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On the Arrow–Lind Theorem

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1. INTRODUCTION

This paper presents an extension of the Arrow–Lind theorem [1] on the asymptotic value of an uncertain public project when the benefits and costs of the project are shared among a large number of agents. The theorem has its origin in the following problem: how should such a project be valued in the absence of a complete set of contingent markets in the private sector? I adopt the Arrow–Lind approach in basing the value of the project on the sum of the individual agents' valuations.

Arrow and Lind value a project by what I call the aggregate sale value. With their assumption that the n agents are identical, this is simply n times the representative agent's sale value of his share of the project's return. This concept of value is appropriate when all agents in the economy are identical since all agents are affected equally by the introduction of a public project. But in a world with different types of agents (income–preference pairs) some agents may be favorably affected while others may be adversely affected. In such a framework there are two natural concepts of value, the aggregate sale value and the aggregate purchase value, the latter being the sum of the individual agents' purchase values—the most that an agent will pay to purchase a given share of the project's return. To extend the Arrow–Lind theorem we need to study the behaviour of both these values.

In Section 2, I show that these two concepts of value are related to the Kaldor–Hicks criteria for a potential Pareto improvement. In Section 3, I show that when the share of the project held by an agent becomes arbitrarily small the individual purchase and sale values coincide and the idiosyncratic risk associated with the project goes to zero (Proposition 1). After introducing a concept of stochastic dependence for a pair of random variables which extends the concept of independence to a concept of positive or negative dependence (Definition 1), I show that the individual sale value is less than (greater than) the expected value of the agent's share if his income

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is positively (negatively) dependent on the return from the public project. It then follows under a uniformity condition (Assumption 2) that as the number of agents is increased ($n \to \infty$) the two aggregate values converge to a common value and this limiting value is less than (greater than) the expected value of the project if the incomes of all but a finite number of agents are positively (negatively) dependent on the return from the public project—with equality in the independent case (Proposition 2). This is our first extension of the Arrow–Lind theorem.

While this proposition leads to a classification of public projects according to the nature of their average stochastic dependence with the incomes of agents in the private sector, Proposition 3 makes this distinction more precise by showing that if we can attribute this stochastic dependence to the fact that the returns on the public project and the incomes of the agents are both influenced by a common random factor such as the business cycle, then the aggregate value of the public project depends not only on whether its returns vary procyclically or countercyclically but on the volatility of these comovements or countermovements with the business cycle. This is our second extension of the Arrow–Lind theorem.

Presented this way the theory of the valuation of public projects becomes closely related to Ross' arbitrage theory of asset pricing [4]. The nonlinear index model introduced in Section 4 to generate the returns on the public project and the incomes of the n agents may be viewed as a simple generalization of the linear index model used by Ross to generate the returns on each of m assets. While in Proposition 3 it is the increase in the number of agents ($n \to \infty$) that leads to risk spreading, which in turn drives the idiosyncratic risk to zero, in Ross' framework it is the increase in the number of assets ($m \to \infty$) and the resultant diversification of risk through the law of large numbers, which drives the idiosyncratic risk to zero. In each case the value of the asset depends only on the resulting nondiversifiable risk and the premium for this risk is proportional to the volatility of the asset's return with respect to the underlying index.

A substantial literature has emerged on various issues raised by the Arrow–Lind theorem—I shall not attempt to enter into these here. Suffice it to say that the most obvious defect of the present extension is that it is confined to the purely static case. In this respect I should mention the recent paper of Wilson [5] which not only presents a discussion of many of the basic issues involved in the valuation of public projects, but also shows some important qualitative results that emerge when the analysis is explicitly extended to a temporal context.

2. Individual and Aggregate Values and the Hicks–Kaldor Criterion

Consider an economy consisting of n agents, where each agent has a random income y_i, $i = 1,\ldots, n$, and receives a share ξ_i of the random return z from a public project. Let $(\Omega, \mathscr{F}, \mathscr{P})$ denote a probability space, Ω being the set of states of nature, $\mathscr{F}$ the σ-field of subsets of Ω, and $\mathscr{P}$ the probability measure on $\mathscr{F}$. We assume that ith agent's preference ordering among random income prospects can be represented by a von Neumann–Morgenstern utility function u_i and define

$$U_i(\xi_i) = \int_\Omega u_i(y_i(\omega) + \xi_i z(\omega))\, d\mathscr{P}(\omega) = Eu_i(y_i + \xi_i z). \tag{1}$$

Assumption 1. (i) $u_i : D_i \to R$ is a concave, strictly increasing and differentiable function on its domain $D_i \subset R$; (ii) the random variables (y_i, z) are restricted so that $U_i(\xi_i)$ is differentiable for all $|\xi_i| < \varepsilon$ for some $\varepsilon > 0$.

A variety of different sufficient conditions can be given for (ii) to hold, depending on the behaviour of $u_i(\cdot)$—I leave the details of this enumeration to the reader.

We would like a unique expression for the money value to the ith agent of the share $\xi_i z$ in the public project. In general no such expression can be found, for there are two natural money values that the ith agent associates with $\xi_i z$ depending on whether we use expected utility before or after receipt of the random return $\xi_i z$ as the reference point for the valuation. Thus we define the *purchase value* $w_i(\xi_i; y_i)$ as the maximum (nonrandom) amount of money that the ith agent will pay for $\xi_i z$ assuming he does not own it,

$$Eu_i(y_i + \xi_i z - w_i) = Eu_i(y_i). \tag{2}$$

Similarly we define the *sale value* $v_i(\xi_i; y_i)$ as the minimum (random) amount of money for which the ith agent will sell $\xi_i z$ assuming he does own it,

$$Eu_i(y_i + v_i) = Eu_i(y_i + \xi_i z). \tag{3}$$

It is clear that w_i and v_i correspond to the compensated and equivalent variations in income in the standard theory of the consumer, when the ith agent instead of facing a change in random income $\xi_i z$ (as above) faces a change in the vector of prices. It will be recalled that the compensated and equivalent variations coincide when there are no income effects. A similar result holds here: using Pratt's result [3, Theorem 2] that $v_i(\xi_i; y_i)$ is independent of y_i when y_i is nonrandom and the ith agent has constant absolute risk aversion, we can show that $w_i(\xi_i; y_i) = v_i(\xi_i; y_i)$ for a general random income y_i if the ith agent has constant abolute risk aversion. While w_i and v_i always have the same sign, positive (negative) if $Eu_i(y_i + \xi_i z) \geqslant (\leqslant) Eu_i(y_i)$, they do not in general coincide in magnitude.

Let $\xi = (\xi_1, ..., \xi_n)$ with $\xi_i > 0$, $i = 1, ..., n$, $\sum_{i=1}^n \xi_i = 1$, $y = (y_1, ..., y_n)$ and consider the move from y to $y + \xi z$ involved in undertaking the public project. In general some agents (the gainers) will be beneficially affected ($w_i > 0$) while others (the losers) will be harmfully affected ($w_i < 0$). The *Kaldor criterion* asserts that the move from y to $y + \xi z$ involves a potential Pareto improvement if the gainers can compensate the losers and still be better off: it is easy to see that this is equivalent to the condition that the *aggregate purchase value* $W_n = \sum_{i=1}^n w_i$ be positive. Consider the move from $y + \xi z$ to y. Let $\hat{w}_i$ denote the purchase value of $-\xi_i z$ starting from $y_i + \xi_i z$, then $\hat{w}_i = -v_i$. The *Hicks condition* requires that the gainers in the move from $y + \xi z$ to y should not be able to compensate the losers and still be better off: this is equivalent to the condition that $\hat{W}_n = \sum_{i=1}^n \hat{w}_i = -\sum_{i=1}^n v_i = -V_n < 0$ or that the *aggregate sale value* V_n be positive. *Thus the Hicks–Kaldor condition for a potential Pareto improvement in moving from y to $y + \xi z$ is equivalent to the condition that the aggregate purchase value W_n and the aggregate sale value V_n be positive.*

The aggregate sale value $V_n = \sum_{i=1}^n v_i$ inherits an important property from the individual sale values v_i, $i = 1, ..., n$. Suppose that instead of considering the single public project represented by the return-share vector (z, ξ) we want to compare it with a second project represented by (z^*, ξ^*). In this case if we define $z_i = \xi_i z$, $z_i^* = \xi_i^* z^*$ and let $Eu_i(y_i + v_i(z_i)) = Eu_i(y_i + z_i)$, then $v_i(z_i)$ represents the ith agent's preference ordering since $v_i(z_i) \geqslant v_i(z_i^*)$ if and only if $Eu_i(y_i + z_i) \geqslant Eu_i(y_i + z_i^*)$. Thus if (z, ξ) is preferred by all agents to (z^*, ξ^*), then $V_n = \sum_{i=1}^n v_i(z_i) > \sum_{i=1}^n v_i(z_i^*) = V_n^*$.

3. Stochastic Dependence and Behaviour of Individual and Aggregate Values

When the proportions ξ_i of the public project z held by the ith agent are sufficiently small, then the behaviour of the individual values w_i and v_i is determined by the nature of the stochastic dependence between y_i and z. To make this idea precise we need an extension of the concept of independence which expresses for an arbitrary pair of random variables (y_i, z) what the concept of positive or negative covariance expresses for a pair of normally distributed random variables. This is supplied by the following definition due to Lehmann [2].

DEFINITION 1. Let (y, z) be a pair of real-valued random variables defined on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$. The pair (y, z) is said to be *positively (negatively) dependent* if for all $(\eta, \xi) \in R^2$

$$\mathscr{P}(\omega \in \Omega \mid y(\omega) \leqslant \eta, z(\omega) \leqslant \xi) \geqslant (\leqslant) \mathscr{P}(\omega \in \Omega \mid y(\omega) \leqslant \eta) \times \mathscr{P}(\omega \in \Omega \mid z(\omega) \leqslant \xi)$$

with strict inequality for some $(\eta, \xi) \in R^2$. For brevity we write (y, z) is p.d. (n.d.) respectively,

It is straightforward to show that if (y, z) is p.d. (n.d.) and $g(\cdot)$ is a real-valued, strictly decreasing function, then $(g(y), z)$ is n.d. (p.d.). Also if (y, z) is p.d. (n.d.), then $\operatorname{cov}(y, z) > 0$ (<0). When (y, z) are joint normally distributed, then (y, z) is p.d. (n.d.) if and only if $\operatorname{cov}(y, z) > 0$ (<0).

Consider the sale value $v_i(\xi_i) = v_i(\xi_i ; y_i)$. Let $G_i(v_i) = Eu_i(y_i + v_i)$, then, under Assumption 1, $G_i(\cdot)$ is a strictly increasing differentiable function so its inverse $G_i^{-1}(\cdot)$ is differentiable. Thus (1) and (3) imply $v_i(\xi_i) = G_i^{-1}(U_i(\xi_i))$ and if $U_i(\cdot)$ is differentiable at ξ_i, then

$$v_i'(\xi_i) = \frac{U_i'(\xi_i)}{G'(v_i(\xi_i))} = \frac{E[u_i'(y_i + \xi_i z) z]}{Eu_i'(y_i + v_i(\xi_i))}. \tag{4}$$

A similar expression is readily derived for the derivative of the purchase value $w_i(\xi_i)$. Since $v_i(0) = w_i(0) = 0$, if $U_i(\cdot)$ is differentiable at $\xi_i = 0$, then

$$v_i'(0) = w_i'(0) = \bar{z} + \frac{\operatorname{cov}(u_i'(y_i), z)}{Eu_i'(y_i)} \tag{5}$$

The differentiability of $v_i(\xi_i)$ and $w_i(\xi_i)$ at $\xi_i = 0$ implies that there exist functions $r_i(\xi_i)$, $s_i(\xi_i)$ where $r_i(\xi_i)/\xi_i$, $s_i(\xi_i)/\xi_i \to 0$ as $\xi_i \to 0$ so that

$$v_i(\xi_i) = \xi_i v_i'(0) + r_i(\xi_i), \qquad w_i(\xi_i) = \xi_i w_i'(0) + s_i(\xi_i). \tag{6}$$

Thus $(v_i(\xi_i) - w_i(\xi_i))/\xi_i \to 0$ as $\xi_i \to 0$. *When the proportion ξ_i is sufficiently small, the ith agent behaves as if he had constant absolute risk aversion and the purchase and sale values coincide.*

For a fixed public project z with mean $\bar{z}$, we define the risk premium of the ith agent $\Delta_i(\xi_i)$ by

$$v_i(\xi_i) = \xi_i \bar{z} - \Delta_i(\xi_i). \tag{7}$$

PROPOSITION 1. *Under Assumption* 1, $\Delta_i(\xi_i) \to \xi_i \delta_i$ *as* $\xi_i \to 0$ *where* $\delta_i = -\operatorname{cov}(u_i'(y_i), z)/Eu_i'(y_i)$. *If the pair* (y_i, z) *is independent,* $\delta_i = 0$; *if the pair* (y_i, z) *is positively (negatively) dependent,* $\delta_i > 0$ (<0).

Proof. The convergence $\Delta_i(\xi_i) \to \xi_i \delta_i$ follows at once from (5) – (7). If (y_i, z) is independent, $\text{cov}(u_i'(y_i), z) = 0$. If (y_i, z) is p.d. (n.d.), since $u_i'(\cdot)$ is strictly decreasing, $(u_i'(y_i), z)$ is n.d. (p.d.). Thus $\text{cov}(u_i'(y_i), z) < 0$ (>0). Since $u_i'(\cdot) > 0$, $\delta_i > 0$ (<0). ∎

There is a simple way of interpreting Proposition 1. Since $v_i(\xi_i)$ is differentiable and $v_i(0) = 0$, by the intermediate value theorem there exists $0 \leqslant \hat{\xi}_i \leqslant \xi_i$ such that $v_i(\xi_i) = \xi_i v_i'(\hat{\xi}_i)$ and v_i' is given by (4). Substituting from (7) shows that the behaviour of the risk premium $\Delta_i(\xi_i)$ is influenced by the stochastic dependence between the pair $(y_i + \hat{\xi}_i z, \xi_i z)$: both the *idiosyncratic risk* arising from the stochastic dependence between the pair $(\hat{\xi}_i z, \xi_i z)$ and the *stochastic dependence risk* arising from the dependence between the pair $(y_i, \xi_i z)$ influence $\Delta_i(\xi_i)$. Proposition 1 asserts that *when an agent holds a sufficiently small proportion of the prospect z, its idiosyncratic risk disappears and the behaviour of the risk premium $\Delta_i(\xi_i)$ is determined by the stochastic dependence between $(y_i, \xi_i z)$.*

It is clear that if we want the aggregate values V_n and W_n to converge as $n \to \infty$ we need a uniformity condition on the types of new agents (income–preference pairs (y_i, u_i)) which are introduced as $n \to \infty$.

ASSUMPTION 2. There exist functions $r(\chi)$, $s(\chi)$ where $r(\chi)/\chi \to 0$, $s(\chi)/\chi \to 0$ as $\chi \to 0$ and $\delta > 0$ such that r_i and s_i defined by (6) satisfy $|r_i(\chi)| \leqslant r(\chi)$, $|s_i(\chi)| \leqslant s(\chi)$, $i = 1, 2, \ldots$, for all sufficiently small values of χ and $|\delta_i| \leqslant \delta$, $i = 1, 2, \ldots$.

Note that this condition is automatically satisfied if there are at most a finite number of distinct types of agents (pairs (y_i, u_i)) in the economy.

PROPOSITION 2. *Under Assumptions* 1 *and* 2, *the aggregate values converge* $V_n = (1/n) \sum_{i=1}^n v_i(1/n) \to V^*$, $W_n = (1/n) \sum_{i=1}^n w_i(1/n) \to W^*$ *as* $n \to \infty$, *furthermore* $V^* = W^* = \bar{z} - \delta^*$, $\delta^* = \lim_{n \to \infty} (1/n) \sum_{i=1}^n \delta_i$. *If the pair* (y_i, z) *is independent for all* $i \notin I$, *where* I *is any finite subset of* $N = \{1, 2, \ldots\}$, *then* $\delta^* = 0$. *If the pair* (y_i, z) *is positively (negatively) dependent for all* $i \notin I$, *then* $\delta^* > 0$ (<0).

Proof. Since, by Assumption 2, $|\delta_i| \leqslant \delta$, $i = 1, 2, \ldots$, the sequence of partial sums $(1/n) \sum_{i=1}^n |\delta_i|$ is increasing and bounded above. Thus $(1/n) \sum_{i=1}^n \delta_i$ is absolutely convergent and $\delta^* = \lim_{n \to \infty} (1/n) \sum_{i=1}^n \delta_i$ exists. By (6) and Assumption 2

$$\left| \sum_{i=1}^n v_i \left(\frac{1}{n} \right) - \frac{1}{n} \sum_{i=1}^n v_i'(0) \right| \leqslant \sum_{i=1}^n \left| r_i \left(\frac{1}{n} \right) \right| \leqslant nr \left(\frac{1}{n} \right) \to 0 \qquad \text{as} \quad n \to \infty.$$

Since, by (5), $(1/n) \sum_{i=1}^n v_i'(0) = \bar{z} - (1/n) \sum_{i=1}^n \delta_i$ and since $\delta^* = \lim_{n \to \infty} (1/n) \sum_{i=1}^n \delta_i = \lim_{n \to \infty} (1/n) \sum_{i=1, i \notin I}^n \delta_i$, $V_n \to V^* = \bar{z} - \delta^*$. Since by (6) and Assumption 2 a similar inequality holds for $\sum_{i=1}^{n} w_i(1/n)$, using (5), $W_n \to W^* = \bar{z} - \delta^*$. The fact that $\delta^* = 0$, >0, (<0) in the independent and positively (negatively) dependent cases follows from Proposition 1. ∎

EXAMPLE 1. Suppose each agent has a constant absolute risk aversion utility function $u_i(\chi) = -e^{-\alpha_i x}$ with $D_i = (-\infty, \infty)$, $0 < \alpha_i \leqslant \alpha$, $i = 1, 2, \ldots$. Let (y_i, z) be joint normally distributed with covariance $\sigma_{y_i z}$, $|\sigma_{y_i z}| \leqslant \sigma$, $i = 1, 2, \ldots$. Then Assumptions 1 and 2 hold and $V^* = \bar{z} - \lim_{n\to\infty} (1/n) \sum_{i=1}^{n} \alpha_i \sigma_{y_i z}$.

EXAMPLE 2. The following example shows that the differentiability of $U_i(\xi_i)$ at $\xi_i = 0$ is essential to Propositions 1 and 2. Let $u_i(\chi) = \ln \chi$ with $D_i(0, \infty)$, $i = 1, 2, \ldots$, let the public project be the sole source of income so that $y_i = 0$ a.s., $i = 1, 2, \ldots$, and let $\ln z$ be normally distributed with mean and variance (μ, σ^2) so that $\bar{z} = e^{\mu + \sigma^2/2}$. Then $G_i(v_i) = \ln v_i$, $U_i(\xi_i) = \ln \xi_i + \mu$ so that $v_i(\xi_i) = G_i^{-1}(U_i(\xi_i)) = e^{\ln \xi_i + \mu} = \xi_i e^{\mu}$. Assumption 1 does not hold since $U_i(\xi_i)$ is not differentiable at $\xi_i = 0$. Thus even though the pair $(y_i, z) = (0, z)$ is independent, $i = 1, 2, \ldots, v_i(\xi_i) = e^{-\sigma^2/2} \xi_i \bar{z} < \xi_i \bar{z}$ and $V^* = e^{-\sigma^2/2} \bar{z} < \bar{z}$. The Arrow–Lind result $V^* = \bar{z}$ in the independent identical agent case is thus no longer valid when $U_i(\xi_i)$ is not differentiable at $\xi_i = 0$.

Note that Proposition 2 holds for more general proportions ξ_i^n satisfying $\sum_{i=1}^{n} \xi_i^n = 1$, $n\,|\xi_i^n - 1/n| \to 0$ as $n \to \infty$ uniformly with respect to i. The interpretation of Proposition 2 is clear. As the number of agents increases, the proportion ξ_i^n held by the ith agent decreases; by Proposition 1, for sufficiently large values of n, the idiosyncratic risk of the public project disappears as far as the ith agent is concerned. Thus all that remains is the stochastic dependence risk so that $w_i(\xi_i^n) = v_i(\xi_i^n) = \xi_i^n(\bar{z} - \delta_i)$. Since this is true for all agents, the aggregate values W_n and V_n converge to $\bar{z} - \delta^*$ where δ^* is the average stochastic dependence risk premium.

When the return from a public project is shared among relatively few agents the fact that the individual purchase and sale values w_i and v_i can differ in magnitude implies (when there are both gainers and losers) that the aggregate values W_n and V_n can differ in sign. Proposition 2 shows that this problem cannot arise when the project is shared among sufficiently many agents, for then the aggregate purchase and sale values coincide. Thus if $W_n > 0$ so that the Kaldor condition for a potential Pareto improvement holds, then $V_n > 0$ so that the combined Hicks–Kaldor condition is satisfied.

4. Volatility and Aggregate Value

The qualitative behaviour of the aggregate value $V^* = W^*$ can be explored further if we introduce an explicit model that serves to explain the observed stochastic dependence between each pair (y_i, z). The natural idea here is that two random variables are positively dependent if they are both increasing functions of a common random variable with their own idiosyncratic noise. More generally I make the following assumption.

Assumption 3. The random variables (y_i, z), $i = 1, 2, \ldots$, are generated by a nonlinear index model

$$z(\omega) = \beta g(x(\omega), \eta(\omega)) + \theta(\omega), \qquad \omega \in \Omega, \quad -\infty < \beta < \infty,$$
$$y_i(\omega) = f_i(x(\omega), \varepsilon_i(\omega)), \qquad \omega \in \Omega, \quad i = 1, 2, \ldots,$$

where $g(\cdot)$ and $f_i(\cdot)$ are measurable functions and $(x, \eta, \theta, \varepsilon_i)$ are independent random variables.

When $g(\cdot, \eta)$ is a monotone function for every value of η, then β is a scalar measure of the extent to which z fluctuates as a result of fluctuations in the common underlying random variable x. I shall call β the *volatility* of z. Under Assumption 3 the marginal risk premium of the ith agent δ_i is a linear function of the volatility of z

$$\begin{aligned}\delta_i(\beta) &= -\frac{\operatorname{cov}(u_i'(f_i(x, \varepsilon_i)), \beta g(x, \eta) + \theta)}{E u_i'(f_i(x, \varepsilon_i))} \\ &= -\frac{\beta \operatorname{cov}(u_i'(f_i(x, \varepsilon_i)), g(x, \eta))}{E u_i'(f_i(x, \varepsilon_i))} = \beta \delta_i(1). \end{aligned} \tag{8}$$

Proposition 3. *Under Assumptions* 1–3, $V^*(\beta) = W^*(\beta) = \bar{z} - \beta \delta^*(1)$, $\delta^*(1) = \lim_{n \to \infty} (1/n) \sum_{i=1}^{n} \delta_i(1)$. *If $g(\cdot, \eta)$ is strictly increasing (decreasing) for every value of η and $f_i(\cdot, \varepsilon_i)$ is strictly increasing for every value of ε_i for all $i \notin I$, where I is any finite subset of $N = \{1, 2, \ldots\}$, then $\delta^*(1) > 0$ (<0). If $f_i(\cdot, \varepsilon_i)$ is strictly decreasing for every value of ε_i for all $i \notin I$, then the sign of $\delta^*(1)$ is reversed.*

Proof. $V^*(\beta) = W^*(\beta) = \bar{z} - \beta \delta^*(1)$ follows at once from (8) and Proposition 2. By a theorem of Lehmann [2, Theorem 1, p. 1138] if $g(\cdot, \eta)$ is strictly increasing (decreasing) for every value of η and $f_i(\cdot, \varepsilon_i)$ is strictly increasing for every value of ε_i, then the pair (y_i, z) is positively (negatively) dependent for $\beta > 0$. Thus, by Proposition 1, $\delta_i(1) > 0$ (<0) and hence, by Proposition 2, $\delta^*(1) > 0$ (<0). If $f_i(\cdot, \varepsilon_i)$ is strictly decreasing for every value of ε_i or if $\beta < 0$, then the sign of the stochastic dependence is reversed. ∎

Thus when $\beta > 0$ and g and f_i $(i \notin I)$ are increasing functions of the common variable x, which we can think of as an index of the business cycle, then (y_i, z) are positively dependent $(i \notin I)$ and an increase in the volatility of the public project reduces its aggregate value V^*. Conversely if g is a decreasing function of x, then (y_i, z) are negatively dependent $(i \notin I)$ and an increase in the volatility of z increases its aggregate value. In the first case an infinitesimal share of z adds risk to each agent's portfolio; an increase in the volatility of z increases this risk. In the second case introducing the public project enables each agent to hedge (at least partially) against the uncertainty of his existing income y_i; an increase in the volatility of z serves to make the hedge more effective. In the case where either g or f_i $(i \notin I)$ are no longer influenced by x, x ceases to create a common dependence and z and y_i $(i \notin I)$ are independent random variables. In this case either $\beta = 0$ or $\delta^*(1) = 0$ and we obtain the Arrow–Lind result $V^* = \bar{z}$.

EXAMPLE 1. Let $u_i(\chi) = -e^{-\alpha_i \chi}$, $D_i = (-\infty, \infty)$, $0 < \alpha_i \leqslant \alpha$, $i = 1, 2, \ldots$, $g(x, \eta) = x$, $f_i(x, \varepsilon_i) = \gamma_i x + \varepsilon_i$, $-\infty < \gamma_i < \infty$, where x and ε_i are independent normally distributed random variables and σ_x^2 is the variance of x. Then $\delta_i(\beta) = \beta \alpha_i \delta_i \sigma_x^2$, $i = 1, 2, \ldots$, and

$$V^*(\beta) = \bar{z} - \beta \delta^*(1) = \bar{z} - \beta \lim_{n \to \infty} \left(\frac{1}{n} \sum_{i=1}^{n} \alpha_i \gamma_i \right) \sigma_x^2.$$

In this case the sign of β determines whether the public project is procyclical ($\beta > 0$), countercyclical ($\beta < 0$), or independent of the business cycle ($\beta = 0$).

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