Propositions 3, 4). Recall that Arrow had shown that if the agent has decreasing absolute risk aversion and increasing relative risk aversion then the demand for the risky asset increases and the wealth elasticity of the demand for cash is greater than one respectively. The two corollaries assert that these conditions are necessary as well as sufficient.

PROPOSITION 3. *$\frac{dy^*}{dw} \geqq 0$ at a given initial wealth level w_0 for all (r, p) such that $y^* > 0$ if and only if*

$$A(a) \geqq A(w_0) \geqq A(b) \quad \text{for all} \quad (a, b) \quad \text{with} \quad a < w_0 < b. \tag{14}$$

PROOF. Let $U(y, w_0)$ denote the integral in (5). Then $\frac{dy^*}{dw_0} \geqq 0$ if and only if $U_{yw_0}(y^*, w_0) \geqq 0$. We thus need to show that

$$U_{yw_0}(y^*, w_0) = -\frac{1}{y^*} Eu'(w)A(w)(w-w_0) \geqq 0 \tag{15}$$

for all random variables $w = w_0 + y^*(r-p)$ such that (7) holds and $y^* > 0$ if and only if (14) holds. Sufficiency follows since (14) implies $A(w)(w-w_0) \leqq A(w_0)(w-w_0)$ for $w \in R$ so that $Eu'(w)A(w)(w-w_0) \leqq A(w_0)Eu'(w)(w-w_0)=0$. To show necessity select any $a < w_0 < b$ as before, letting w take the values a and b with probabilities $1-\pi$ and π, with π given by (9) so that (7) holds. Then (15) reduces to $A(b)-A(a) \leqq 0$ from which (14) follows. Q. E. D.

If we require that the demand for the risky asset increase at all wealth levels then we get the following condition.

COROLLARY. $\frac{dy^*}{dw} \geqq 0$ *for all* w_0 *and for all* (r, p) *such that* $y^* > 0$ *if and only if* $A(a) \geqq A(b)$, *for all* $b > a$ *for all* $a \in R$.

How about the condition that the wealth elasticity of demand for cash be greater than one?

PROPOSITION 4. $\frac{w_0 dx^*}{x^* dw_0} \geqq 1$ *at a given initial wealth level* w_0 *for all* (r, p) *such that* $x^* > 0$, $y^* > 0$ *if and only if*

$$R(a) \leqq R(w_0) \leqq R(b) \quad \text{for all} \quad (a, b) \quad \text{with} \quad a < w_0 < b. \tag{16}$$

PROOF. Let $U(y, w_0)$ denote the integral in (5) then $\frac{w_0 dx^*}{x^* dw_0} - 1 = \frac{1}{x^*}\left[w_0 - pw_0 \cdot \frac{dy^*}{dw_0} - x^*\right]$. Using $\frac{dy^*}{dw_0} = -\frac{U_{yw_0}}{U_{yy}}$ and inserting the expressions for U_{yw_0} and U_{yy} we find after a little manipulation that

$$\frac{w_0 dx^*}{x^* dw_0} - 1 = \alpha Eu'(w)R(w)(w - w_0) \tag{17}$$

where $\alpha = \frac{p}{x^* y^* U} > 0$. Thus we need to show that the integral in (17) is nonnegative for all random variables $w = x^* + y^* r$ satisfying (3), (7) and $x^* > 0$, $y^* > 0$ if and only if (16) holds. Applying the method used in the proof of Proposition 3 completes the proof.

We can also require that the wealth elasticity of demand for cash be greater than one at all wealth levels: this leads to the following condition.

COROLLARY. $\frac{w_0 dx^*}{x^* dw_0} \geqq 1$ *for all* w_0 *and for all* (r, p) *such that* $x^* > 0$, $y^* > 0$ *if and only if* $R(a) \leqq R(b)$, *for all* $b > a$, *for all* $a \in R$.

How does an increase in the random return affect the demand for the risky asset? In short, suppose there is a stochastically dominant shift in r, under what conditions does the demand for the risky asset increase?

PROPOSITION 5. *For a fixed initial wealth* w_0, $y^*(\hat{r}) \geqq y^*(r)$ *for all* $\hat{r}$ *and* r *satisfying* $\hat{r} \gtrsim r$ *if and only if* $P(\gamma, w_0) \leqq 1$ *for all* $\gamma > 0$.

PROOF. We use the approach explained at the end of Section 2. Let $U(y, t)$ denote the integral in (5) with r replaced by $r_t = r + t(\hat{r} - r)$, $t \in [0, 1]$ and let y_t^* denote the solution to (5) for $t \in [0, 1]$. Then $\frac{dy_t^*}{dt} \geqq 0$ if and only if $U_{yt}(y_t^*, t) \geqq 0$ where

$$U_{yt}(y_t^*, t) = -Eu'(w^t)(P(w^t - w_0, w_0) - 1)(\hat{r} - r) \tag{18}$$

$w^t = w_0 + y_t^*(r_t - p)$. Since $\hat{r} \gtrsim r$ implies $\hat{r} - r \geqq 0$ almost surely, sufficiency follows from $P(\gamma, w_0) \leqq 1$ for all $\gamma > 0$. To show necessity note that $\hat{r} \gtrsim r$ implies $r_\tau \gtrsim r_t$ for $\tau > t$ so that y_t^* is an increasing function of t. Thus $\frac{dy_t^*}{dt} \geqq 0$, $t \in [0, 1]$ and in particular $U_{yt}(y_0^*, 0) \geqq 0$. Choose $a < w_0 < b$ as before and let w take on the two values a and b with probabilities $1-\pi$ and π with π given by (9) so that (7) holds. Choose y_0^* and a random variable r which takes on the two values r_1, r_2 with the above probabilities so that $a = w_0 + y_0^*(r_1 - p)$, $b = w_0 + y_0^*(r_2 - p)$. Now choose the random variable $\hat{r}$ so that $\hat{r}_1 - r_1 = 1$, $\hat{r}_2 - r_2 = 0$ then $U_{yt}(y_0^*, 0) \geqq 0$ reduces to $(1-\pi)u'(a)(P(a - w_0, w_0) - 1) \leqq 0$. Since $a < w_0$ is arbitrary, the result follows. Q. E. D.

The conditions in Propositions 2 and 5 are the same: let us note what they become when we require that they hold for all initial wealth levels w_0.

COROLLARY. *$y^*(\hat{r}) \geqq y^*(r)$ for all $w_0 \geqq 0$ and for all $\hat{r}$ satisfying $\hat{r} \gtrsim r$ if and only if $R(w) \leqq 1$ for all $w \geqq 0$.*

PROOF. *$P(\gamma, w_0) \leqq 1$ for all $\gamma > 0$, for all $w_0 \geqq 0$ if and only if $R(w) \leqq 1$ for all $w \geqq 0$.*

Thus, the agent's demand for the risky asset increases for all initial wealth levels when there is a stochastically dominant shift in its return if and only if his relative risk aversion is bounded above by one. Note that an agent whose utility function has decreasing absolute risk aversion and relative risk aversion which is increasing and bounded above by one will make a portfolio decision (x^*, y^*) which satisfies each of the comparative statics properties of this section.

4. LIQUIDITY PREFERENCE THEORY

The object of this section is to indicate briefly one way in which the results of the preceeding section can be applied. Recall that the Hicks-Tobin [1935, 1958] reformulation of the Keynesian theory of liquidity preference is based on the idea that even though the yield on money is less than the expected yield on alternative risky assets, a positive amount will be held since money provides a hedge against the risk that is incurred when investing in these alternative assets. In his classical formulation of the portfolio problem, Tobin however was not able to provide precise conditions under which the demand for money would be inversely related to the rate of interest as postulated by Keynes. Arrow [1965] in reexamining the problem within the framework of the same model showed that a sufficient condition for the inverse relationship to hold is that each agent have decreasing absolute risk aversion. The theorems of the previous section allow us to extend this result and to obtain a related result in a somewhat less restrictive model than that considered by Arrow and Tobin.

Let the two assets be cash and consols and let each consol have a face value of one dollar and pay a coupon rate of δ dollars per year. The agent is assumed to know the current market interest rate α but to be uncertain about next period's interest rate β which thus is taken to be a random variable. It is assumed that there is an increasing function $\phi(\cdot)$ which reflects the way the market values the consol with the property that the current market price of the consol is given by $p=\phi\left(\frac{\delta}{\alpha}\right)$, while next period's price will be $\phi\left(\frac{\delta}{\beta}\right)$. Assuming that the consol is sold at the end of the period, the net return on each consol purchased is given by $r-p=\delta+\phi\left(\frac{\delta}{\beta}\right)-\phi\left(\frac{\delta}{\alpha}\right)$, where it is assumed that $E(r)-p>0$. The risk from investing in the consol arises from the fluctuation in the interest rate which induces an uncertain capital gain or loss.

Tobin [1958, p. 67] and Arrow [1965, pp. 106–107] make the rather restrictive assumption that the coupon rate δ always coincides with the current market interest rate α (in short, they do not allow for variations in p in the budget constraint (3)). A change in α must thus be accompanied by a change in δ or a change in the instrument itself. Secondly, they assume that next period's interest rate β is distributed in such a way that the percentage change in the interest rate $\frac{\beta-\alpha}{\alpha}$ does not depend on α. With these two assumptions Arrow [1965, p. 107] showed that if absolute risk aversion is decreasing then the demand for money is a decreasing function of the interest rate α. Proposition 1 allows us to refine this result by showing the demand for money is a decreasing function of the interest rate under a weaker restriction on absolute risk aversion.

COROLLARY. *If* (i) $\delta=\alpha$, (ii) *the distribution of* $\frac{\alpha}{\beta}$ *does not depend on* α, (ii) $A(b)-\frac{1}{b-w_0}<A(a)-\frac{1}{a-w_0}$ *for all* (a, b) *with* $a<w_0<b$, *then the demand for money is a strictly decreasing function of the current market interest rate* α.

PROOF. Since $r-p=\alpha+\phi\left(\frac{\alpha}{\beta}\right)-\phi(1)$ and the distribution of $\frac{\alpha}{\beta}$ does not depend on α, an increase in α leads to a uniform upward shift in the net return. By Proposition 1 $\frac{dy^*}{d\alpha}>0$. From the budget constraint $x+\phi(1)y=w_0$, $\frac{dx^*}{d\alpha}<0$.

Q. E. D.

The next result is an application of Proposition 2. The basic Tobin-Arrow model can be made less restrictive by allowing α to differ from the coupon rate δ so that the price of the consol $p=\phi\left(\frac{\delta}{\alpha}\right)$ varies when α changes and by dropping the distributional assumption on $\frac{\beta}{\alpha}$. The following result then assures us that the demand for money is a decreasing function of the interest rate within this more general model.

COROLLARY. *If* $P(\gamma, w_0)<1$ *for all* $\gamma>0$, *then the demand for money is a strictly decreasing function of the current market interest rate* α.

PROOF. Since $\frac{dx^*}{d\alpha} = -\frac{dx^*}{dp}\phi'\left(\frac{\delta}{\alpha}\right)\frac{\delta}{\alpha^2}$, since $\phi'>0$ and $\frac{dx^*}{dp}>0$ by Proposition 2, $\frac{dx^*}{d\alpha}<0$. Q. E. D.

Thus, in the more general model the condition that ensures that the demand for money is a decreasing function of the interest rate is that the agent's partial relative risk aversion be bounded above by 1. The same condition also enables us to give a definite answer to the following question. How does an upward shift in the agent's expectations regarding next period's interest rate affect his holding of money balances?

COROLLARY. *If $P(\gamma, w_0)<1$ for all $\gamma>0$ and if the agent's expectations regarding the future interest rate shift from β to $\hat{\beta}$ with $\hat{\beta}\succ\beta$, then the demand for money increases $x^*(\hat{\beta})>x^*(\beta)$.*

PROOF. From Proposition 5 $y^*(\hat{\beta})<y^*(\beta)$, applying the budget constraint gives $x^*(\hat{\beta})>x^*(\beta)$.

5. THE RAMSEY PROBLEM

The next application can be thought of as a portfolio problem at the aggregate social level with the added feature that the return on the risky investment is nonlinear — hence, the appearance of an elasticity term on the production side as well. The model, which was first introduced by Mirman [1971], allows us to capture the essence of the Ramsey problem under uncertainty without the technical complication created by the presence of an infinite horizon. The model is as follows. The community inherits an initial stock of capital $k_0>0$. This stock can be used either for consumption in the first period or as an investment input for producing output for the second period — all of this output being used for consumption. The investment input $k\geqq 0$ leads however to an uncertain output $f(k, \theta)$ next period, θ being a random variable which characterises the uncertain state of technology. For simplicity we assume that f is homogeneous of degree one and twice differentiable with $f>0$, $f_k>0$, $f_\theta>0$, $f_{kk}<0$ and hence $f_{k\theta}>0$, for all (k, θ) strictly positive. The community has utility function $u(c)$ for consumption in each period which is twice differentiable and satisfies $u'>0$, $u''<0$ for all $c>0$. In addition f_k and u' satisfy the boundary conditions

$$(19)\quad f_k(\ ,\theta)\longrightarrow\infty \text{ as } k\longrightarrow 0 \quad \text{for all } \theta>0,\ u'(c)\longrightarrow\infty \text{ as } c\longrightarrow 0.$$

The community chooses aggregate investment k so as to maximize the expected discounted utility of consumption

$$(20)\qquad \max_{k\in[0,k_0]}\ (u(k_0-k)+\beta Eu(f(k,\theta))$$

where $\beta>0$ is the discount factor. Define the elasticity of substitution $\sigma(k, \theta)=\frac{f_k f_\theta}{f f_{k\theta}}$ and the elasticity of the marginal utility of consumption $R(c)=-\frac{u''(c)}{u'(c)}c$.

The following is a most natural question to ask. Under what conditions will an improvement in technology lead the community to increase its saving (investment)?

PROPOSITION 6. *If $\hat{\theta} \succ \theta$ and if the elasticity condition*

$$(21) \qquad R(f)\sigma < (>) 1 \quad \text{for all} \quad (k, \theta) \quad \text{strictly positive}$$

is satisfied then $k^*(\hat{\theta}) > (<) k^*(\theta)$.

PROOF. Using the approach outlined at the end of Section 2 let $U(k, t)$ denote the sum in (20) with θ replaced by $\theta_t = \theta + t(\hat{\theta} - \theta)$, $t \in [0, 1]$. A standard argument using the conditions in (19) shows that for each $t \in [0, 1]$ there exists $k_t^* \in (0, k_0)$ satisfying the first order condition

$$(22) \qquad U_k(k_t^*, t) = -u'(k_0 - k_t^*) + \beta E u'(f(k_t^*, \theta_t)) f_k(k_t^*, \theta_t) = 0.$$

It thus suffices to show that $\frac{dk_t^*}{dt} > (<) 0, t \in [0, 1]$. But this holds if and only if

$$U_{kt}(k_t^*, t) = -\beta E \Delta (R(f)\sigma - 1)(\hat{\theta} - \theta) > (<) 0, \quad t \in [0, 1]$$

where $\Delta = u' f_{k\theta}$. Since $\hat{\theta} \succ \theta$, $\hat{\theta} - \theta > 0$ on a set of positive measure so that (21) implies $U_{kt}(k_t^*, t) > (<) 0$, $t \in [0, 1]$, from which the result follows. Q. E. D.

How does one provide an economic interpretation for the elasticity condition $R(f)\sigma < 1$ in (21)? How does it throw light on the forces which are at work in inducing the community to save either more or less? The first order condition (22) implies that k_t^* satisfies the equilibrium condition of equating the supply price of capital $u'(k_0 - k_t^*)$ with the demand price $\beta E u' f_k$. When t increases by a small amount, θ_t increases leading to two effects. First, by increasing output f the marginal utility u' is decreased (the extent of this decrease depending on R). Secondly, since $f_{k\theta} > 0$, the increase in θ raises the marginal product of capital f_k (the extent of this increase depending on σ). The first effect decreases the demand price tending to lower k_t^* — with more output next period there is less to save. The second effect raises the demand price and hence k_t^* — with capital more productive next period there is more of an incentive to invest. The balance between these two opposing tendencies is precisely determined by the elasticity condition and this in turn determines the direction in which saving moves.

Note that within the framework of the same model Mirman [1971] has analysed the related question of the effect of a mean preserving spread in the distribution of θ on the investment decision k^* in the case where θ enters multiplicatively, $f(k, \theta) = \theta h(k)$. It can be argued however that the shift in θ that we consider is perhaps in many respects the more natural one to analyse.

6. PRODUCTION AND THE DISTRIBUTION OF PRICE

The last example considers the portfolio problem faced by the manager of an individually owned firm when the price of the firm's output is random. If the price increases how does the firm's output respond? The basic model is as follows. Let $c(x)$ denote the cost of producing the output $x \geqq 0$ and let p denote the random price. We assume that $c(x)$ is convex, increasing and differentiable for $x \geqq 0$ and that in addition to $c(0)=0$ it satisfies

$$c'(0) < p \text{ with probability one and } c'(x) \longrightarrow \infty \text{ as } x \longrightarrow \infty. \tag{23}$$

Suppose the production decision must be made before the price of the output is known. Since the firm's profit is random and since the firm is individually owned the firm's production decision is a portfolio problem for the manager. We thus assume that output is chosen so as to maximise the expected utility of profit

$$\max_{x \in R^+} Eu(px - c(x)) \tag{24}$$

where $u'>0$, $u''<0$ for all $w \geqq 0$. We are interested in the following question: how does the optimal output vary when the random price p shifts to a strictly dominant price $\hat{p}$? If p and $\hat{p}$ were nonrandom the answer would be clear: the firm would increase its output. But in the stochastic case when the random price increases from p to $\hat{p}$, even though the price is greater in every state of nature, the variability of $\hat{p}$ may be greater than that for p. If the agent's risk aversion is too great then the incentive to increase output arising from the upward shift in the random price (the return effect) may be offset by the incentive to reduce output arising from the increased variability in profit (the risk effect). Our final result shows that if the agent has less than unit relative risk aversion then the return effect dominates.

PROPOSITION 7. *Let p and $\hat{p}$ be nonnegative and bounded above with probability one. If $\hat{p} \succ p$ and if $R(w) < 1$ for all $w \geqq 0$, then $x^*(\hat{p}) > x^*(p)$.*

PROOF. Let $U(x, t)$ denote the integral in (24) when p is replaced by $p_t = p + t(\hat{p} - p)$, $t \in [0, 1]$. By (23) and the boundedness of p and $\hat{p}$, $U_x(0, t) > 0$ and $U_x(x, t) < 0$ for sufficiently large x for all $t \in [0, 1]$. Thus for each $t \in [0, 1]$ there exists $x_t^* > 0$ such that

$$U_x(x_t^*, t) = Eu'(p_t x_t^* - c(x_t^*))(p_t - c'(x_t^*)) = 0.$$

Since $\dfrac{dx_t^*}{dt} > 0$ for all $t \in [0, 1]$ if and only if for all $t \in [0, 1]$

$$U_{xt}(x_t^*, t) = E[u''(w^t)(p_t x_t^* - c'(x_t^*)x_t^*) + u'(w^t)](\hat{p} - p) > 0$$

where $w^t = p_t x_t^* - c(x_t^*)$ and since $\hat{p} - p > 0$ on a set of positive measure, it suffices to show that

$$u''(w^t)(p_t x_t^* - c'(x_t^*)x_t^*) + u'(w^t) > 0 \quad \text{for all } t \in [0, 1]. \tag{25}$$

If $p_t x_t^* - c'(x_t^*)x_t^* \leqq 0$ then (25) holds. If $p_t x_t^* - c'(x_t^*)x_t^* > 0$ then $w^t > 0$ since

$c'(x)x \geqq c(x)$, for all $x \geqq 0$. Thus, establishing (25) reduces to showing that $R(w^t) \cdot \left(\frac{p_t x_t^* - c'(x_t^*)x_t^*}{p_t x_t^* - c(x_t^*)}\right) < 1$ which follows from $R < 1$ and $c'(x)x \geqq c(x)$ for all $x \geqq 0$.

Q. E. D.

Note that within the framework of the same model Rothschild and Stiglitz [1971] have analysed the related question of the effect of a mean preserving spread in the distribution of the price p on the firm's output decision x^*.

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