Anchoring expectations of inflation[☆]

Michael Magill [a,*], Martine Quinzii [b]

[a] *University of Southern California, United States*
[b] *University of California, Davis, United States*

HIGHLIGHTS

- We consider a monetary economy with a Ricardian fiscal policy.
- We model explicitly agents' expectations of future inflation as probability distributions on fixed support.
- Monetary policy consists of fixing short and longer-term interest rates.
- We show equivalence of sequential, extensive form of equilibrium and reduced-form equilibrium.
- We give conditions for the existence and uniqueness of equilibrium.

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ABSTRACT

This paper studies the existence and uniqueness of equilibrium in a monetary model in which the fiscal policy is Ricardian. The innovation of the paper is to model agents' expectations as endogenous probabilities which are determined in equilibrium. Since economies with a Ricardian fiscal policy typically exhibit indeterminacy of equilibrium when the monetary policy instrument is the short-term interest rate, we augment the instruments of monetary policy to the interest rates on a family of bonds of maturities $1, \ldots, T$ and derive conditions under which this ensures uniqueness of equilibrium.

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1. Introduction

This paper provides an abstract framework for studying how a monetary policy can anchor agents' expectations of inflation in a regime where the monetary–fiscal policy is Ricardian so that agents' expectations of inflation are a priori indeterminate. The new element in our analysis lies in the way we model expectations as an equilibrium sequence of probability distributions, and in our focus on the consistency conditions that must be satisfied between bond prices and agents' expectations of inflation.

The indeterminacy in the model arises from the forward-looking nature of expectations of inflation: agents' borrow-ing–saving decisions today, which determine the demand for goods and current equilibrium prices, depend on their expecta-tions of prices in the future. The choice of a short-term nominal in-terest rate by the monetary authority serves to tie down the mean

value of inflation next period but not its probability distribution. Furthermore the natural transversality condition which could po-tentially tie down the forward-looking inflation process at infinity is removed as a separate equilibrium condition when the fiscal pol-icy is Ricardian, since this means that taxes automatically adapt to asymptotically pay off the government debt (in present-value terms).

Since expectations of inflation play an essential role in a monetary equilibrium, we introduce them explicitly into the model. We consider a cash-in-advance model with a simple saving–consumption–labor choice of the type studied by Schmitt-Grohe and Uribe (2000) with certainty and by Nakajima and Pole-marchakis (2005) with uncertainty, in which agents believe that inflation next period can take one of a finite number of values—a discretization of agents' beliefs. In a rational expectations model the probability distribution on this support of inflation rates must be compatible with the prices of the nominal securities traded on the financial markets, in particular the prices of government bonds. The monetary authority, aware of this fact, determines the prices of (the interest rates on) bonds of different maturities $1, \ldots, T$ with a view to "anchoring" the stochastic process of beliefs to a process of its choice. Since the short-term nominal interest rate influences

the labor (production) decision which in turn influences the real interest rate, restrictions must be placed on the candidate inflation processes to be sure that they are compatible with non-negative nominal interest rates. This is how the so-called problem of the zero lower bound enters in the analysis, as a problem of existence of equilibrium. Thus the first step of our analysis is to define the set of inflation processes to which the monetary authority can consider anchoring agents' expectations because they are compatible with the existence of an equilibrium.

Having established conditions for the existence of an equilibrium we then turn to the problem of uniqueness of equilibrium. The standard approach to analyzing uniqueness (determinacy) in a monetary model is to examine whether, given short-term interest rates chosen by the monetary authority, there is a unique inflation process which is compatible with equilibrium. When the fiscal policy is Ricardian uniqueness does not hold since the short-term interest rate only ties down the expected value of inflation.[1] This is why we assume that the monetary authority uses more instruments than the short-term interest rate and fixes the interest rates (prices) on a family of bonds of different maturities, i.e. a part of the term structure of interest rates.[2] We find conditions under which a term-structure policy is associated with a unique inflation process, that is when there is a unique inflation process for which the bond prices satisfy the equilibrium pricing equations. When uniqueness holds we say that the monetary policy anchors agents' expectations, since from the knowledge of the interest rates that will hold in the future, agents can only adopt one compatible belief. This of course requires transparency of the monetary policy and rationality of investors.

Uniqueness is first studied for an economy in which all agents are identical and there is no fundamental uncertainty, so that all uncertainty comes from the agents' beliefs about inflation. The results are then extended to an economy subject to real (productivity) shocks satisfying a Markov process in which agents are heterogeneous. The conditions for uniqueness are expressed as rank conditions on the prices of the bonds, which essentially require that the term structure of interest rates vary with current inflation. This in turn implies that the inflation process to be anchored must be such that the probability distribution of next-period inflation rates varies systematically with current inflation. While there is considerable flexibility in the belief process a monetary authority can induce, it cannot induce a deterministic belief of immediate return to some chosen target inflation rate, or even beliefs which are independent of current inflation.

The sharp characterization of conditions for both existence and uniqueness of equilibrium hinges on a crucial transformation of the equilibrium from an extensive to a reduced form. The extensive form expresses the conditions which must be satisfied period by period and involves a large number of variables – the real and financial decisions of the agents and the government and the associated prices – making it essentially intractable. The reduced-form equilibrium involves the minimum number of variables needed to characterize an equilibrium and the constraints which are retained are expressed in the more condensed present-value form.

Section 2 introduces the model of a monetary economy in which the fiscal policy is Ricardian and agents' expectations of inflation are described by a sequence of probability distributions which are endogenously determined in equilibrium, and establishes the basic theorem on the equivalence between an extensive-form and a reduced-form equilibrium. Section 3 establishes conditions for the existence and uniqueness of equilibrium for an economy with no fundamental uncertainty and a representative agent. Section 4 shows how these results can be extended to an economy subject to real productivity shocks in which agents are heterogeneous. Section 5 gives a brief review of the related literature on the determinacy of equilibrium.

2. Monetary economy

Consider a monetary economy in discrete time over an infinite horizon with a finite number of agents, in which money serves not only as a unit of account but also as the medium of exchange. The objects of trade are goods and securities and the purchase (payment) for either must be made using money. There are two sets of agents: $\mathcal{H}$, the finite set of households in the private sector, and a government, the monetary–fiscal authority. The government issues money and nominal (government) bonds of different maturities and imposes taxes on the agents. There are two sources of uncertainty in the economy: the first is real, the second nominal. Real uncertainty is modeled as usual by exogenous shocks which affect the productivity of the economy. Nominal uncertainty on the other hand is endogenous: prices today depend on agents' consumption–savings decisions since these determine the quantity of money they use for current consumption, and these decisions in turn depend on the agents' expectations about future prices. Thus we model the nominal uncertainty as a process for agents' beliefs about inflation requiring that in a rational expectations equilibrium the beliefs coincide with the actual process.

Uncertainty and event-tree. To simplify the modeling of the agents' belief process about inflation, we discretize the support of the possible inflation rates, identifying each subinterval with a particular inflation rate in the subinterval (typically the midpoint except for the two extremes which represent the upper and lower tails).[3] We then approximate a probability distribution on the inflation rates at date t by the discrete probability distribution on the finite support consisting of these representative inflation rates. Since we are interested in stationary equilibria we take the same support $\Pi = \{\pi_1, \ldots, \pi_S\}$ at all dates, letting $\mathcal{S} = \{1, \ldots, S\}$ denote the index set of the inflation rates. Since the belief process can be influenced by actions taken by the monetary authority, the sequence of probability distributions on the support Π forms part of the endogenous variables in an equilibrium.

The real uncertainty arises from the fluctuations in the productivities of the agents in their production activities: let $\mathcal{G} = \{1, \ldots, G\}$ denote the index set for the possible real shocks. The process of real shocks is exogenously given by a family of probabilities $\mathbf{A} = (A_t)_{t \geq 0}$ on the histories $(g_0, \ldots, g_t)$ of the shocks up to date t, with the property that

$$\sum_{(g_0, \ldots, g_t) \in \mathcal{G}^t} A_t(g_0, \ldots, g_t) = 1,$$

$$A_t(g_0, \ldots, g_t) = \sum_{g \in \mathcal{G}} A_{t+1}(g_0, \ldots, g_t, g)$$

where $\mathcal{G}^t$ denotes the set of partial histories of the real shocks to date t. Let $\mathbb{D}_t = \mathcal{S}^t \times \mathcal{G}^t$ denote the set of partial histories of the

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[1] When taxes are exogenously given in real terms, so that they do not adjust to accommodate the government debt as in the Ricardian case, the transversality condition for the government's debt becomes an equilibrium condition which serves to tie down prices and leads to the uniqueness of equilibrium. This explains the determinacy in monetary models without a fiscal policy (i.e. with zero taxes), e.g. Dubey and Geanakoplos (1992, 2003, 2006) or Lucas and Stokey (1987).

[2] A related approach to establishing the uniqueness of equilibrium in the state-of-nature model of Nakajima and Polemarchakis (2005) is given in Adao et al. (2010). In Magill and Quinzii (2013) it is shown that there is another type of monetary policy consisting of fixing the expected value of the future short-term rate for several periods ahead which also ensures the uniqueness of equilibrium.

[3] In order that the tails, in particular the upper tail, can be represented by a finite value, the probability of very high inflation rates must be negligible.

combined inflation–productivity shocks with typical element $\xi_t = ((s_0, g_0), \ldots, (s_t, g_t))$. $\mathbb{D}_t$ denotes the set of nodes or date-events at date t: the union of all such date-events defines the event-tree

$$\mathbb{D} = \bigcup_{t=0}^{\infty} \mathbb{D}_t$$

consisting of all possible date-events ξ into the indefinite future. Any date-event ξ has a date $t(\xi)$, a unique predecessor ξ^- at date $t(\xi) - 1$ and a set ξ^+ of immediate successors at date $t(\xi) + 1$. If $\xi = ((s_0, g_0), \ldots, (s_t, g_t))$ then

$$\xi^- = ((s_0, g_0), \ldots, (s_{t-1}, g_{t-1})),$$
$$\xi^+ = \{((s_0, g_0), \ldots, (s_t, g_t), (s, g)) \mid (s, g) \in \mathcal{S} \times \mathcal{G}\}.$$

For any node $\xi \in \mathbb{D}$, let $\mathbb{D}(\xi)$ denote the subtree originating at ξ and let $\mathbb{D}_T(\xi)$ denote the nodes of the subtree $\mathbb{D}(\xi)$ at date T.

A belief process $\boldsymbol{B}$ on the event-tree $\mathbb{D}$ is a family of probabilities $\boldsymbol{B} = (B_t)_{t \geq 0}$ on the partial histories $\mathbb{D}_t$ such that

$$\sum_{\xi \in \mathbb{D}_t} B_\xi = 1, \qquad B_\xi = \sum_{\xi' \in \xi^+} B_{\xi'}, \quad \forall \xi \in \mathbb{D}_t$$

where we use the shorthand notation B_ξ to denote $B_{t(\xi)}(\xi)$. The agents' beliefs must be consistent with the exogenously given probabilities $A = (A_t)_{t \geq 0}$ for the productivity shocks, which are assumed to be independent of the realized inflation rates. Thus we say that a belief process $\boldsymbol{B} = (B_t)_{t \geq 0}$ on $\mathbb{D}$ is *compatible with A* if for all $t \geq 0$

$$\sum_{(s_0, \ldots, s_t) \in \mathcal{S}^t} B_t((s_0, g_0), \ldots, (s_t, g_t)) = A_t(g_0, \ldots, g_t),$$
$$\forall (g_0, \ldots, g_t) \in \mathcal{G}^t.$$

Agents' characteristics and actions. To incorporate production into the model in the simplest way which at the same time captures the idea that production is influenced by the nominal interest rate set by the monetary authority, we follow Schmitt-Grohe and Uribe (2000) and Nakajima and Polemarchakis (2005) using a cash-in-advance model in which consumption is a 'cash good' and leisure is a 'credit good' in the terminology of Lucas and Stokey (1987). We assume that there are H agents (indexed by $h \in \mathcal{H}$) and that at each date-event $\xi \in \mathbb{D}$ an agent has an endowment e^h of 'time' which can be used either for leisure (ℓ_ξ^h) or to produce labor services (L_ξ^h), with $e^h = \ell_\xi^h + L_\xi^h$, $\xi \in \mathbb{D}$. The agent sells the labor services L_ξ^h to a firm which uses them to produce $y_\xi = a_\xi^h L_\xi^h$ units of output, the productivity of the agent's labor services depending only on the real and not on the inflation shock.[4] If c_ξ^h is the agent's consumption of the good at node ξ, then the pair $x_\xi^h = (c_\xi^h, \ell_\xi^h)$ generates the flow utility $u^h(c_\xi^h, \ell_\xi^h)$ where the functions u^h, $h \in \mathcal{H}$, satisfy the following conditions.

Assumption $\mathcal{U}$ (*Preferences*). For each $h \in \mathcal{H}$ the function $u^h : \mathbb{R}_+ \times [0, e^h] \to \mathbb{R}$ has the following properties:

1. increasing and differentiably strictly concave:
   $u_c > 0, \; u_\ell > 0$
   $u_{cc} < 0, \; u_{\ell\ell} < 0, \; u_{cc}^h u_{\ell\ell}^h - (u_{c\ell}^h)^2 > 0$
2. supermodular: $u_{c\ell}^h \geq 0$
3. Inada conditions: $\begin{aligned} u_c^h(c, \ell) &\to \infty \text{ as } c \to 0, \; \forall \ell \in (0, e^h) \\ u_\ell^h(c, \ell) &\to \infty \text{ as } \ell \to 0, \; \forall c > 0 \end{aligned}$
4. asymptotic satiation: $\begin{aligned} u_c^h(c, \ell) &\to 0 \text{ as } c \to \infty, \; \forall \ell \in (0, e^h) \\ u_\ell^h(c, \ell) &\to 0 \text{ as } \ell \to e^h, \; \forall c > 0. \end{aligned}$

Given a belief process $\boldsymbol{B}$, a consumption–leisure process[5] $\boldsymbol{x}^h = (c_\xi^h, \ell_\xi^h)_{\xi \in \mathbb{D}}$ on the event-tree $\mathbb{D}$ generates the lifetime expected utility

$$U^h(\boldsymbol{x}^h) = \sum_{\xi \in \mathbb{D}} \delta^{t(\xi)} B_\xi u^h(x_\xi^h), \quad 0 < \delta < 1. \tag{1}$$

In addition to the consumption–leisure decision x_ξ^h the agent makes a portfolio decision at each date-event which finances the consumption stream. To describe the transactions we use the standard timing of the cash-in-advance model, each period being divided into three subperiods. In the first subperiod, securities are traded and taxes are paid to the government; in the second the available money balances are used to purchase the consumption good at the current money price p_ξ and in the final subperiod firms pay agents for their labor services.

Thus in the first subperiod the agent decides on the money $\tilde{m}_\xi^h$ to lay aside to finance the purchase of consumption $p_\xi c_\xi^h$, and on the holding of the securities, which consist of zero-coupon nominal (government) bonds of maturities $\tau = 1, \ldots, T$ and a collection of private-sector short-lived securities in zero net supply indexed by $j = T + 1, \ldots, J$ with payoffs $V_{\xi'}^j$ (in units of money) at the immediate successors $\xi' \in \xi^+$. Let $\mathcal{J}_g$ denote the set of T government bonds, let $\mathcal{J}_p$ denote the set of private-sector securities and let $\mathcal{J} = \mathcal{J}_g \cup \mathcal{J}_p$ be the set of all securities. We assume that the combined set of securities is sufficiently rich to ensure complete markets (full spanning at each node ξ of the event-tree $\mathbb{D}$). Let $q_\xi = (q_\xi^j)_{j \in \mathcal{J}}$ denote the vector of (money) prices of the securities and let $z_\xi^h = (z_\xi^{hj})_{j \in \mathcal{J}}$ denote the agent's portfolio at node ξ, the first T components consisting of the agent's holdings of the government bonds. Since a τ-period bond purchased at node ξ becomes a $\tau - 1$ period bond at each of the successors $\xi' \in \xi^+$ and since the 1-period bond at node ξ pays 1 (dollar) at each successor, the payoffs at $\xi' \in \xi^+$ of the T bonds purchased at node ξ are given by the vector $(1, q_{\xi'}^1, \ldots, q_{\xi'}^{T-1})$. Given that we focus on the bond market, we let $\hat{q}_\xi = (1, q_\xi^1, \ldots, q_\xi^{T-1}, V_\xi^j, j = T + 1, \ldots, J)$ denote the payoff at node ξ of all the securities traded at node ξ^-. The $SG \times J$ matrix of payoffs of the securities traded at node ξ at the successors ξ^+ is denoted by

$$\left[\hat{q}_{\xi^+} \right] \equiv \left[\hat{q}_{\xi'}^j \right]_{\substack{j \in \mathcal{J} \\ \xi' \in \xi^+}}.$$

The condition that markets are complete is equivalent to the property that rank$[\hat{q}_{\xi^+}] = SG$, or that $[\hat{q}_{\xi^+}]$ is invertible for all $\xi \in \mathbb{D}$. We consider only price processes $\boldsymbol{q}$ which do not offer arbitrage opportunities, so that each agent has a solution to the problem of choosing an optimal portfolio. For any no-arbitrage $\boldsymbol{q}$ there exists a process $\boldsymbol{P} = (P_\xi)_{\xi \in \mathbb{D}}$, where P_ξ / P_{ξ_0} is the present value at date 0 of a promise to deliver one unit of money at node ξ, such that $P_\xi q_\xi^j = \sum_{\xi' \in \xi^+} P_{\xi'} \hat{q}_{\xi'}^j$. Given the assumption of complete markets, $\boldsymbol{P}$ is unique up to normalization.

Let $m_{\xi^-}^h$ denote the money balances brought into node ξ; since the agent receives the payoff $\hat{q}_\xi z_{\xi^-}^h$ on the portfolio $z_{\xi^-}^h$ purchased at the preceding node, he has the wealth $w_\xi^h = m_{\xi^-}^h + \hat{q}_\xi z_{\xi^-}^h$ available in the first subperiod of node ξ to buy a new portfolio z_ξ^h of the securities and to pay the taxes θ_ξ^h which are due. The agent lays aside enough money balances $\tilde{m}_\xi^h \geq p_\xi c_\xi^h$ to purchase the planned consumption c_ξ^h on the goods market of the second subperiod. Thus

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the agent chooses $(\widetilde{m}_\xi^h, z_\xi^h)$ so that

$$\widetilde{m}_\xi^h + \theta_\xi^h + q_\xi z_\xi^h = m_{\xi-}^h + \hat{q}_\xi z_{\xi-}^h, \quad \xi \in \mathbb{D} \tag{2}$$

$$\widetilde{m}_\xi^h \geq p_\xi c_\xi^h, \quad \xi \in \mathbb{D}. \tag{3}$$

Let ω_ξ denote the wage at node ξ. In the last subperiod of node ξ, the firm pays the agent $\omega_\xi a_\xi^h L_\xi^h$ for the labor services rendered at node ξ: this money and the unspent balances

$$m_\xi^h = \omega_\xi a_\xi^h L_\xi^h + \left(\widetilde{m}_\xi^h - p_\xi c_\xi^h\right), \quad \xi \in \mathbb{D} \tag{4}$$

are transferred to each of the successors $\xi' \in \xi^+$ of node ξ.

Since the agent is not willing to lend to any other agent or the government "at infinity", and since no agent is willing to lend to him "at infinity", he is obliged to confine his portfolio strategies to those for which the transversality condition

$$\lim_{T \to \infty} \sum_{\xi \in \mathbb{D}_T(\tilde{\xi})} P_\xi q_\xi z_\xi^h = 0, \quad \forall \tilde{\xi} \in \mathbb{D} \tag{5}$$

is satisfied, where $\boldsymbol{P} = (P_\xi)_{\xi \in \mathbb{D}}$ is the valuation of income compatible with the price process $\boldsymbol{q}$ (see Magill and Quinzii (1994)).

Monetary and fiscal policy. Whether or not there is a problem with the determinacy of agents' expectations of inflation depends on the way monetary and fiscal policy interact. We are interested in a regime where the monetary policy is dominant and the fiscal policy adapts to ensure that the government debt does not grow too fast, since this is the natural setting in which the monetary policy has a role in tying down agents' expectations of inflation.

To introduce the fiscal policy in the simplest way we assume that the government inherits a debt at date 0 which it needs to finance. There is no government expenditure on goods, so that what we call "taxes" is actually the excess of revenues over government expenditure, i.e. the primary surplus. At each date-event the government can issue money and bonds of several maturities, and can tax (or make transfers to) the agents. Let M_ξ denote the money outstanding at node ξ and let $(Z_\xi^j, j \in \mathcal{J}_g)$ denote the portfolio of bonds of different maturities issued at node ξ: to keep the government's portfolio commensurate[6] with that of the agents in the private sector it is convenient to set $Z_\xi^j = 0$ for $j \in \mathcal{J}_p$ and let $Z_\xi = (Z_\xi^j, j \in \mathcal{J})$. Let θ_ξ denote the taxes at node ξ and let

$$W_\xi = M_{\xi-} + \hat{q}_\xi Z_{\xi-}, \quad \xi \in \mathbb{D}$$

denote the government liabilities at the beginning of node ξ, inherited from the preceding node. These liabilities need to be covered by taxes θ_ξ, and open market operations (M_ξ, Z_ξ) satisfying

$$M_\xi + \theta_\xi + q_\xi Z_\xi = M_{\xi-} + \hat{q}_\xi Z_{\xi-}, \quad \xi \in \mathbb{D}. \tag{6}$$

We do not address the choice of an optimal fiscal/monetary policy but rather focus on characterizing policies which are compatible with the existence of a unique equilibrium. We can think of the fiscal authority as deciding the taxes θ_ξ while the monetary authority decides the bond prices $(q_\xi^j, j \in \mathcal{J}_g)$ and accommodates the private sector demand for money and government bonds. However we as-

sume that the fiscal policy is secondary to the monetary policy in the sense that for any path $(\boldsymbol{M}, \boldsymbol{Z}, \boldsymbol{q}) = (M_\xi, Z_\xi, q_\xi)_{\xi \in \mathbb{D}}$ of monetary policy, taxes are set to ensure that the government's liabilities do not grow "too fast", which is formalized by the property that the transversality condition holds for the government's liabilities: this is the analogue of condition (5) for any private sector agent, namely that $\sum_{\xi \in \mathbb{D}_T(\tilde{\xi})} P_\xi W_\xi \to 0$ as $T \to \infty$ for every $\tilde{\xi} \in \mathbb{D}$. Sargent (1982) gave an intuitive example of a fiscal policy which has this property – any additional government expense must be financed by an increase in current or future taxes to pay for it – and thus called it a "Ricardian" fiscal policy. Here we do not have government expenses, just a debt to finance, so we adopt the more abstract rule considered by Benhabib et al. (2001): there is some α_ξ with $0 \leq \alpha_\xi \leq 1$ such that

$$\frac{r_\xi^1}{1 + r_\xi^1} M_\xi + \theta_\xi = \alpha_\xi W_\xi, \quad \xi \in \mathbb{D}. \tag{7}$$

At each node of the event-tree the tax θ_ξ plus the seignorage revenue $\frac{r_\xi^1}{1+r_\xi^1} M_\xi$ reimburses a fraction α_ξ of the current liabilities W_ξ. For example if $\alpha_\xi = \frac{r_\xi^1}{1+r_\xi^1}$ then the rule becomes

$$\theta_\xi + \frac{r_\xi^1}{1 + r_\xi^1}(M_\xi - M_{\xi-}) = \frac{r_\xi^1}{1 + r_\xi^1} \hat{q}_\xi Z_{\xi-}, \quad \xi \in \mathbb{D}$$

so that the tax plus the seignorage revenue on the issue of new money covers the interest on the debt $\hat{q}_\xi Z_{\xi-}$ inherited at node ξ: this is a slightly modified version of the "balanced budget requirement" of Schmitt-Grohe and Uribe (2000). As we shall see in the proof of Theorem 3, the following assumption implies that a rule of the type (7) ensures that the present value of the government liabilities tends asymptotically to zero.

Assumption RC (*Ricardian Condition*). The fiscal policy is such that the process $(\alpha_\xi)_{\xi \in \mathbb{D}}$ in (7) satisfies $\underline{\alpha} \leq \alpha_\xi \leq 1$ for all $\xi \in \mathbb{D}$ for some $\underline{\alpha} > 0$.

In a multi-agent economy the total tax θ_ξ needs to be shared by the agents: we consider a simple proportional rule of the form

$$\theta_\xi^h = \gamma^h \theta_\xi, \quad \xi \in \mathbb{D}, \ h \in \mathcal{H} \tag{8}$$

for some $\boldsymbol{\gamma} = (\gamma_h)_{h \in \mathcal{H}} \in \Delta^H$, the simplex in $\mathbb{R}^H$. Agents should not be required to pay more taxes than they can possibly pay with their income, since this would lead to nonexistence of equilibrium: in Theorem 15 below, which establishes the existence of an equilibrium in a multi-agent economy, we give an assumption which ensures that each agent's after-tax income is positive.

An economy is described by the agents' characteristics, their initial holdings of money and bonds, and by the fiscal policy of the government. Thus we let

$$\mathcal{E}(\boldsymbol{u}, \delta, \boldsymbol{e}, \boldsymbol{a}, \boldsymbol{A}, \boldsymbol{m}_{-1}, \boldsymbol{z}_{-1}, \boldsymbol{\alpha}, \boldsymbol{\gamma})$$

(often shortened to $\mathcal{E}$) denote an economy in which agents' preferences and endowments are given by $(\boldsymbol{u}, \delta, \boldsymbol{e}, \boldsymbol{a}) = (u^h, \delta, e^h, (a_\xi^h)_{\xi \in \mathbb{D}})_{h \in \mathcal{H}}$, $\boldsymbol{A}$ is the process of real shocks, initial money and bond holdings are $(\boldsymbol{m}_{-1}, \boldsymbol{z}_{-1}) = (m_{-1}^h, z_{-1}^h)_{h \in \mathcal{H}}$, the initial liabilities of the government being $(M_{-1}, Z_{-1}) = \sum_{h \in \mathcal{H}} (m_{-1}^h, z_{-1}^h)$, and $(\boldsymbol{\alpha}, \boldsymbol{\gamma})$ describes the government fiscal policy.

With a Ricardian fiscal policy, if the sole instrument of monetary policy consists in fixing the short-term interest rate then the

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[6] Since there are no transaction costs we follow the same convention for the government's portfolio as for the agents' portfolios, assuming that the government closes the portfolio $Z_{\xi-}$ at node ξ and issues the new portfolio Z_ξ. The restriction $Z_\xi^j = 0$ if $j \in \mathcal{J}_p$ is not essential and the results that follow do not depend on it. Traditionally the monetary authority (central bank) has not traded private-sector securities, but this has changed with the "unconventional" QE policies.

equilibrium is indeterminate[7]: in the simplest representative-agent case this is an immediate consequence of Proposition 4 below (see also Nakajima and Polemarchakis (2005)).

Our objective is to show that, under an assumption of stationarity, the monetary authority can tie down agents' expectations of inflation to a process of its choice if it uses more instruments than just the short-term interest rate. We are thus led to study a monetary policy which consists in fixing the prices of bonds of maturities $1, \ldots, T$ as function of current inflation, which we call a *generalized interest-rate rule* or a *term-structure rule*. We show that under appropriate conditions such a policy leads to a unique equilibrium. In Magill and Quinzii (2013) we show (in the setting of an exchange economy) that such a uniqueness result can also be obtained by a monetary policy which consists in fixing the current interest rate and the expected values of the short-term interest rate for the next T periods, which we call a *forward-guidance rule*. For brevity the formal analysis of this paper – the relation between extensive and reduced-form equilibrium, existence and uniqueness of equilibrium – is restricted to the case of a term-structure rule. We leave it to the reader to adapt the arguments to the case of a forward-guidance rule. The conditions for the existence of equilibrium are the same for both types of policy and the restrictions on an inflation process required to obtain a unique equilibrium are essentially the same, the differences being explained in Magill and Quinzii (2013).

Thus in this paper a monetary policy consists in fixing the prices $(q_\xi^j, j \in \mathcal{J}_g)_{\xi \in \mathbb{D}}$ of the government bonds for every node in the event-tree. Such a policy must be compatible with the expectations process $\boldsymbol{B}$ for inflation since nominal interest rates depend on real interest rates and expected inflation. Thus the beliefs enter as an integral part of the description of an equilibrium, to express the property that they are compatible with the bond prices chosen by the monetary authority. In the spirit of inflation targeting we interpret $\boldsymbol{B}$, or more precisely the inflation component of $\boldsymbol{B}$, as being chosen by the monetary authority, the monetary policy $\boldsymbol{q}$ ensuring that agents adopt $\boldsymbol{B}$ as their beliefs. Uniqueness of equilibrium – the property that there is only one process $\boldsymbol{B}$ compatible with the monetary policy – is then essential to ensure that the monetary authority can "anchor" agents' expectations of inflation to $\boldsymbol{B}$.

An equilibrium consists of a monetary–fiscal policy for the government, which includes an inflation process to direct agents' expectations and the associated bond prices, consumption–labor choices by agents as well as their associated money and portfolio holdings, production plans for firms, and money prices for labor, consumption good and securities across the event-tree, which are mutually compatible. Let $\ell^1(\mathbb{D})$ denote the space of summable sequences on the event-tree $\mathbb{D}$, $\ell^1(\mathbb{D}) = \{\boldsymbol{P} \in \mathbb{R}^\mathbb{D} \mid \sum_{\xi \in \mathbb{D}} |P_\xi| < \infty\}$.

Definition 1. An *(extensive-form) equilibrium* of $\mathcal{E}$ is a triple consisting of choices by the government and the agents, and prices

$$\left[\left(\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_p}, \bar{\boldsymbol{M}}, \bar{\boldsymbol{Z}}, \bar{\boldsymbol{\theta}} \right), \left(\bar{\boldsymbol{x}}, \widetilde{\bar{\boldsymbol{m}}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{L}} \right), \left(\bar{\boldsymbol{P}}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{\omega}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_p} \right) \right]$$

such that

(i) $\bar{\boldsymbol{B}}$ is compatible with the exogenous process $\boldsymbol{A}$;

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[7] This well known in macroeconomics since it was first pointed out by Sargent and Wallace (1975) and subsequently gave rise to a substantial literature which sought to obtain determinacy by assuming an interest-rate rule satisfying the Taylor principle with a more than one-for-one response of the short-term interest rate to inflation. Uniqueness results can be obtained if the criterion for uniqueness is weakened to local uniqueness around a steady state (see e.g. Woodford (2003) and Benhabib et al. (2001)). Benhabib et al. (2001) show that the result on local uniqueness is sensitive to the way preferences for money are modeled, and also show that local uniqueness does not in general imply global uniqueness.

(ii) for every node $\xi = ((s_0, g_0), \ldots, (s_t, g_t)) \in \mathbb{D}$, $\bar{p}_\xi = (1 + \pi_{s_1}) \ldots (1 + \pi_{s_t})$;

(iii) $\bar{P}_\xi \bar{q}_\xi^j = \sum_{\xi' \in \xi^+} \bar{P}_{\xi'} \hat{\bar{q}}_{\xi'}^j$, $\forall \xi \in \mathbb{D}$, $j \in \mathcal{J}$, $(\bar{P}_\xi \bar{p}_\xi)_{\xi \in \mathbb{D}} \in \ell^1_+(\mathbb{D})$;

(iv) $\bar{q}^\tau \le 1$, $\tau = 1, \ldots, T$;

(v) $(\bar{\boldsymbol{x}}^h, \widetilde{\bar{\boldsymbol{m}}}^h, \bar{\boldsymbol{z}}^h)$ maximizes $\sum_{\xi \in \mathbb{D}} \delta^{t(\xi)} \bar{B}_\xi u^h(x_\xi^h)$ subject to (2)–(5) with prices $(\bar{\boldsymbol{P}}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{\omega}}, \bar{\boldsymbol{q}})$;

(vi) $(\bar{y}_\xi, \bar{L}_\xi)$ maximizes $\bar{p}_\xi y - \bar{\omega}_\xi L$, subject to $y = L$, for all $\xi \in \mathbb{D}$;

(vii) $(\bar{\boldsymbol{M}}, \bar{\boldsymbol{Z}}, \bar{\boldsymbol{\theta}})$ satisfies (6)–(8) and RC;

(viii) $\bar{y}_\xi = \sum_{h \in \mathcal{H}} \bar{c}_\xi^h$, $\bar{L}_\xi = \sum_{h \in \mathcal{H}} a_\xi^h \bar{L}_\xi^h$, $\bar{y}_\xi = \bar{L}_\xi$, $\xi \in \mathbb{D}$;

(ix) $\sum_{h \in \mathcal{H}} \widetilde{\bar{m}}_\xi^h = \bar{M}_\xi$, $\sum_{h \in \mathcal{H}} \bar{z}_\xi^h = \bar{Z}_\xi$, $\xi \in \mathbb{D}$.

As can be seen from (ii) we take as given the root node $\xi_0 = (s_0, g_0)$ so that the initial inflation and real shock are well defined, setting the price $p_{\xi_0} = p_{-1}(1 + \pi_{s_0})$ equal to 1. This normalization is used since with an interest rate policy the price level at date 0 is not determined. The analysis focuses on whether the inflation process is determinate. (iv) ensures that the nominal interest rates on the bonds of different maturities are non-negative. If q_ξ^τ is the price of the τ-period bond, the associated interest rate or *yield to maturity* r_ξ^τ is defined by $q_\xi^\tau = \frac{1}{(1+r_\xi^\tau)^\tau}$ and $q_\xi^\tau \le 1$ is equivalent to $r_\xi^\tau \ge 0$.

Reduced-form equilibrium. The sequential structure of an extensive-form equilibrium makes it a complex object to analyze directly. For analytical purposes a simpler form of equilibrium can be obtained by expressing all budget equations (both those of agents and the government) in present-value form, eliminating the financial variables – money and portfolios – while retaining the present value of taxes and the bond prices to capture the government fiscal and monetary policy. To simplify the analysis of equilibrium we assume that the portfolios z_{-1}^h inherited from date -1 are composed only of short-lived bonds, so that the wealth w_0^h of each agent $h \in \mathcal{H}$ at the beginning of date 0, $w_0^h = m_{-1}^h + z_{-1}^h$, is exogenously given and does not depend on the security prices at date 0. We also simplify the description of a reduced-form equilibrium by taking (i) and (ii) of Definition 1 as given, namely that the process $\bar{\boldsymbol{B}}$ is compatible with the exogenous process $\boldsymbol{A}$ for the real shocks and that the spot (money) price of the good is given by $\bar{p}_\xi = (1 + \pi_{s_1}) \ldots (1 + \pi_{s_t})$ for any $\xi = ((s_0, g_0), \ldots, (s_t, g_t)) \in \mathbb{D}$.

Definition 2. A *reduced-form equilibrium* of the economy $\mathcal{E}$ consists of a pair

$$\left(\left(\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_g}, \bar{\Theta} \right), \bar{\boldsymbol{x}}, \bar{\boldsymbol{P}} \right)$$

such that

(1) $(\bar{P}_\xi \bar{p}_\xi)_{\xi \in \mathbb{D}} \in \ell^1_+(\mathbb{D})$;

(2) $\bar{\boldsymbol{x}}^h \in \text{argmax} \left\{ \sum_{\xi \in \mathbb{D}} \delta^{t(\xi)} \bar{B}_\xi u^h(x_\xi^h) \,\middle|\, \sum_{\xi \in \mathbb{D}} \bar{P}_\xi \bar{p}_\xi \left(c_\xi^h - \frac{a_\xi^h(e^h - \ell_\xi^h)}{1 + \bar{r}_\xi^1} \right) \right.$

$\left. + \gamma^h \bar{P}_{\xi_0} \bar{\Theta} = \bar{P}_{\xi_0} w_0^h \right\}$;

(3) $\sum_{h \in \mathcal{H}} \bar{c}_\xi^h = \sum_{h \in \mathcal{H}} \bar{a}_\xi^h(e^h - \bar{\ell}_\xi^h)$, $\xi \in \mathbb{D}$;

(4) $\bar{P}_\xi \bar{q}_\xi^\tau = \sum_{\xi' \in \xi^+} \bar{P}_{\xi'} \bar{q}_{\xi'}^{\tau-1}$, $q_\xi^0 = 1$, $\tau = 1, \ldots, T$, $\xi \in \mathbb{D}$;

(5) $\bar{q}_\xi^\tau \le 1$, $\tau = 1, \ldots, T$, $\xi \in \mathbb{D}$.

The number of variables in a reduced-form equilibrium is much smaller than in an extensive form equilibrium since portfolios, money holdings and period-by-period taxes no longer appear. The sequence of node-by-node budget constraints for each agent is replaced by a single inter-temporal budget constraint so that the optimal lifetime consumption/labor decision of an agent

only depends on the present-value prices $(\bar{P}_\xi \bar{p}_\xi)_{\xi \in \mathbb{D}}$, the agent's share γ^h of the present value of taxes $\bar{\Theta}$, and the sequence of short-term interest rates $(\bar{r}_\xi^1)_{\xi \in \mathbb{D}}$ because of the cash-in-advance constraint. Furthermore there is no inter-temporal constraint for the government since the Ricardian condition (RC) ensures that the present value of taxes plus seigniorage reimburse the government's date 0 liabilities, for all values of prices and interest rates over the event-tree. Although the prices of the private-sector securities are not present (they can be reconstructed from the security payoffs and the present-value prices) the prices $(q^j)_{j \in \mathcal{J}_g}$ of the government bonds appear in a reduced-form equilibrium since they are chosen by the monetary authority. The compatibility conditions (4) can be viewed either as a restriction on the bond prices $(q^j)_{j \in \mathcal{J}_g}$ given P, or as a restriction on the present-value price P given $(q^j)_{j \in \mathcal{J}_g}$: in the analysis that follows we use the latter interpretation. In the next theorem we show that all the variables of an extensive form equilibrium can be recovered from the knowledge of the reduced-form equilibrium.

Theorem 3 (*Equivalence of Extensive and Reduced-form Equilibrium*). *If financial markets are complete,[8] $\left(rank \left[\hat{\bar{q}}_{\xi+} \right] = SG \text{ for all } \right.$ $\xi \in \mathbb{D} \Big)$ then $\left((\bar{B}, (\bar{q}^j)_{j \in \mathcal{J}_g}, \bar{\Theta}) \bar{x}, \bar{P} \right)$ is a reduced-form equilibrium of $\mathcal{E}$ if and only if there exist money holdings, portfolios, taxes and prices such that $\left[\left(\bar{B}, (\bar{q}^j)_{j \in \mathcal{J}_g}, (\bar{M}, \bar{Z}, \bar{\theta}) \right), \left(\bar{x}, \widetilde{\bar{m}}, \bar{z}, \ \bar{y}, \bar{L} \right), \left(\bar{P}, \bar{p}, \right.$ $\bar{\omega}, (\bar{q}^j)_{j \in \mathcal{J}_p} \Big) \Big]$ is an extensive-form equilibrium of $\mathcal{E}$.*

Proof. (See the Appendix). It is straightforward to show that an extensive-form equilibrium must satisfy the conditions of a reduced-form equilibrium. To show the converse, the first step for recovering the variables which no longer appear in a reduced-form equilibrium is to reconstruct the prices of the private-sector securities using the no-arbitrage equations $\bar{P}_\xi \bar{q}_\xi^\tau = \sum_{\xi' \in \xi+} \bar{P}_{\xi'} \bar{q}_{\xi'}^{\tau-1}$ and to show that if a sequence of taxes (θ_ξ) has the present value $\sum_{\xi \in \mathbb{D}} \bar{P}_\xi \theta_\xi = \bar{\Theta}$ then the opportunity sets of an agent in the extensive-form and the reduced-form equilibrium are the same. The second step is to show how the government portfolio-tax policy can be reconstructed when the agents' demand for money is known. The last step is to show that the market clearing conditions are satisfied on the government bond and private securities' markets.

Theorem 3 plays a crucial role in the analysis that follows: for an equilibrium expressed in reduced form has a sufficiently simple structure to permit the existence and uniqueness of equilibrium to be established. The basic ideas and the analytical approach are most easily explained in the simplest setting of an economy with identical agents (representative agent) and no shock to productivity. This is the case studied in the next section. In Section 4 we show how the analysis can be extended to an economy with heterogeneous agents and real shocks.

3. Identical agents and no real shocks

Consider the special case of the economy in Section 2 in which there are no real shocks ($G = 1$), all agents have identical preferences and endowments, $u^h = u, a^h = 1, e^h = 1$ for all $h \in \mathcal{H}$, and the only securities are the government bonds: $\mathcal{J}_p = \emptyset$. If we write the equilibrium in per-capita terms, then the equilibrium is formally equivalent to the equilibrium of Definition 1 with $H = 1$,

$\gamma^h = 1, \mathcal{J}_p = \emptyset$. In the next proposition we show that the variables which characterize an equilibrium can be further reduced to just the belief process, the bond price and consumption processes, because the present-value prices P and the present value of taxes $\bar{\Theta}$ can be deduced from these variables. To ensure the summability of present-value prices $(\bar{P}_\xi \bar{p}_\xi)_{\xi \in \mathbb{D}}$ the consumption sequence is assumed to be uniformly bounded away from zero.

Proposition 4 (*Equilibrium Equations*). *If $\mathcal{E}$ is an economy with identical agents, then a reduced-form equilibrium for which $\bar{c}_\xi \geq \varepsilon, \forall \xi \in \mathbb{D}$ for some $\varepsilon > 0$ is characterized by the pair $((\bar{B}, \bar{q}), \bar{c})$ satisfying the system of equations*

(a) $\frac{u_c(\bar{c}_\xi, 1 - \bar{c}_\xi)}{u_\ell(\bar{c}_\xi, 1 - \bar{c}_\xi)} = 1 + \bar{r}_\xi^1, \xi \in \mathbb{D}$

(b1) $\bar{B}_\xi u_c(\bar{c}_\xi, 1 - \bar{c}_\xi) \bar{q}_\xi^\tau = \delta \sum_{\xi' \in \xi+} \bar{B}_{\xi'} \frac{u_c(\bar{c}_{\xi'}, 1 - \bar{c}_{\xi'})}{1 + \pi_{\xi'}} \bar{q}_{\xi'}^{\tau-1}, \tau = 1, \ldots, T, \bar{q}_{\xi'}^0 = 1, \xi \in \mathbb{D}$

(b2) $\bar{q}_\xi^\tau \leq 1, \tau = 1, \ldots, T, \xi \in \mathbb{D}$.

Proof. $(\Longrightarrow)$ Let $\left((\bar{B}, \bar{q}, \bar{\Theta}), \bar{x}, \bar{P} \right)$ be a reduced-form equilibrium. The FOCs for the maximum problem (2) of Definition 2 for the agent imply that there exists $\lambda > 0$ such that for all $\xi \in \mathbb{D}$

$$\bar{B}_\xi \ \delta^{t(\xi)} \ u_c(\bar{c}_\xi, \bar{\ell}_\xi) = \lambda \ \bar{P}_\xi \ \bar{p}_\xi \tag{9}$$

$$\bar{B}_\xi \ \delta^{t(\xi)} \ u_\ell(\bar{c}_\xi, \bar{\ell}_\xi) = \lambda \ \frac{\bar{P}_\xi \ \bar{p}_\xi}{1 + \bar{r}_\xi^1} \tag{10}$$

where, by Assumption $\mathcal{U}$ the consumption/leisure decision is always interior. Market clearing implies $\bar{\ell}_\xi = 1 - \bar{c}_\xi$ and (a) follows by taking the ratio of (9) and (10). Replacing $\bar{P}_\xi$ by its value given in (9), (4) of Definition 2 implies that (b1) is satisfied, and since (b2) is the same as (5) of Definition 2, a reduced-form equilibrium satisfies (a), (b1), (b2).

$(\Longleftarrow)$ Let $((\bar{B}, \bar{q}), \bar{c})$ satisfy (a), (b1), (b2). For all $\xi \in \mathbb{D}$ define $\bar{\ell}_\xi = 1 - \bar{c}_\xi, \bar{P}_\xi = \bar{B}_\xi \delta^{t(\xi)} \frac{u_c(\bar{c}_\xi, \bar{\ell}_\xi)}{p_\xi}$, so that $\bar{P}_{\xi_0} = u_c(\bar{c}_0, \bar{\ell}_0)$. Since $\bar{c}$ is uniformly bounded away from 0, $(\bar{P}_\xi \bar{p}_\xi)_{\xi \in \mathbb{D}} \in \ell_1(\mathbb{D})$. (9) is satisfied with $\lambda = 1$ and since (a) holds, (10) also holds with $\lambda = 1$. Define $\bar{\Theta}$ by $\sum_{\xi \in \mathbb{D}} \bar{P}_\xi \bar{p}_\xi \bar{c}_\xi \frac{\bar{r}_\xi^1}{1 + \bar{r}_\xi^1} + \bar{P}_{\xi_0} \bar{\Theta} = \bar{P}_{\xi_0} w_0^h$. Then $\bar{x}$ satisfies the budget constraint in (2) of Definition 2, and since the FOCs are satisfied it solves the maximum problem (2) of Definition 2. Thus $\left((\bar{B}, \bar{q}, \bar{\Theta}), \bar{x}, \bar{P} \right)$ is a reduced-form equilibrium. □

In the representative-agent case the system of equations characterizing an equilibrium has a strikingly simple form: it reduces to the representative agent's FOCs for optimal consumption/leisure and portfolio choice. In particular the representative-agent inter-temporal budget equation is not included since we want the equations to be independent: the agent's inter-temporal budget constraint is the mirror image of that of the government and, by the Ricardian property, the present value of taxes always adjust so that the inter-temporal equation of the government is satisfied. In the proof of Proposition 4 the agent's budget constraint is used to define the present value of the taxes which are part of a reduced-form equilibrium, once it is recognized that the nominal present-value prices $\bar{P}_\xi$ are the probability of the node times the marginal utility of one unit of money at this node.

It is clear from the FOC (a) that the "real" allocation only depends on the current short-term nominal interest rate r_ξ^1 and that the process of inflation per se has no direct real effect. It is however useful to study the conditions which ensure that the process of inflation can be anchored by a monetary policy in this model with flexible prices since it is simpler to analyze than a model with staggered prices and imperfect competition in which

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[8] If there is only one agent, or equivalently all agents are identical, the condition of complete markets is not necessary for the theorem to hold.

inflation has a direct real effect. It is standard practice in monetary theory to study the determinacy of equilibrium in the model with flexible prices (Schmitt-Grohe and Uribe, 2000; Nakajima and Polemarchakis, 2005) since the conclusions typically extend to the more complex model with rigidities (Nakajima and Polemarchakis, 2005; Adao et al., 2010).

It follows from (a, b1) in Proposition 4 that a monetary policy which only determines the price of the short-term bond cannot fully anchor agents' expectations of inflation. The choice of a sequence of short-term interest rates $(r_\xi^1)_{\xi \in \mathbb{D}}$ determines the real allocation $(c_\xi)_{\xi \in \mathbb{D}}$ through the equations (a). But any process $\boldsymbol{B}$ which satisfies

$$\frac{1}{1 + \bar{r}_\xi^1} = \delta \sum_{\xi' \in \xi^+} B_{\xi\xi'} \frac{u_c(\bar{c}_{\xi'}, 1 - \bar{c}_{\xi'})}{u_c(\bar{c}_\xi, 1 - \bar{c}_\xi)} \frac{1}{1 + \pi_{\xi'}}, \quad \xi \in \mathbb{D}$$

where $B_{\xi\xi'}$ is the conditional probability at node ξ of the successor node $\xi' \in \xi^+$, satisfies the equilibrium equations. Since a single equation at node ξ cannot determine S values $(B_{\xi\xi'})_{\xi' \in \xi^+}$ there are infinitely many inflation processes compatible with the sequence $(r_\xi^1)_{\xi \in \mathbb{D}}$ of short-term interest rates. However if the monetary authority fixes the prices of more bonds the restrictions on the processes $\boldsymbol{B}$ imposed by equations (b1) increase and, as we shall see, by fixing the prices of S bonds[9] at each node it is possible to anchor expectations of inflation to an appropriately chosen process $\boldsymbol{B}$.

The concept of equilibrium assumes that the government is committed to the bond prices $(\bar{q}^j, j \in \mathcal{J})$ and that agents know them over the whole event-tree. The latter assumption only seems reasonable if there is a systematic rule by which bond prices are set at each node. In the analysis that follows we focus on simple rules for which bond prices depend only on current inflation: this is in the spirit of Taylor rules for the short-term interest rate studied in New-Keynesian models (see Woodford (2003)). It is clear from Proposition 4(a) that if the security prices only depend on current inflation, then the agents' consumption, which is determined by the nominal short-term interest rate, also only depends on current inflation. Since (b1) is a system of first-order difference equations, if the bond prices and the consumption only depend on current inflation and if $\bar{\boldsymbol{B}}$ is the only belief compatible with the system of equations (b1), then it has to be Markov. We thus restrict attention to Markov processes for inflation characterized by a Markov matrix $[B_{ss'}]_{s,s' \in \mathcal{S}}$: if $\xi = (s_0, s_1, \ldots, s)$ and $\xi' = (s_0, s_1, \ldots, s, s')$ then $B_{\xi\xi'} = B_{ss'}$.

As a special case of Proposition 4 we characterize a Markov reduced-form equilibrium $((\bar{B}, \bar{q}), \bar{c})$ as a Markov matrix $\bar{B} = [\bar{B}_{ss'}]_{s,s' \in \mathcal{S}}$, bond prices $\bar{q} = (\bar{q}_s^\tau, \tau = 1, \ldots, T, s \in \mathcal{S})$ and a vector of consumption $\bar{c} = (\bar{c}_s, s \in \mathcal{S})$ which only depend on current inflation.[10]

Corollary 5 (*Stationary Equilibrium Equations*). *If $\mathcal{E}$ is an economy with identical agents, then a Markov reduced-form equilibrium is characterized by a pair $((\bar{B}, \bar{q}), \bar{c}) \in \mathbb{R}_+^{SS} \times \mathbb{R}_+^{TS} \times \mathbb{R}_+^S$ satisfying the reduced-form equilibrium equations*

(a) $\frac{u_c(\bar{c}_s, 1 - \bar{c}_s)}{u_\ell(\bar{c}_s, 1 - \bar{c}_s)} = 1 + \bar{r}_s^1, \quad s \in \mathcal{S};$

(b1) $\bar{q}_s^\tau = \delta \sum_{s' \in S} \frac{\bar{B}_{ss'}}{1 + \pi_{s'}} \frac{u_c(\bar{c}_{s'}, 1 - \bar{c}_{s'})}{u_c(\bar{c}_s, 1 - \bar{c}_s)} \bar{q}_{s'}^{\tau-1}, \quad \tau = 1, \ldots, T, \bar{q}_{s'}^0 = 1, s \in S;$

(b2) $\bar{q}_s^\tau \leq 1, \tau = 1, \ldots, T, s \in \mathcal{S}.$

We assume that the objective of the monetary authority is to direct agents' expectations to an inflation process $\boldsymbol{B}$ of its choice, assuming that this belief once adopted by the agents is self-fulfilling. To characterize the inflation processes among which the monetary authority can choose, two questions need to be answered:

(i) *existence*: for which matrices B does there exist $(\boldsymbol{q}, \boldsymbol{c})$ such that $((B, \boldsymbol{q}), \boldsymbol{c})$ is an equilibrium?
(ii) *uniqueness*: what additional properties must a matrix B satisfy if the equilibrium is to be unique?

Existence of equilibrium. If the monetary authority is to induce a matrix of beliefs B it must set the term-structure rule in such a way that equations (b) of Corollary 5 are satisfied. Since these equations involve both the expectations matrix B and the stochastic discount factor which, by equations (a), is determined by the short-term interest rates $r^1 = (r_s^1, s \in \mathcal{S})$, there is a fixed-point problem which needs to be solved. To study this problem it is convenient to use the S gross returns

$$R_s = 1 + r_s^1, \quad s \in \mathcal{S}$$

on the short-term nominal bond as the basic variables. We first show that for a given scalar return R there is a unique solution to the agent's consumption/leisure choice problem in any given state.

Lemma 6 (*Equilibrium Consumption*). *If u satisfies Assumption $\mathcal{U}$ (1)–(3) then*

(i) *for all $R > 0$, the equation*

$$\frac{u_c(c, 1 - c)}{u_\ell(c, 1 - c)} = R \tag{11}$$

has a unique solution $c(R)$, where $c(R)$ is a strictly decreasing function of R.

(ii) $\Phi(R) \equiv u_c(c(R), 1 - c(R))$ *is strictly increasing on $(0, \infty)$.*

(iii) $\widetilde{\Phi}(R) \equiv \frac{\Phi(R)}{R}$ *is strictly decreasing on $(0, \infty)$.*

Proof. (i) Let $h(c) \equiv \frac{u_c(c, 1-c)}{u_\ell(c, 1-c)}$. Then $h'(c) = \frac{1}{u_\ell^2}(u_{cc}u_\ell - u_{c\ell}u_\ell - u_{\ell c}u_c + u_{\ell\ell}u_c)$. Since $u_{cc} < 0, u_{\ell\ell} < 0, u_c > 0, u_\ell > 0, u_{c\ell} \geq 0$, it follows that $h'(c) < 0$ and h is decreasing. By the Inada condition $h(c) \to \infty$ as $c \to 0$ and $h(c) \to 0$ as $c \to 1$. Thus (11) has a unique solution $c(R)$. Differentiating $h(c(R)) = R$ gives $h'(c(R))c'(R) = 1$: $h' < 0$ implies $c'(R) < 0$.

(ii) $\Phi'(R) = (u_{cc} - u_{c\ell})c'(R) > 0$ by (i).

(iii) $\widetilde{\Phi}'(R) = \frac{1}{R^2}\left((u_{cc} - u_{c\ell})c'(R)R - u_c\right)$. Using $c'(R) = 1/h'(c(R))$ where $h'(c)$ has been calculated in (i), and $R = u_c/u_\ell$, we obtain $\widetilde{\Phi}'(R) = \frac{u_c^2}{D}(u_{c\ell} - u_{\ell\ell}) < 0$, where the derivatives are calculated at $c(R)$ and $D = R^2(u_{cc}u_\ell - u_{\ell c}u_c - u_{c\ell}u_\ell + u_{\ell\ell}u_c)$. Since $D < 0$ and $u_{c\ell} \geq 0$, $\widetilde{\Phi}'(R) < 0$. $\quad\square$

In view of Lemma 6 the FOCs (a) and the FOCs (b1) for the short-term nominal bond of Corollary 5, can be combined into the system of equilibrium equations

$$\frac{1}{R_s} = \delta \sum_{s' \in \mathcal{S}} \frac{B_{ss'}}{1 + \pi_{s'}} \frac{\Phi(R_{s'})}{\Phi(R_s)}, \quad s \in \mathcal{S}. \tag{12}$$

For each $s \in \mathcal{S}$ this is the 'true' stochastic Fisher equation relating the price $q_s^1 = \frac{1}{R_s}$ of the short-term nominal bond to the price of the *real* bond

$$\frac{1}{1 + r_s^{real}} = \delta \sum_{s' \in \mathcal{S}} B_{ss'} \frac{\Phi(R_{s'})}{\Phi(R_s)}, \quad s \in \mathcal{S}$$

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[9] Equation (b1) when applied to long-term bonds ($\tau \geq 2$) is usually viewed as a statement that agents' expectations of inflation determine the prices of long-term bonds: see e.g. Söderlind and Svensson (1997) and Goodfriend (1991, 1998) for a study of the way agents' expectations can be deduced from the prices of bonds, and more generally futures and options on interest rates. We reverse this logic and point out that fixing the prices of a sufficient number of bonds restricts the expectations of inflation which are compatible with these prices on the bond market.

[10] Although the growth of money demand $\frac{M_{\xi'}}{M_\xi} = \frac{(1 + \pi_{\xi'})\bar{c}_{\xi'}}{\bar{c}_s}$ is Markov, the financial variables of the extensive-form equilibrium or their rates of growth are not necessarily Markov, especially if the government's debt reimbursement policy α_ξ is not Markov.

and inflation $(\pi_{s'})_{s' \in \mathcal{S}}$ next period, when the current inflation is π_s. Since the nominal interest rate determines the real wage and hence output and consumption, it affects the real interest rate and the system of equations (12) only implicitly defines the nominal interest rates associated with an expectations matrix B. If the condition $R_s \geq 1, s \in \mathcal{S}$ (non-negative nominal interest rates), were omitted then Eqs. (12) would always have a solution (this can be deduced from the fixed point argument given below). However when the condition $R_s \geq 1, s \in \mathcal{S}$, is imposed, conditions of compatibility have to be imposed on the matrix B.

Definition 7. A Markov matrix B is said to be *compatible with non-negative interest rates* if the system of equations (12) has a solution $\boldsymbol{R} \geq \mathbf{1} = (1, \ldots, 1)$.

Note that if $\bar{\boldsymbol{R}} \geq \mathbf{1}$ is a solution of (12) then $\bar{q}_s^1 = \frac{1}{R_s} \leq 1, s \in \mathcal{S}$, and (b2) of Corollary 5 is satisfied for $\tau = 1$. (b1) of same corollary, which can be written as

$$\bar{q}_s^\tau = \delta \sum_{s' \in \mathcal{S}} \frac{B_{ss'}}{1 + \pi_{s'}} \frac{\Phi(R_{s'})}{\Phi(R_s)} \bar{q}_{s'}^{\tau-1}, \quad s \in \mathcal{S}, \ \tau = 2, \ldots, T$$

then gives by successive substitution the prices of the bonds of higher maturities and the inequality $\bar{q}_s^\tau \leq 1$ is transferred to these prices. Thus all the conditions of Corollary 5 are satisfied and there exists a reduced-form equilibrium.

Using the function $\widetilde{\Phi}$ defined in Lemma 6(iii), (12) can be written as

$$\widetilde{\Phi}(R_s) = \delta \sum_{s' \in \mathcal{S}} \frac{B_{ss'}}{1 + \pi_{s'}} \Phi(R_{s'}), \quad s \in \mathcal{S}. \tag{13}$$

Since $\widetilde{\Phi}$ is decreasing, if for each s the right side of (13) lies in the image of $\widetilde{\Phi}$ (a condition for this is given below) then $\widetilde{\Phi}$ can be inverted and (13) is equivalent to the system of equations

$$R_s = \widetilde{\Phi}^{-1}\left(\delta \sum_{s' \in \mathcal{S}} \frac{B_{ss'}}{1 + \pi_{s'}} \Phi(R_{s'})\right) \equiv \Psi_s(R_1, \ldots, R_S), \quad s \in \mathcal{S} \tag{14}$$

where Ψ_s is decreasing for each $s \in S$. Let $\boldsymbol{\Psi} = (\Psi_1, \ldots, \Psi_S)$ denote the vector-valued map which associates with $\boldsymbol{R}$ the new vector of returns $\boldsymbol{\Psi}(\boldsymbol{R})$. An equilibrium $\bar{\boldsymbol{R}}$ is a fixed point of $\boldsymbol{\Psi}: \bar{\boldsymbol{R}} = \boldsymbol{\Psi}(\bar{\boldsymbol{R}})$.

Since a vector of nominal returns must satisfy $\boldsymbol{R} \geq \mathbf{1} = (1, \ldots, 1)$ and since $\boldsymbol{\Psi}$ is decreasing, the minimal return vector $\mathbf{1}$ maps into the maximal return vector $\boldsymbol{R}^{\max} = (R_1^{\max}, \ldots, R_S^{\max}) = \boldsymbol{\Psi}(\mathbf{1})$. Consider the rectangular subset of the non-negative orthant of $\mathbb{R}^S$

$$K = \left\{\boldsymbol{R} \in \mathbb{R}_+^S \mid \mathbf{1} \leq \boldsymbol{R} \leq \boldsymbol{R}^{\max}\right\}.$$

If $\boldsymbol{R}^{\max} \geq \mathbf{1}$ then $K \neq \phi$, and if $\boldsymbol{\Psi}(\boldsymbol{R}^{\max}) = \boldsymbol{\Psi}(\boldsymbol{\Psi}(\mathbf{1})) \geq \mathbf{1}$ then $\boldsymbol{\Psi}(K) \subset K$ so that Brouwer's Theorem can be applied.

It remains to give conditions which ensure that the two properties $K \neq \phi$ and $\boldsymbol{\Psi}(K) \subset K$ are satisfied. The maximum achievable consumption c^* occurs when the nominal interest rate is zero, $c^* = c(1)$: this is also what an agent's consumption would be without a cash-in-advance constraint. $\boldsymbol{R}^{\max} = \boldsymbol{\Psi}(\mathbf{1})$ is equivalent to

$$\frac{1}{R_s^{\max}} = \delta \sum_{s' \in \mathcal{S}} \frac{B_{ss'}}{1 + \pi_{s'}} \frac{u_c(c^*, 1 - c^*)}{u_c\left(c(R_{s'}^{\max}), 1 - c(R_{s'}^{\max})\right)}, \quad s \in \mathcal{S}, \tag{15}$$

(15) must have a solution for each s, and this solution must satisfy $R_s^{\max} \geq 1$. Condition (1) in Theorem 8 below ensures that these two properties hold: the first inequality implies that $R_s^{\max}$ exists and the second inequality ensures that it is greater than or equal to 1. The right side of (15) involves an upper bound on the real interest rate since it assumes that the consumption in each state s' next period is maximal (at c^*) while it is minimal today (at $c(R_s^{\max})$). Thus $R_s^{\max} \geq 1$ requires that the nominal interest rate, which is essentially the real interest rate plus the expected rate of inflation, be

positive when the real interest rate is at its highest possible value. This is clearly a necessary condition. The condition $E_s^B\left(\frac{\delta}{1+\pi}\right) \leq 1$, which ensures that $R_s^{\max} \geq 1$ requires that high deflation rates are not given too much weight.

To express a condition which ensures that $\boldsymbol{\Psi}(\boldsymbol{\Psi}(\mathbf{1})) = \boldsymbol{\Psi}(\boldsymbol{R}^{\max}) \geq \mathbf{1}$, consider the vector $\boldsymbol{R}^{\min} = (R_1^{\min}, \ldots, R_S^{\min})$ where $R_s^{\min}$ is defined by

$$\frac{1}{R_s^{\min}} = \delta \sum_{s' \in \mathcal{S}} \frac{B_{ss'}}{1 + \pi_{s'}} \frac{u_c\left(c(R_{s'}^{\max}), 1 - c(R_{s'}^{\max})\right)}{u_c(c^*, 1 - c^*)}, \quad s \in \mathcal{S}. \tag{16}$$

The return $R_s^{\min}$ would occur if consumption today were maximal (at c^*) and consumption tomorrow were expected to be at its minimal value $c(R_{s'}^{\max})$ in each state s': this gives a lower bound on the real interest rate in each state. Condition (2) in Theorem 8 requires that the nominal interest rate is positive even when the real interest rate is at this lower bound, a more demanding requirement than condition (1).

Theorem 8 (*Existence of Stationary Equilibrium*). *If B is an $S \times S$ Markov matrix such that*

(1) $\lim_{R_s \to \infty} \frac{u_c(c(R_s), 1 - c(R_s))}{R_s u_c(c^*, 1 - c^*)} < E_s^B\left(\frac{\delta}{1+\pi}\right) \leq 1, \ s \in \mathcal{S}$,
(2) $R_s^{\min} \geq 1, \ s \in \mathcal{S}$ *where $R_s^{\min}$ is defined by (16) and $R_s^{\max}$ is defined by (15),*

then there exists a stationary equilibrium of $\mathcal{E}$, i.e. B is compatible with non-negative interest rates.

Proof. Eq. (15) is equivalent to

$$\frac{\widetilde{\Phi}(R_s^{\max})}{\widetilde{\Phi}(1)} = a_s, \qquad a_s = E_s^B\left(\frac{\delta}{1+\pi}\right).$$

Since $\widetilde{\Phi}$ is decreasing, if $a_s \leq 1$ (which is the second inequality in (1)) then the solution, if it exists, will satisfy $R_s^{\max} \geq 1$. The equation will have a solution if $a_s \widetilde{\Phi}(1) > \inf_{R \geq 1} \widetilde{\Phi}(R) = \lim_{R \to \infty} \widetilde{\Phi}(R)$, which is the first inequality in (1). This proves that, when (1) is satisfied, $K \neq \phi$.

It remains to show that $\boldsymbol{\Psi}(\boldsymbol{R}^{\max}) \geq \mathbf{1}$ to ensure $\boldsymbol{\Psi}(K) \subset K$. For each state $s \in \mathcal{S}$

$$\Psi_s(\boldsymbol{R}^{\max}) = \widetilde{\Phi}^{-1}\left(\delta \sum_{s' \in \mathcal{S}} \frac{B_{ss'}}{1 + \pi_{s'}} \Phi(R_{s'}^{\max})\right) \geq 1$$

$$\Longleftrightarrow \delta \sum_{s' \in \mathcal{S}} \frac{B_{ss'}}{1 + \pi_{s'}} \Phi(R_{s'}^{\max}) \leq \widetilde{\Phi}(1) = \frac{\Phi(1)}{1}$$

$$\Longleftrightarrow \frac{1}{R_s^{\min}} \leq 1.$$

By Brouwer's Theorem $\boldsymbol{\Psi}$ has a fixed point $\bar{\boldsymbol{R}}$ in K which defines a positive short-term interest rate compatible with the expectations matrix B. $\quad\square$

Inflation targeting formalizes the idea that a monetary authority induces an inflation process which is mean reverting towards a target and thus spends most of its time around the target. More formally it is natural to define the target associated with an inflation process B as the mean inflation rate $E^\rho(\pi)$ under its invariant measure ρ, defined by $\rho B = \rho$. Let us show that there is no inflation process compatible with non-negative interest rates for which the target is the deflation rate $\pi^* = \delta - 1$ associated with the Friedman rule.

Corollary 9 (*Impossibility of Targeting Friedman Rule*). *Any expectations matrix compatible with non-negative interest rates satisfies $E^\rho(\pi) > \pi^*$, where ρ (defined by $\rho B = \rho$) is the invariant measure associated with B.*

Proof. As mentioned earlier and as is clear from the proof of Theorem 8, while condition (2) is only a sufficient condition for the existence of equilibrium, the inequalities $E_s^B\left(\frac{\delta}{1+\pi}\right) \leq 1$, $s \in \mathcal{S}$ are necessary. Since the function $x \to \frac{1}{1+x}$ is convex, $E_s^B\left(\frac{\delta}{1+\pi}\right) \geq \frac{\delta}{1+E_s^B(\pi)}$, with a strict inequality if $\text{var}(\pi) > 0$, which we assume. Thus

$$1 > \frac{\delta}{1+E_s^B(\pi)} \iff E_s^B(\pi) > \delta - 1,$$
$$\iff B_s\pi > \delta - 1, \quad s \in \mathcal{S}$$

where B_s is row s of the matrix B. Thus $B\pi \gg (\delta-1)\mathbf{1}$. Since B^∞, the limit of B^n when n tends to infinity, has positive terms in each row $B^\infty B\pi = B^\infty \pi \gg (\delta-1)B^\infty \mathbf{1} = (\delta-1)\mathbf{1}$. Since each row of B^∞ is equal to ρ, $E^\rho(\pi) > \delta - 1 = \pi^*$. $\square$

The impossibility of targeting the negative inflation rate $\pi^* = \delta - 1$ comes from the zero lower bound on the nominal interest rate which places an important constraint on monetary policy—both in theory and in practice.[11] If there was no uncertainty about inflation ($\text{var}(\pi) = 0$) the deflation rate π^* would be associated with a zero nominal interest rate. However as soon as there is uncertainty about inflation next period ($\text{var}(\pi) > 0$) each row of the belief matrix must put more weight on inflation rates above π^* (for which $\frac{\delta}{1+\pi_s} < 1$) than on inflation rates below π^* (for which $\frac{\delta}{1+\pi_s} > 1$) to be sure that the necessary condition $E_s^B\left(\frac{\delta}{1+\pi}\right) \leq 1$ is satisfied. In short when there is variability of inflation the target rate cannot be set too low if the nominal interest rate is to stay non-negative.

Uniqueness of equilibrium. Theorem 8 gives restriction on an inflation process $\bar{B}$ which ensures that there is an associated short-term interest rate (or bond price) satisfying the FOCs (a) and (b1) of Corollary 5 which is non-negative (i.e. satisfies (b2)). If the monetary policy is restricted to determining the short-term interest rate $\bar{\mathbf{r}}^1 = (r_s^1)_{s\in\mathcal{S}}$ then for the same short-term interest rate process, there are many inflation processes also satisfying (b1). However if the monetary authority fixes more bond prices, then this reduces the number of inflation processes satisfying these equations. If the matrix $\bar{B}$ satisfying (b1) is unique, then we say that the monetary authority has *anchored* agents' expectations to $\bar{B}$ by the generalized interest-rate rule, or bond-pricing rule $((\bar{q}_s^\tau)_{s\in\mathcal{S}}, \tau = 1, \ldots, T)$.

It is useful to note that for a representative-agent economy the variables characterizing an equilibrium can be further reduced to just the pair $(\bar{B}, \bar{\mathbf{q}})$ since, in view of Lemma 6(i), the consumption stream $\bar{c}$ can be deduced from the short-term interest rate.

Definition 10. An interest-rate (bond-pricing) rule $\bar{\mathbf{q}} = (\bar{\mathbf{q}}^1, \ldots, \bar{\mathbf{q}}^T)$ anchors agents' expectations to the inflation process $\bar{B}$ if $\bar{B}$ is the unique matrix such that $(\bar{B}, \bar{\mathbf{q}})$ is a reduced-form equilibrium.

Given a reduced-form equilibrium $(\bar{B}, \bar{\mathbf{q}})$ we can define the vector of marginal utilities of the representative agent at the equilibrium

$$\bar{\boldsymbol{\Phi}} = (\bar{\Phi}_s)_{s\in\mathcal{S}} = \left(\Phi(\bar{R}_s)\right)_{s\in\mathcal{S}} = \left(u_c(c(\bar{R}_s), 1 - c(\bar{R}_s))\right)_{s\in\mathcal{S}} \quad (17)$$

and, for any belief matrix B the present-value matrix

$$\Gamma = [\Gamma_{s,s'}]_{s,s'\in\mathcal{S}}, \quad \text{with} \quad \Gamma_{s,s'} = \frac{B_{s,s'}\bar{\Phi}_{s'}}{(1+\pi_{s'})\bar{\Phi}_s}, \quad s, s' \in \mathcal{S}$$

where $\Gamma_{s,s'}$ is the present value in inflation state s of a promise to pay one dollar in inflation state s' next period. A matrix B is compatible with the bond prices $\bar{\mathbf{q}}$ if the equations (b1) are satisfied, i.e.

$$\bar{q}_s^\tau = \sum_{s'\in\mathcal{S}} \Gamma_{s,s'}\bar{q}_{s'}^{\tau-1}, \quad s \in \mathcal{S}, \ \tau = 1, \ldots, T \iff \bar{\mathbf{q}}^\tau = \Gamma\bar{\mathbf{q}}^{\tau-1},$$
$$\tau = 1, \ldots, T \quad (18)$$

the price $\bar{q}_s^\tau$ of a τ-bond in inflation state s being the present value of a $\tau - 1$-bond at each of the successors (with $\bar{q}^0 = \mathbf{1}$). (18) can be written as the matrix system of equations

$$[\bar{\mathbf{q}}^1, \ldots, \bar{\mathbf{q}}^T] = \Gamma[\mathbf{1}, \bar{\mathbf{q}}^1, \ldots, \bar{\mathbf{q}}^{T-1}]$$
$$\iff Q = \Gamma\widehat{Q} \quad (19)$$

with $Q = [\bar{\mathbf{q}}^1, \ldots, \bar{\mathbf{q}}^T]$ denoting the matrix of bond prices across the inflation states and the matrix $\widehat{Q} = [\mathbf{1}, \bar{\mathbf{q}}^1, \ldots, \bar{\mathbf{q}}^{T-1}]$ denoting their next period payoffs. The matrix Γ can be written as

$$\Gamma = \bar{D}_1 B \bar{D}_2 \quad (20)$$

with

$$\bar{D}_1 = \begin{bmatrix} \frac{1}{\bar{\Phi}_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\bar{\Phi}_S} \end{bmatrix}$$
$$\bar{D}_2 = \begin{bmatrix} \frac{\delta\bar{\Phi}_1}{1+\pi_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{\delta\bar{\Phi}_S}{1+\pi_S} \end{bmatrix}. \quad (21)$$

Since $\bar{D}_1$ and $\bar{D}_2$ are diagonal matrices—with strictly positive diagonal terms, there is a one to one relation between Γ and B. Viewing Γ as the $S \times S$ matrix of unknowns in the linear system of Eq. (19), uniqueness of Γ implies uniqueness of B satisfying the FOC (b1) in equilibrium, and this requires that $\widehat{Q}$ is invertible.[12] This leads to the following result.

Theorem 11 (*Uniqueness of Equilibrium*). Let $(\bar{B}, \bar{\mathbf{q}})$ be a reduced-form equilibrium. The bond-pricing rule $\bar{\mathbf{q}} = (\bar{\mathbf{q}}^1, \ldots, \bar{\mathbf{q}}^T)$ anchors the inflation expectations matrix $\bar{B}$
(i) if $T \geq S$ and $\text{rank}[\widehat{Q}] = \text{rank}[\mathbf{1}, \bar{\mathbf{q}}^1, \ldots, \bar{\mathbf{q}}^{S-1}] = S$ or
(ii) if $T \geq S - 1$ and $\text{rank}[\widetilde{Q}] = \text{rank}[\frac{1+\pi}{\Phi}, \mathbf{1}, \bar{\mathbf{q}}^1, \ldots, \bar{\mathbf{q}}^{S-2}] = S$.

Proof. (i) follows from the fact that B must satisfy the equilibrium pricing equations (19). (ii) comes from using the Markov property $B\mathbf{1} = \mathbf{1}$: for each s the vector of probabilities $B_s = (B_{ss'})_{s'\in\mathcal{S}}$ has only $S - 1$ parameters so that typically the prices of $S - 1$ bonds should suffice to determine B.

$$B\mathbf{1} = \mathbf{1} \iff \bar{D}_1 B \bar{D}_2 \bar{D}_2^{-1} \mathbf{1} = \bar{D}_1 \mathbf{1}$$
$$\iff \Gamma\left[\frac{1+\pi}{\delta\Phi}\right] = \left[\frac{1}{\Phi}\right] \quad (22)$$

with the convention that $[\mathbf{x}]$ denotes the column vector $(x_1, \ldots, x_S)^\top$. If the prices of T bonds are determined in a reduced-form equilibrium, a Markov matrix B is compatible with these bond prices if it satisfies the equations

$$\Gamma\left[\frac{1+\pi}{\delta\Phi}\right] = \left[\frac{1}{\Phi}\right], \qquad Q = \Gamma\widehat{Q} \iff \left[\frac{\delta}{\Phi}, Q\right] = \Gamma\widetilde{Q}$$

with $\widetilde{Q} = \left[\frac{1+\pi}{\Phi}, \widehat{Q}\right]$ from which (ii) follows. $\square$

Condition (ii) indicates that it may suffice to fix the price of $S - 1$ bonds to make the equilibrium determinate. However the condition that the vector $\left[\frac{1+\pi}{\Phi}\right]$ is independent of the payoffs $[\mathbf{1}, \bar{\mathbf{q}}^1, \ldots, \bar{\mathbf{q}}^{S-2}]$ of the bonds is not easy to interpret or verify a priori. In the Appendix we examine the case where $S = 2$, showing that conditions (i) and (ii) are distinct. Although (i) is stronger in the sense that it calls for fixing prices of S rather than $S - 1$ bonds, the rank condition (i) does not imply the rank condition (ii). When $S = 2$, (i) is useful only when (ii) is not satisfied. In this case fixing the price of one more bond serves to ensure uniqueness.

As indicated in Section 2 we have in mind a case where $(\pi_1, \ldots, \pi_S)$ is a discretization of the support of inflation rates deemed possible by the agents. If the resulting equilibrium is to be a reasonable approximation of an equilibrium with continuous support for inflation then the discretization will require more than two points. As S increases the difference between fixing the prices of S or $S - 1$ bonds becomes less important and condition (i) becomes attractive because it focuses on the conditions that must be satisfied by the term structure generated by $\bar{B}$ if the bond pricing rule is to anchor $\bar{B}$. The main message of condition (i) is that the term structure generated by $\bar{B}$ must vary systematically across the current inflation states ($s \in \mathcal{S}$) and this in turn implies that the probabilities $(B_{ss'})_{s' \in \mathcal{S}}$ must vary systematically with current inflation s. The simplest case for understanding the restrictions on $\bar{B}$ implied by the rank condition (i) is the case where the pricing is risk neutral.

Suppose therefore that the agent's utility function is quasi-linear in consumption

$$u(c, \ell) = c + v(\ell)$$

where v is increasing, differentiable, strictly concave and satisfies $v'(\ell) \to \infty$ when $\ell \to 0$ and $v'(\ell) \to 0$ when $\ell \to 1$. The FOC (a) in Corollary 5

$$v'(1 - c) = \frac{1}{1 + r_s} = \frac{1}{R_s}$$

defines the optimal consumption $\bar{c}(R_s)$ as a function of the current nominal gross return R_s of the short-term bond. Since $\Phi(R_s) = u_c(\bar{c}(R_s), 1 - \bar{c}(R_s)) = 1$ the bond pricing equations (19) become

$$Q = \Gamma \widehat{Q}, \quad \text{with } \Gamma = \delta B \operatorname{diag}\left(\frac{1}{1+\pi}\right)$$

$$= \begin{bmatrix} \dfrac{\delta B_{11}}{1 + \pi_1} & \cdots & \dfrac{\delta B_{1S}}{1 + \pi_S} \\ \vdots & \vdots & \vdots \\ \dfrac{\delta B_{S1}}{1 + \pi_1} & \cdots & \dfrac{\delta B_{SS}}{1 + \pi_S} \end{bmatrix}. \tag{23}$$

Corollary 12 (*B Full Rank*). *If the bond prices are given by* (23) *and* rank $[\mathbf{1}, \bar{\mathbf{q}}^1, \ldots, \bar{\mathbf{q}}^{T-1}] = S$, *then the inflation expectations matrix B must be of full rank* (rank $B = S$).

Proof. Since $\bar{\mathbf{q}}^{\tau} = \Gamma \bar{\mathbf{q}}^{\tau-1}$, $\tau = 1, \ldots, S - 1$ the vectors $\bar{\mathbf{q}}^1, \ldots, \bar{\mathbf{q}}^{T-1}$ are in the image of Γ. Since $\Phi_s = 1$ for all $s \in \mathcal{S}$, the matrix $\bar{D}_1$ is the identity matrix and (22) reduces to $\Gamma[\mathbf{1}+\pi] = \delta \mathbf{1}$ so that $\mathbf{1}$ is in the image of Γ. Since the image of Γ contains S independent vectors rank $\Gamma = S$, and since rank $B = $ rank Γ, the result follows. $\quad \square$

The condition that B is of full rank means that the rows of B must be systematically different: this means that the probability distribution $(B_{ss'})_{s' \in \mathcal{S}}$ of next period inflation s' given the current inflation changes systematically when current inflation s changes. No i.i.d. process can satisfy condition (i) of Theorem 11 since for any i.i.d. process of inflation the nominal interest rate is constant—the

real interest rate is constant equal to $\delta - 1$ and the expectation of inflation is independent of current inflation: thus the term structure is flat and identical in all inflation states. No information concerning the inflation process can be obtained from the bond prices beyond the fact that inflation is some i.i.d. process with a mean determined by the short-term interest rate. In particular it is not possible to anchor expectations of an immediate return next period to a target inflation rate π_{s^*} when current inflation deviates from the target ($s \neq s^*$). For the matrix of such an inflation process would be of the form

$$B = \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}$$

with all rows identical. In this case all inflation processes with the same expected next period inflation

$$\sum_{s' \in \mathcal{S}} \frac{B_{ss'}}{1 + \pi_{s'}} = \frac{1}{1 + \pi_{s^*}}, \quad s \in \mathcal{S}$$

give the same bond prices so that the equilibrium is indeterminate. This example suggests that a monetary authority targeting an inflation rate π_{s^*} should choose an inflation process moving more sluggishly to the target with some permanence – the main weights in inflation state s being on $B_{s,s}$ and $B_{s,s+1}$ if $s < s^*$ or on $B_{s,s}$ and $B_{s,s-1}$ if $s > s^*$ – the fluctuations in inflation rates representing the price to pay for being able to anchor agents' random beliefs.

If B and $\widehat{Q} = [\mathbf{1}, \bar{\mathbf{q}}^1, \ldots, \bar{\mathbf{q}}^{T-1}]$ are both of full rank then the matrix of bond prices $Q = [\bar{\mathbf{q}}^1, \ldots, \bar{\mathbf{q}}^T]$ is also of full rank. Since any row of Q

$$[\bar{q}_s^1, \ldots, \bar{q}_s^T] = \left[\frac{1}{1 + r_s^1}, \ldots, \frac{1}{(1 + r_s^T)^T}\right]$$

is the equivalent in terms of bond prices of the term structure of interest rates in inflation state s, in order that B can be anchored, different inflation rates must result in different term structures.

With risk aversion the proof of Corollary 12 does not go through since by (22) $B\mathbf{1} = \mathbf{1}$ implies

$$\Gamma\left[\frac{1 + \pi}{\bar{\Phi}}\right] = \delta\left[\frac{1}{\bar{\Phi}}\right].$$

We did not find an argument proving that $\mathbf{1}$ is in the image of Γ or that the vector $\left[\frac{1}{\bar{\Phi}}\right]$ is linearly independent of $(\bar{\mathbf{q}}^1, \ldots, \bar{\mathbf{q}}^{T-1})$, so that we can only conclude that a necessary condition for condition (i) is that rank $(B) \geq S - 1$. This however is similar in spirit to the earlier full rank condition since it requires that for all but perhaps one row of B the probability distribution of next period inflation vary systematically with current inflation, so that the term structure $r_s^1, \ldots, r_s^{S-1}$ varies systematically with s. If condition (ii) is used rather than condition (i) then the linear independence of $[\mathbf{1}, \bar{\mathbf{q}}^1, \ldots, \bar{\mathbf{q}}^{S-2}]$ implies that rank $\bar{B} \geq S - 2$ so that the term structure $r_s^1, \ldots, r_s^{S-2}$ varies systematically with s.

4. Heterogeneous agents and real shocks

In the previous section we characterized the beliefs that a monetary authority can induce as equilibrium beliefs in an economy with homogeneous agents where the only source of uncertainty comes from the stochastic beliefs of agents about future inflation. We saw that even in an economy with no real uncertainty, the presence of stochastic beliefs regarding inflation can lead to stochastic self-fulfilling equilibria, and that a monetary authority can induce a unique equilibrium of its choice in the class of inflation processes satisfying the conditions of Theorems 8 and 11 by appropriate use of a term-structure policy. In this section

we show how the analysis can be extended to an economy with heterogeneous agents in which there is an additional source of uncertainty coming from the exogenous process of shocks to the agents' productivities. To this end we revert to the general model introduced in Section 2, restricted as in Section 3 to the stationary Markov setting. Thus we let $\eta = (s, g) \in \mathcal{S} \times \mathcal{G}$ identify the current inflation π_s and the real shock g which determines the productivities $a_g^h = a_\eta^h$ of the agents $h \in \mathcal{H}$. The exogenous shocks are assumed to have a Markov structure described by a $G \times G$ matrix $A = (A_{gg'})_{g,g' \in \mathcal{G}}$ and we consider only Markovian beliefs $B = [B_{\eta\eta'}]_{\eta, \eta' \in \mathcal{S} \times \mathcal{G}}$ on the support $\mathcal{S} \times \mathcal{G}$ which are compatible with A.

An equilibrium of an economy with heterogeneous agents and a cash-in-advance constraint is not Pareto optimal: there is thus no social welfare function which is maximized at an equilibrium, and in this sense, no representative agent. However what is really needed to derive the properties of an equilibrium with security markets is a common stochastic discount factor for pricing the securities which only depends on the aggregate state of the economy. In Proposition 13 we show that we can exploit the property that the marginal rates of substitution of the agents are equalized – they all face the same prices and the nominal interest rate distorts the real wage in the same way for all agents – to derive a *social marginal utility of consumption* at equilibrium, denoted by $\left(\varPhi_\eta\right)_{\eta \in \mathcal{S} \times \mathcal{G}}$, which when discounted to date 0, leads to the *real stochastic discount factor* for pricing the securities. Lemma 14 will show that this social marginal utility of consumption is a function of the income distribution, the real shock and the nominal short-term interest rate. Once the function $\varPhi$ is introduced, many of the constructions of the previous section can be extended to the multi-agent case, for $\varPhi_\eta$ plays a role akin to the marginal utility u_c of the representative agent in the previous section. The fixed-point argument however needs to be extended to include not only the vector of returns $\boldsymbol{R}$ on the short-term bond, but also the vector of weights for the agents characterizing the distribution of income in the economy.

The first step of the analysis is given by Proposition 13 which provides the heterogeneous-agent generalization of Corollary 5 of the previous section: it characterizes a reduced-form equilibrium of the economy in which the bond prices $q_\eta = (q_\eta^1, \ldots, q_\eta^T)$ and the consumption–leisure decisions (c_η^h, ℓ_η^h) only depend on the current state η, assuming that the agents adopt B as their beliefs. The maximization of each agent in (2) of Definition 2 is replaced by the corresponding first-order conditions (a1, a2) and the budget equation (a4). The first-order conditions are expressed as the statement that the marginal utility of consumption of each agent is proportional to the social marginal utility of consumption $\varPhi_\eta$, the vector of coefficients of proportionality $\boldsymbol{\nu} = (\nu^h)_{h \in \mathcal{H}}$ in the simplex $\Delta^H \subset \mathbb{R}^H$ capturing the relative wealth of the agents. These weights are determined by the lifetime budget equations of the agents which can be expressed (in (a4)) as functions of the variables $(\varPhi_\eta, c_\eta, \ell_\eta, r_\eta^1)_{\eta \in \mathcal{S} \times \mathcal{G}}$ which are state, and not path, dependent. Let $R_\eta = 1 + r_\eta^1$ denote the gross return on the short-term bond in state η. Then the following equations characterize a stationary reduced-form equilibrium of the heterogeneous-agent economy.

Proposition 13 (*Stationary Equilibrium Equations*). *Under Assumption $\mathcal{U}$, a stationary reduced-form equilibrium is characterized by a pair $\left((\bar{B}, \bar{q}, \bar{\Theta}), (\bar{\nu}, \bar{x}, \bar{\varPhi})\right)$ satisfying the following system of equations:*

(a1) $\bar{\nu}^h u_c^h(\bar{c}_\eta^h, \bar{\ell}_\eta^h) = \bar{\varPhi}_\eta, \; \eta \in \mathcal{S} \times \mathcal{G}, \; h \in \mathcal{H};$

(a2) $\bar{\nu}^h u_\ell^h(\bar{c}_\eta^h, \bar{\ell}_\eta^h) = \frac{a_\eta^h \bar{\varPhi}_\eta}{\bar{R}_\eta}, \; \eta \in \mathcal{S} \times \mathcal{G}, \; h \in \mathcal{H};$

(a3) $\sum_{h \in \mathcal{H}} \bar{c}_\eta^h = \sum_{h \in \mathcal{H}} a_\eta^h(e^h - \bar{\ell}_\eta^h), \; \eta \in \mathcal{S} \times \mathcal{G};$

(a4) $\sum_{\eta \in \mathcal{S} \times \mathcal{G}} [I - \delta\bar{B}]_{\eta_0\eta}^{-1} \bar{\varPhi}_\eta \left(\bar{c}_\eta^h - \frac{a_\eta^h(e^h - \bar{\ell}_\eta^h)}{\bar{R}_\eta}\right) + \bar{\varPhi}_{\eta_0} \gamma^h \bar{\Theta} = \bar{\varPhi}_{\eta_0} w_0^h, \; h \in \mathcal{H};$

(b1) $\bar{\varPhi}_\eta \bar{q}_\eta^\tau = \delta \sum_{\eta' \in \mathcal{S} \times \mathcal{G}} \frac{\bar{B}_{\eta\eta'}}{1 + \pi_{\eta'}} \bar{\varPhi}_{\eta'} \bar{q}_{\eta'}^{\tau-1}, \; \bar{q}_\eta^0 = 1, \; \eta \in \mathcal{S} \times \mathcal{G}, \; \tau = 1, \ldots, T;$

(b2) $\bar{q}_\eta^\tau \le 1, \; \tau = 1, \ldots, T, \; \eta \in \mathcal{S} \times \mathcal{G}.$

Proof. Consider a reduced-form equilibrium $\left((\bar{B}, (\bar{q}^j)_{j \in \mathcal{J}_g}, \bar{\Theta}), \bar{x}, \bar{P}\right)$ as in Definition 2. First note that under the Markov assumption $\bar{B}_\xi$ is given by

$$\bar{B}_\xi = \bar{B}_{\eta_0\eta_1}, \ldots, \bar{B}_{\eta_{t-1}\eta_t} \quad \text{if } \xi = (\eta_0, \ldots, \eta_t).$$

Assuming that the present value of after-tax income $\bar{P}_{\xi_0}(w_0^h - \gamma^h\bar{\Theta}) + \sum_\xi \bar{P}_\xi \bar{p}_\xi a_\xi^h e^h / \bar{R}_\xi$ is positive, in view of Assumption $\mathcal{U}$, the solution to the maximum problem of agent h is interior, i.e. $c_\xi^h > 0$ and $0 < \ell_\xi^h < e^h$ for all $\xi \in \mathbb{D}$, and is characterized by the FOCs for optimal consumption/leisure and the present-value budget equation. The FOCs for the maximum problem of agent h are

$$\delta^{t(\xi)} \bar{B}_\xi u_c^h(\bar{c}_\xi^h, \bar{\ell}_\xi^h) = \bar{\lambda}^h \bar{P}_\xi \bar{p}_\xi$$

$$\delta^{t(\xi)} \bar{B}_\xi u_\ell^h(\bar{c}_\xi^h, \bar{\ell}_\xi^h) = \bar{\lambda}^h a_\xi^h \frac{\bar{P}_\xi \bar{p}_\xi}{\bar{R}_\xi} \qquad (24)$$

for some $\bar{\lambda}^h > 0$ and all ξ such that $\bar{B}_\xi > 0$. If the equilibrium is Markov, then $\bar{c}_\xi^h = \bar{c}_\eta^h, \bar{\ell}_\xi^h = \bar{\ell}_\eta^h, \bar{R}_\xi = \bar{R}_\eta$ for all ξ such that $\xi = (\eta_0, \ldots, \eta)$. Thus for each η there exists $\bar{\varPhi}_\eta$ such that

$$\frac{\bar{P}_\xi \bar{p}_\xi}{\delta^{t(\xi)} \bar{B}_\xi} = \bar{\varPhi}_\eta, \quad \text{for all } \xi = (\eta_0, \ldots, \eta_{t(\xi)})$$

$$\text{with } \eta_{t(\xi)} = \eta. \qquad (25)$$

Define the weight $\bar{\nu}^h$ of agent h as the reciprocal of the marginal utility of income: since $\bar{P}$ in Definition 2 can be normalized, we may assume that the weights are chosen in the simplex ($\sum_{h \in \mathcal{H}} \nu^h = 1$), which amounts to having $\bar{P}_{\xi_0} \sum_h \frac{1}{u_c^h(\bar{c}_{\xi_0}^h, \bar{\ell}_{\xi_0}^h)} = 1$. Expressing the FOCs (24) using (25) leads to the first-order conditions (a1) and (a2).

Using (25) the FOCs (4) for the bond portfolios in Definition 2 can be written as

$$\frac{\delta^{t(\xi)} \bar{B}_\xi}{\bar{p}_\xi} \bar{\varPhi}_\eta \bar{q}_\eta^\tau = \sum_{\eta' \in \mathcal{S} \times \mathcal{G}} \frac{\delta^{t(\xi)+1} \bar{B}_{\xi'}}{\bar{p}_{\xi'}} \bar{\varPhi}_{\eta'} \bar{q}_{\eta'}^\tau,$$

$$\text{for } \xi = (\eta_0, \ldots, \eta), \; \xi' = (\eta_0, \ldots, \eta, \eta').$$

Since $\bar{B}_{\xi'}/\bar{B}_\xi = \bar{B}_{\eta\eta'}$ and $\bar{p}_{\xi'}/\bar{p}_\xi = 1 + \pi_{\eta'}$, in the Markov case the FOCs for optimal bond portfolios reduce to (b1) in Proposition 13.

Finally, using (25) and grouping the terms in the budget equation of agent h which correspond to the same current inflation–productivity state η at the same date, the budget equation of agent h can be written as

$$\sum_{t=0}^\infty \delta^t \sum_{\eta \in \mathcal{S} \times \mathcal{G}} \left(\sum_{\xi \in \mathbb{D}_t} \sum_{\xi = (\eta_0, \ldots, \eta)} \bar{B}_{\eta_0\eta_1}, \ldots, \bar{B}_{\eta_{t-1}\eta}\right)$$

$$\times \bar{\varPhi}_\eta \left(c_\eta^h - \frac{a_\eta^h(e^h - \ell_\eta^h)}{R_\eta}\right) - \gamma^h \bar{\varPhi}_{\eta_0} \Theta = \bar{\varPhi}_{\eta_0} w_0^h.$$

The probability in parentheses is the probability of being in state η in t periods starting from η_0 and by the Markov property is equal to $[\bar{B}^t]_{\eta_0\eta}$, where $[\bar{B}^t]$ is the tth power of the matrix $\bar{B}$. Since $\delta < 1$, $\sum_{t=0}^\infty \delta^t [\bar{B}^t] = [I - \delta\bar{B}]^{-1}$, leading to the budget equation (a4). $\quad \square$

Comparing the characterization of a stationary equilibrium for the multi-agent economy in Proposition 13 with that of the representative-agent economy in Corollary 5, (a1)–(a3) replace condition (a) in Corollary 5. (a4) is the lifetime budget constraint of each agent written in present-value form. Summing these budget constraints gives an equation for determining the value of $\bar{\Theta}$. When there is only one agent the budget equation can be omitted if $\bar{\Theta}$ is omitted from the equilibrium variables. When there are heterogeneous agents both the lifetime budget constraints and $\bar{\Theta}$ have to be retained in the definition of an equilibrium. (b1, b2) of Proposition 13 are equivalent to (b1, b2) of Corollary 5, the marginal utility of the representative agent being replaced by the social marginal utility Φ.

Existence of stationary equilibrium. The next step is to characterize the expectations matrices B for which there exist a stationary reduced-form equilibrium: as in the previous section (Definition 7) we call any such matrix a *compatible* expectations matrix, where 'compatible' means compatible with a non-negative interest rate. To generalize Theorem 8 and obtain conditions for a matrix B to be a compatible expectations matrix we need the equivalent of Lemma 6 for the multi-agent case. We show that under Assumption $\mathcal{U}$, for a fixed vector of positive weights $\mathbf{v}$ and a fixed vector of productivities $\mathbf{a} = (a^h)_{h \in \mathcal{H}}$, the first-order conditions (a1)–(a2) and the market-clearing equations (a3) uniquely define the consumption and leisure of each agent and the social marginal utility of consumption.

Lemma 14 (**$\mathbf{v}$-equilibrium Consumption**)**.** *Let u^h satisfy Assumption $\mathcal{U}$ for all $h \in \mathcal{H}$.*

(i) *For any $(\mathbf{v}, \mathbf{a}, R) \in \Delta^H \times \mathbb{R}^H_{++} \times \mathbb{R}_{++}$ the equations*
   (a1) $v^h u^h_c(c^h, \ell^h) = \Phi$ *if $v^h > 0$, $c^h = 0$, if $v^h = 0$, $h \in \mathcal{H}$;*
   (a2) $v^h u^h_\ell(c^h, \ell^h) = \frac{a^h \Phi}{R}$ *if $v^h > 0$, $\ell^h = 0$, if $v^h = 0, h \in \mathcal{H}$;*
   (a3) $\sum_{h \in \mathcal{H}} c^h = \sum_{h \in \mathcal{H}} a^h(e^h - \ell^h)$
   have a unique solution $\left(c^h(\mathbf{v}, \mathbf{a}, R), \ell^h(\mathbf{v}, \mathbf{a}, R), \Phi(\mathbf{v}, \mathbf{a}, R) \right)$ continuous on $\Delta^H \times \mathbb{R}^H_{++} \times \mathbb{R}_{++}$.
(ii) $\Phi(\mathbf{v}, \mathbf{a}, R)$ *is strictly increasing in R.*
(iii) $\widetilde{\Phi}(\mathbf{v}, \mathbf{a}, R) \equiv \frac{\Phi(\mathbf{v}, \mathbf{a}, R)}{R}$ *is strictly decreasing in R.*

Proof. See the Appendix. ∎

Lemma 14 is the multi-agent analogue of Lemma 6 in the previous section: it permits equations (a1)–(a3), and (b1) of Proposition 13 for $\tau = 1$ to be combined into the system of equations

$$\frac{1}{R_\eta} = \delta \sum_{\eta' \in \mathscr{S} \times \mathscr{G}} \frac{B_{\eta\eta'}}{1 + \pi_{\eta'}} \frac{\Phi(\bar{\mathbf{v}}, \mathbf{a}_{\eta'}, R_{\eta'})}{\Phi(\bar{\mathbf{v}}, \mathbf{a}_\eta, R_\eta)}, \quad \eta \in \mathscr{S} \times \mathscr{G} \qquad (26)$$

which has the same form as the equilibrium equations (12) for the single-agent economy. Finding a reduced-form equilibrium for the single-agent economy reduced to finding a solution $\bar{R} \geq \mathbf{1}$ to Eqs. (12). For an economy with heterogeneous agents, in addition to solving the system of equations (26) we must find the present value of taxes which when combined with seignorage pays off the government's initial liabilities, and the vector $\bar{\mathbf{v}}$ of relative weights which are compatible with the distribution of wealth implied by the budget equations (a4) in Proposition 13. Thus for a heterogeneous-agent economy finding an equilibrium reduces to finding $(\bar{\Theta}, \bar{\mathbf{v}}, \bar{R})$ such that the budget equations (a4) and the bond pricing equations (26) are satisfied.

The conditions which imply the existence of a reduced-form equilibrium thus naturally reduce to two sets of conditions: the first set is analogous to conditions (1) and (2) in Theorem 8 which ensure that there is a solution $\bar{R}$ to the short-term bond-pricing equations (26) satisfying $\bar{R} \geq \mathbf{1}$; the second ensures that the *tax burden* is shared among the agents in a way which is

commensurate with their wealth, so that each agent can afford positive consumption and leisure in all states $\eta \in \mathscr{S} \times \mathscr{G}$ after paying his/her share of the present value of the taxes $\bar{\Theta}$.

To give conditions which ensure (26) has a solution in the right domain, we need to bound the possible values of R. For each $\mathbf{v} \in \Delta^H$, define $R^{\max}_\eta(\mathbf{v})$ as the solution of the equation

$$\widetilde{\Phi}(\mathbf{v}, \mathbf{a}_\eta, R^{\max}_\eta) = \frac{\Phi(\mathbf{v}, \mathbf{a}_\eta, R^{\max}_\eta)}{R^{\max}_\eta}$$

$$= \delta \sum_{\eta' \in \mathscr{S} \times \mathscr{G}} B_{\eta\eta'} \frac{\Phi(\mathbf{v}, \mathbf{a}_{\eta'}, 1)}{1 + \pi_{\eta'}}, \quad \eta \in \mathscr{S} \times \mathscr{G} \qquad (27)$$

(condition (1) in Theorem 14 below ensures that the equation has a solution) and then for each $\eta \in \mathscr{S} \times \mathscr{G}$ define

$$R^{\max}_\eta = \max_{\mathbf{v} \in \Delta^H} R^{\max}_\eta(\mathbf{v}).$$

As before, conditions which ensure $\mathbf{R}^{\max} \geq \mathbf{1}$ place restrictions on the matrix B given the inflation/technology/preference characteristics of the economy. We then define, for each $\mathbf{v} \in \Delta^H$, $R^{\min}_\eta(\mathbf{v})$ by

$$\frac{1}{R^{\min}_\eta(\mathbf{v})} = \delta \sum_{\eta' \in \mathscr{S} \times \mathscr{G}} \frac{B_{\eta\eta'}}{1 + \pi_{\eta'}} \frac{\Phi(\mathbf{v}, \mathbf{a}_{\eta'}, R^{\max}_{\eta'})}{\Phi(\mathbf{v}, \mathbf{a}_\eta, 1)}, \quad \eta \in \mathscr{S} \times \mathscr{G}, \qquad (28)$$

$R^{\min}_\eta(\mathbf{v})$ gives a lower bound on the nominal interest rate since it corresponds to the lowest possible real interest rate, and hence the assumption that $R^{\min}_\eta(\mathbf{v}) \geq 1$ (condition 2 below) imposes stronger restrictions on B.

To understand the tax-sharing assumption consider each agent's present-value budget equation in the original form given in (2) of Definition 2. Summing the budget equations of the households implies that when the market clearing equations (3) in Definition 2 hold, then

$$\sum_{\xi \in \mathbb{D}} \bar{P}_\xi \frac{\bar{r}^1_\xi M_\xi}{1 + \bar{r}^1_\xi} + \bar{P}_{\xi_0} \bar{\Theta} = \bar{P}_{\xi_0} \sum_{h \in \mathcal{H}} w^h_0 = \bar{P}_{\xi_0} W_0 \qquad (29)$$

where $M_\xi = \bar{p}_\xi \sum_{h \in \mathcal{H}} \bar{c}^h_\xi$. (29) expresses the property that the government *asymptotically withdraws its initial liabilities* $\bar{P}_{\xi_0} W_0$ (which correspond to the initial wealth of the private sector) by a combination of *seignorage* (the first term on the left side) and *direct taxes* ($\bar{P}_{\xi_0} \bar{\Theta}$). Since $\bar{r}^1_\xi \geq 0$, it follows from (29) that

$$\bar{\Theta} \leq W_0. \qquad (30)$$

We want to be sure that each agent h has a positive after-tax present value of income

$$\gamma^h \bar{P}_{\xi_0} \bar{\Theta} < \bar{P}_{\xi_0} w^h_0 + \sum_{\xi \in \mathbb{D}} \frac{\bar{P}_\xi \bar{p}_\xi a^h_\xi e^h}{\bar{R}_\xi}, \quad h \in \mathcal{H}$$

and, to ensure that it holds in equilibrium, we require that it holds for the 'lowest' possible values of $\frac{\bar{P}_\xi \bar{p}_\xi}{\bar{R}_\xi}$ and the highest possible $\bar{P}_{\xi_0}$. In the stationary case this can be expressed using the highest returns $R^{\max}_\eta$ and leads to condition (3) in the following theorem.

Theorem 15 (*Existence of Stationary Equilibrium*)**.** *Let $\mathcal{E}$ be an economy in which the agents' utility functions satisfy Assumption $\mathcal{U}$. If B is a Markov matrix such that*

(1) $\lim_{R \to \infty} \frac{\Phi(\mathbf{v}, \mathbf{a}_\eta, R)}{R} < \delta \sum_{\eta' \in \mathscr{S} \times \mathscr{G}} B_{\eta\eta'} \frac{\Phi(\mathbf{v}, \mathbf{a}_{\eta'}, 1)}{1 + \pi_{\eta'}} \leq \Phi(\mathbf{v}, \mathbf{a}_\eta, 1)$, $\forall \eta \in \mathscr{S} \times \mathscr{G}, \ \forall \mathbf{v} \in \Delta^H$
(2) $R^{\min}_\eta(\mathbf{v}) \geq 1, \ \forall \eta \in \mathscr{S} \times \mathscr{G}, \ \forall \mathbf{v} \in \Delta^H$

and if the tax burden is distributed among agents so that $\gamma = (\gamma^h)_{h \in \mathcal{H}} \in \Delta^H$ *satisfies*

(3) $\gamma^h W_0 < w_0^h + \sum_{\eta \in \mathcal{S} \times \mathcal{G}} [I - \delta B]_{\eta_0 \eta}^{-1} \frac{\Phi(\nu, a_\eta, R_\eta^{\max}) \, a_\eta^h e^h}{R_\eta^{\max} \, \Phi(\nu, a_{\eta_0}, R_{\eta_0}^{\max})}, \ \forall \, \nu \in \Delta^H, \ \forall \, h \in \mathcal{H}$

where $R_\eta^{\min}$ *and* $R_\eta^{\max}$ *are defined by* (28) *and* (27) *, then there exists a stationary equilibrium of* $\mathcal{E}$*, and B is compatible with non-negative interest rates.*

Proof. See the Appendix.

Conditions (1) and (2) of Theorem 15 ensure that the short-run nominal interest rate is always non-negative. These conditions are of the same nature as those in Theorem 8. However fluctuations between high and low productivity of labor tend to augment the fluctuations between high and low consumption (already present in Section 3) leading to more negative real interest rates. Other things equal in a setting of inflation targeting this would imply a need for a higher target inflation rate to obtain the existence of an equilibrium.

Uniqueness of equilibrium. If B is an expectations matrix satisfying the conditions of Theorem 15, then the monetary authority can find a short-term interest rate policy r^1 which is compatible with B and the real side of the economy. To examine the additional conditions required to ensure that B is the only matrix which is solution of the bond-pricing equations (b1) of Proposition 13, let

$$\bar{\nu} = (\nu^h(B))_{h \in \mathcal{H}}, \qquad \bar{R} = (\bar{R}_\eta)_{\eta \in \mathcal{S} \times \mathcal{G}} = (R_\eta(B))_{\eta \in \mathcal{S} \times \mathcal{G}}$$

denote the vector of relative weights of the agents and the vector of equilibrium returns on the short-term bond across the states $\eta \in \mathcal{S} \times \mathcal{G}$. By Proposition 13(a1, a2) there is at the equilibrium a vector

$$\bar{\Phi} = (\bar{\Phi}_\eta)_{\eta \in \mathcal{S} \times \mathcal{G}} = (\Phi_\eta(\bar{\nu}, a_\eta, \bar{R}_\eta))_{\eta \in \mathcal{S} \times \mathcal{G}}$$

to which all agents' gradients are collinear and which thus represents the vector of social marginal utilities of income across the states. $\bar{\Phi}$ is the generalization of the vector of marginal utilities of income of the representative agent (17) in Section 3. Clearly if the monetary authority only determines the equilibrium prices of the short-term bond then there are many expectations matrices $\widetilde{B} = \left(\widetilde{B}_{\eta \eta'} \right)_{\eta \eta' \in \mathcal{S} \times \mathcal{G}}$ which satisfy the short-term bond pricing equations

$$\frac{1}{1 + \bar{r}^1} = \delta \sum_{\eta' \in \mathcal{S} \times \mathcal{G}} \frac{\widetilde{B}_{\eta \eta'}}{1 + \pi_{\eta'}} \frac{\bar{\Phi}_{\eta'}}{\bar{\Phi}_\eta}, \quad \eta \in \mathcal{S} \times \mathcal{G} \tag{31}$$

and are compatible with the exogenous transitions $A = (A_{gg'})_{gg' \in \mathcal{G}}$ for the real shocks. If the monetary authority determines the prices of additional longer-term bonds of maturities $\tau = 2, \dots, T$, then a matrix of expectations $\widetilde{B}$ will need to satisfy the additional bond-pricing equations

$$\bar{\Phi}_\eta \bar{q}_\eta^\tau = \delta \sum_{\eta' \in \mathcal{S} \times \mathcal{G}} \frac{\widetilde{B}_{\eta \eta'}}{1 + \pi_{\eta'}} \bar{\Phi}_{\eta'} \bar{q}_{\eta'}^{\tau - 1},$$

$$\eta \in \mathcal{S} \times \mathcal{G}, \ \tau = 2, \dots, T. \tag{32}$$

A naive attempt to apply the reasoning underlying Theorem 11 would suggest the following criterion for uniqueness: if the monetary authority determines the prices of $S \times G$ bonds and the payoff matrix is invertible, then $\widetilde{B} = B$ is the unique solution of Eqs. (31) and (32). This however is not a satisfactory criterion: for a standard model such as the Smets and Wouters (2007) model, which is a workhorse for policy evaluation, uses between 10 and 20 exogenous shocks. If we use a grid of 10 inflation intervals, this would imply determining the prices of 100–200 bonds. It is hard to

imagine a central bank even entertaining the idea of implementing such a policy. A more careful analysis reveals however that a policy restricted to determining the prices of only S or $S - 1$ bonds suffices to anchor agents' expectations under a mild condition on the Markov transition matrix, which formalizes the idea that the technology shock process is exogenous to the model.

Assumption MT (*Markov Transitions*). There exists a family $(N^g)_{g \in \mathcal{G}}$ of $S \times S$ Markov matrices such that the transition matrix B is given by

$$B_{\eta \eta'} = N_{ss'}^g A_{gg'}, \quad \forall \, \eta = (s, g), \ \eta' = (s', g') \in \mathcal{S} \times \mathcal{G}. \tag{33}$$

MT asserts that the transition probabilities for the real shocks are not influenced by either the current or the future inflation rates (s, s'). However agents' expectations of future inflation s' can be influenced not only by the current inflation s but also by the real shock g. Under this assumption the bond-pricing equations can be written as

$$\bar{q}_{sg}^\tau = \delta \sum_{s' \in \mathcal{S}} \frac{N_{ss'}^g}{1 + \pi_{s'}} \sum_{g' \in \mathcal{G}} A_{gg'} \frac{\bar{\Phi}_{s'g'}}{\bar{\Phi}_{sg}} \bar{q}_{s'g'}^{\tau - 1},$$

$$\bar{q}_{sg}^0 = 1, \quad \forall \, (s, g) \in \mathcal{S} \times \mathcal{G}, \ \tau = 1, \dots, T. \tag{34}$$

Define the risk-neutral probability $\rho_{s'}^g$ of the real shocks tomorrow conditional on the current real shock being g and the inflation tomorrow being s'

$$\rho_{s'}^g(g') = \frac{A_{gg'} \bar{\Phi}_{s'g'}}{\sum_{g'' \in \mathcal{G}} A_{gg''} \bar{\Phi}_{s'g''}}, \quad \forall \, g' \in \mathcal{G}, \ \forall \, s' \in \mathcal{S}, \ \forall \, g \in \mathcal{G}$$

and the associated expected values of the bond prices in inflation state s'

$$\widehat{E}^g(\bar{q}_{s'}^\tau) = \sum_{g' \in \mathcal{G}} \rho_{s'}^g(g') \bar{q}_{s'g'}^\tau, \quad \forall \, s' \in \mathcal{S}, \ \forall \, g \in \mathcal{G}, \ \tau = 1, \dots, T.$$

If we also consider the average marginal utility of income tomorrow if the inflation rate is s'

$$\widehat{\Phi}_{s'}^g = \sum_{g' \in \mathcal{G}} A_{gg'} \bar{\Phi}_{s'g'}, \quad \forall \, s' \in \mathcal{S}, \ \forall \, g \in \mathcal{G}$$

when the real shock is g today, then the bond-pricing equations (34) decompose into a system of $S \times T$ equations for each current real shock $g \in \mathcal{G}$

$$\bar{q}_{sg}^\tau = \delta \sum_{s' \in \mathcal{S}} \frac{N_{ss'}^g}{1 + \pi_{s'}} \frac{\widehat{\Phi}_{s'}^g}{\bar{\Phi}_{sg}} \widehat{E}^g(\bar{q}_{s'}^{\tau - 1}), \qquad \widehat{E}^g(\bar{q}_{s'}^0) = 1,$$

$$\forall \, s \in \mathcal{S}, \ \tau = 1, \dots, S. \tag{35}$$

For each $g \in \mathcal{G}$ this system of equations is similar to the recursive equations (18) in the previous section, the marginal utility of income and the payoff of the bond next period in state s' being replaced by averages over the possible shocks g'. If for each $g \in \mathcal{G}$ we define the analogue of the diagonal matrices in (21), and the present-value matrix in (20)

$$\bar{D}_1^g = \text{diag}\left(\frac{1}{\bar{\Phi}_{sg}} \right)_{s \in \mathcal{S}} \qquad \widehat{D}_2^g = \text{diag}\left(\frac{\delta \widehat{\Phi}_{s'}^g}{1 + \pi_{s'}} \right)_{s' \in \mathcal{S}},$$

$$\Gamma^g = \bar{D}_1^g N^g \widehat{D}_2^g$$

and define the $S \times S$ matrices of bond prices and their average payoffs

$$Q^g = \begin{bmatrix} \bar{q}_{1g}^1 & \cdots & \bar{q}_{1g}^T \\ \vdots & \ddots & \vdots \\ \bar{q}_{Sg}^1 & \cdots & \bar{q}_{Sg}^T \end{bmatrix}$$

$$\widehat{Q}^g = \begin{bmatrix} 1 & \widehat{E}^g(\bar{q}_1^1) & \cdots & \widehat{E}^g(\bar{q}_1^{T-1}) \\ \vdots & \vdots & & \vdots \\ 1 & \widehat{E}^g(\bar{q}_S^1) & \cdots & \widehat{E}^g(\bar{q}_S^{T-1}) \end{bmatrix}$$

then (35) is equivalent to the G matrix equations

$$\Gamma^g \widehat{Q}^g = Q^g, \quad g \in \mathcal{g}. \tag{36}$$

Since $\bar{D}_1^g$ and $\bar{D}_2^g$ are positive diagonal matrices there is a one-to-one relation between Γ^g and N^g. Thus if $\widehat{Q}^g$ is invertible then (36) has a unique solution Γ^g and hence a unique solution N^g for each $g \in \mathcal{g}$. If the relation $N^g \mathbf{1} = \mathbf{1}$ for $g \in \mathcal{g}$ is taken into account then Γ^g satisfies the additional condition

$$\Gamma^g [\widehat{D}_2^g]^{-1} \mathbf{1} = \bar{D}_1^g \mathbf{1} \iff \Gamma^g \left[\frac{\mathbf{1}+\boldsymbol{\pi}}{\delta \widehat{\boldsymbol{\Phi}}^g}\right] = \left[\frac{\mathbf{1}}{\bar{\boldsymbol{\Phi}}^g}\right], \quad g \in \mathcal{g}$$

so that Γ^g can also be determined by the matrix equation

$$\Gamma^g \widetilde{Q}^g = \left[\frac{\delta}{\bar{\boldsymbol{\Phi}}^g}, Q^g\right], \quad \text{with } \widetilde{Q}^g = \left[\frac{\mathbf{1}+\boldsymbol{\pi}}{\widehat{\boldsymbol{\Phi}}^g}, \widehat{Q}^g\right].$$

This leads to the following generalization of Theorem 11.

Theorem 16 (*Uniqueness of Equilibrium*)*. Let* $(\bar{B}, \bar{\boldsymbol{q}}, \bar{\Theta})$, $(\bar{\boldsymbol{v}}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{\Phi}})$ *be a stationary reduced form equilibrium.* $\bar{B}$ *is the only matrix satisfying MT which is compatible with the term-structure policy* $\bar{\boldsymbol{q}}$ *involving bonds of maturities* $1, \ldots, T$

(i) *if* $T \geq S$ *and* $rank[\widehat{Q}^g] = rank\left[\mathbf{1}, \widehat{E}^g(\boldsymbol{q}^1), \ldots, \widehat{E}^g(\boldsymbol{q}^{S-1})\right] = S$, *for all* $g \in \mathcal{g}$ *or*

(ii) *if* $T \geq S - 1$ *and* $rank[\widetilde{Q}^g] = rank\left[\frac{\mathbf{1}+\boldsymbol{\pi}}{\widehat{\boldsymbol{\Phi}}^g}, \mathbf{1}, \widehat{E}^g(\boldsymbol{q}^1), \ldots, \widehat{E}^g(\boldsymbol{q}^{S-2})\right] = S$, *for all* $g \in \mathcal{g}$.

The conditions for uniqueness are similar in spirit to that in the simplified model in Section 3. Condition (i) requires that for a given value g of the real shock the rows of the matrix $\widehat{Q}^g$

$$\left[1 \ \widehat{E}^g\left(\bar{q}_{s'}^1\right) \cdots \widehat{E}^g\left(\bar{q}_{s'}^{T-1}\right)\right], \quad s' \in \mathcal{s}$$

i.e. the term-structure next period (averaged over the real shocks) must vary systematically with the realized inflation s'. This in turn implies by (35) that the probability distribution $[N_{s's''}^g]_{s'' \in \mathcal{s}}$ of the subsequent period inflation must vary systematically with s'. Since the criterion applies separately for each real shock the term structure can be adapted in a flexible way to vary with the real shock $g \in \mathcal{g}$ which influences the evolution of the real interest rate. In particular if the monetary authority chooses to implement a mean-reverting process which returns towards a target inflation rate, then the speed of reversion to the target rate in the matrix $[N_{ss'}^g]$ can vary systematically with the different values of g, but as we have seen earlier, the return must be stochastic and gradual since an immediate return to target in not compatible with the uniqueness of equilibrium,

5. Related literature

The study of the determinacy of equilibrium has a long tradition in general equilibrium. Initially the focus was on conditions needed to obtain the *uniqueness* of equilibrium—but the conditions obtained turned out to be far too restrictive. All this was changed by the fundamental paper of Debreu (1970) which proposed an approach of great generality and power. The idea was to study equilibria as solutions of systems of equations for which the generic dimension of the manifold of solutions are given by the Transversality Theorem: uniqueness is replaced by *local uniqueness* i.e. the condition that the dimension be zero so that equilibria consist of at most a finite number of points. At the time determinacy was understood as the existence of a finite number of equilibrium allocations and relative prices, since price levels are not determined in an abstract GE model.

Under the influence of Savage (1954), Arrow (1953) used the state-of-nature approach for defining a random variable to extend general equilibrium to uncertainty, and made the transition to the concept of a sequential equilibrium by introducing a collection of nominal contracts—an "Arrow" security being a promise to pay one unit of account if state *s* occurs. The price level in each state was indeterminate, but since there were complete markets the indeterminacy was purely nominal and did not affect the real allocation, so it passed unnoticed. Until Cass (1989) revisited the model and had the important insight that if the payoffs of the securities are nominal and the financial markets are incomplete then real equilibrium outcomes are indeterminate: Cass interpreted this striking finding as a critique of the viability of rational expectations equilibrium. The result caused quite a stir and led to the study of the degree of indeterminacy of equilibrium in models with nominal contracts and incomplete markets (Balasko and Cass, 1989; Geanakoplos and Mas-Colell, 1989). Magill and Quinzii (1992) argued that since the indeterminacy in the allocations came from the indeterminacy of the price levels which left the purchasing power of the security payoffs indeterminate, the introduction of an additional monetary component into the standard model to determine the price levels would restore the determinacy of equilibrium: using Clower's (1967) constraint they transformed the indeterminacy result into the result that monetary policy has real effects when markets are incomplete.

This changed the focus towards finding more sophisticated ways of embedding the monetary component into the GEI model. In a series of papers Dubey and Geanakoplos (1992, 2003, 2006) studied a form of cash-in-advance model in which agents borrow money from a bank to finance their transactions, which yields determinate prices and equilibrium allocations when either money supplies or interest rates are fixed. A similar model was explored by Drèze and Polemarchakis (1999, 2001) (DP) with the important difference that the bank is seen as a private bank which redistributes the seignorage revenue as profit to its shareholders, and this reintroduces indeterminacy. In an informal discussion DP (2001) introduced the notion of a "comprehensive" monetary policy, defined as a policy which would determine the price level (at every node of the event-tree) and in their model this required more than fixing the short-term interest rate. They suggested either fixing the short-term interest rate and the quantity of money at each node or the short-term interest rate and the prices of Arrow securities, noting that both solutions would be difficult to implement. Controlling monetary aggregates has been found to be difficult and using a policy based on prices of Arrow securities would require the creation of liquid markets contingent on all the exogenous states which are taken to drive uncertainty in the economy. As DP indicate, such markets do not exist and are almost certain never to exist. Our approach seeks to address these issues: the instruments we propose are the prices of securities which

the monetary and fiscal authorities trade on a regular basis and the policy is contingent on observed inflation rates rather than unspecified states of the economy.

Bloise et al. (2005) show how DP (1999) can be extended to an equilibrium model over an infinite horizon with monetary and fiscal policy, drawing the analysis closer to the macro literature. They show that Woodford's (1994) well-known result on determinacy with a non-Ricardian policy (also known as the fiscal theory of the price level) is fragile even if one accepts the tenuous thought experiment of fixing taxes in real terms for the indefinite future: for example, to obtain indeterminacy it suffices to allow the fiscal policy to raise more revenue than needed to finance the debt and to redistribute the surplus; or, if the government trades other securities than the short-term bond, it suffices to leave the composition of the government portfolio undetermined (see also McMahon et al., 2012). The indeterminacy characteristic of a Ricardian policy was also explored in Nakajima and Polemarchakis (2005) in a representative-agent model drawn from macro, using the tools of the general equilibrium approach to count the dimension of indeterminacy. This paper forms the point of departure of our analysis.

In this class of monetary models with a Ricardian fiscal policy, the indeterminacy does not come from the incompleteness of the security markets, as in the earlier models mentioned above, but comes rather from the lack of instruments for determining price levels. Since models with complete markets are easier to study than those with incomplete markets, the paper of NP and ours are in a setting with complete markets. Where we differ from NP is in the representation of uncertainty. In a related paper (Magill and Quinzii, 2009) we have emphasized the importance for the realism of GE models under the uncertainty of replacing the state-of-nature representation of random variables used in current GE models (since its introduction by Arrow (1953)) by an approach based on probability distributions over realized outcomes which are observable. The idea of adding long-term interest rates as instruments to determine price levels was proposed independently by Adao et al. (2010) in a model similar to NP: however since the interest rates are node dependent they do not constitute an "interest-rate rule" that could be learned by agents in the private sector.

The approach that we adopt in which the term-structure rule is based on realized inflation makes it a natural generalization of the standard short-term interest-rate rule used in macroeconomics to solve the indeterminacy problem—a problem which macroeconomists discovered a long time ago in a macro setting (Sargent and Wallace, 1975). The subsequent macro literature has converged on the idea of using a short-term interest-rate rule which satisfies the Taylor Principle: for this implies that there is only one equilibrium trajectory which is locally stable, and this trajectory is selected as the candidate equilibrium. Whether this way of selecting one out of many candidate equilibria is a legitimate way of obtaining "determinacy" is the subject of debate (see Woodford (2003) and Cochrane (2011)).

Acknowledgments

Appendix

A.1. Proof of Theorem 3

Step 1. Show that the extensive and reduced-form budget sets are the same. The budget set $\mathbb{B}^h(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta})$ of agent h in an extensive-form equilibrium is given by

$$
\mathbb{B}^h(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta})
$$
$$
= \left\{ \boldsymbol{x}^h \in (\ell_+^\infty(\mathbb{D}))^2 \left| \begin{array}{l} \exists \, \boldsymbol{z}^h \in \mathbb{R}^{\mathbb{D}J} \quad \text{such that} \ \forall \, \xi \in \mathbb{D} \\ p_\xi c_\xi^h + \gamma^h \theta_\xi + q_\xi z_\xi^h = p_{\xi-} L_{\xi-}^h + \hat{q}_\xi z_{\xi-}^h \\ \lim_{T \to \infty} \sum_{\xi' \in \mathbb{D}_T(\xi)} P_{\xi'} q_{\xi'} z_{\xi'}^h = 0 \end{array} \right. \right\}
$$

where $\ell_\infty^+(\mathbb{D})$ is the space of non-negative bounded sequences on $\mathbb{D}$, and where $p_{\xi_0^-} L_{\xi_0^-}^h + \hat{q}_{\xi_0} z_{\xi_0^-}^h = w_0^h$ denotes the agent's initial wealth at date 0. While the extensive-form budget set is defined by an infinite sequence of budget constraints (one at each node of the event-tree) the reduced-form budget set of agent h, denoted by $B^h(\boldsymbol{P}, \boldsymbol{p}, \boldsymbol{r}^1, \Theta)$ is defined by a single present-value budget equation

$$
B^h(\boldsymbol{P}, \boldsymbol{p}, \boldsymbol{r}^1, \Theta) = \left\{ \boldsymbol{x}^h \in (\ell_+^\infty(\mathbb{D}))^2 \right.
$$
$$
\left| \sum_{\xi \in \mathbb{D}} P_\xi p_\xi \left(c_\xi^h - \frac{L_\xi^h}{1 + r_\xi^1} \right) + \gamma^h P_{\xi_0} \Theta = P_{\xi_0} w_0^h \right\}
$$

where $\boldsymbol{r}^1 = (r_\xi^1)_{\xi \in \mathbb{D}}$ is the sequence of short-term interest rates over the event-tree $\mathbb{D}$. Let $\ell_1(\mathbb{D}) = \{ \boldsymbol{P} \in \mathbb{R}^{\mathbb{D}} | \sum_{\xi \in \mathbb{D}} |P_\xi| < \infty \}$ denote the space of summable sequences on $\mathbb{D}$. We want to show that if

(i) rank $[\hat{q}_{\xi+}] = SG, \ \forall \, \xi \in \mathbb{D}$ (complete markets),
(ii) $P_\xi q_\xi^j = \sum_{\xi' \in \xi+} P_{\xi'} \hat{q}_{\xi'}^j, j \in \mathcal{J}, \ \forall \, \xi \in \mathbb{D}$ (no-arbitrage security prices),
(iii) $(P_\xi p_\xi)_{\xi \in \mathbb{D}} \in \ell_+^1(\mathbb{D})$ (summable prices),
(iv) $\sum_{\xi' \in \mathbb{D}_T(\xi)} P_{\xi'} \theta_{\xi'} \to 0$ as $T \to \infty$, $\Theta = \sum_{\xi \in \mathbb{D}} P_\xi \theta_\xi$, (summable taxes),

then $\mathbb{B}^h(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}) = B^h(\boldsymbol{P}, \boldsymbol{p}, \boldsymbol{r}^1, \Theta)$ for all $h \in \mathcal{H}$.

($\Longrightarrow$) We show $\mathbb{B}^h(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}) \subset B^h(\boldsymbol{P}, \boldsymbol{p}, \boldsymbol{r}^1, \Theta)$. Pick $\boldsymbol{x}^h \in \mathbb{B}^h$. Multiplying the budget equation at node ξ by P_ξ for all $\xi \in \mathbb{D}$, summing over the event-tree up to date T and using the no-arbitrage condition (ii) gives (remember that $\mathbb{D}^T$ denotes the set of nodes up to date T and $\mathbb{D}_T$ the set of nodes at date T)

$$
\sum_{\xi \in \mathbb{D}^T} P_\xi p_\xi c_\xi^h + \gamma^h \sum_{\xi \in \mathbb{D}^T} P_\xi \theta_\xi + \sum_{\xi \in \mathbb{D}_T} P_\xi q_\xi z_\xi^h
$$
$$
= \sum_{\xi \in \mathbb{D}^{T-1}} \left(\sum_{\xi' \in \xi+} P_{\xi'} \right) p_\xi L_\xi^h + P_{\xi_0} w_0^h.
$$

Since no-arbitrage applied to the short-term bond implies that, for all $\xi \in \mathbb{D}$, $\frac{P_\xi}{1 + r_\xi^1} = \sum_{\xi' \in \xi+} P_{\xi'}$, this equation can be written as

$$
\sum_{\xi \in \mathbb{D}^{T-1}} P_\xi p_\xi \left(c_\xi^h - \frac{L_\xi^h}{1 + r_\xi^1} \right) + \gamma^h \sum_{\xi \in \mathbb{D}^T} P_\xi \theta_\xi + \sum_{\xi \in \mathbb{D}_T} P_\xi p_\xi c_\xi^h
$$
$$
+ \sum_{\xi \in \mathbb{D}_T} P_\xi q_\xi z_\xi^h = P_{\xi_0} w_0^h.
$$

Since $(P_\xi p_\xi)_{\xi \in \mathbb{D}} \in \ell^1(\mathbb{D})$ and $c^h \in \ell_+^\infty(\mathbb{D})$, $\lim_{T \to \infty} \sum_{\xi \in \mathbb{D}_T} P_\xi p_\xi c_\xi^h = 0$, and by the Transversality Condition for $\mathbb{B}^h$, $\lim_{T \to \infty} \sum_{\xi \in \mathbb{D}_T} P_\xi q_\xi z_\xi^h = 0$, so that $\boldsymbol{x} \in B^h(\boldsymbol{P}, \boldsymbol{p}, \boldsymbol{r}^1, \Theta)$.

($\Longleftarrow$) We show $\mathbb{B}^h(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}) \supset B^h(\boldsymbol{P}, \boldsymbol{p}, \boldsymbol{r}^1, \Theta)$. Pick $\boldsymbol{x}^h \in B^h(\boldsymbol{P}, \boldsymbol{p}, \boldsymbol{r}^1, \Theta)$. We need to find portfolios z^h such that the sequential budget constraints are satisfied at each node and the Transversality Condition is satisfied. We define the portfolio z_ξ^h by the requirement that it brings enough wealth to the successors of ξ to finance the excess present value of expenditure over after-tax income on the subtrees originating at each of these nodes. In view of the assumption of complete markets such a portfolio exists and is defined by

$$\left[\sum_{\xi'' \in \mathbb{D}(\xi')} P_{\xi''} p_{\xi''} \left(c_{\xi''}^h - \frac{L_{\xi''}^h}{1 + r_{\xi''}^1} \right) \right.$$
$$\left. + \gamma^h \sum_{\xi'' \in \mathbb{D}(\xi')} P_{\xi''} \theta_{\xi''} - P_{\xi'} p_\xi L_\xi^h \right]_{\xi' \in \xi^+} = \left[P \circ_{\xi^+} \hat{q} \right] z_\xi^h,$$
$$\forall \, \xi \in \mathbb{D} \tag{37}$$

where $\left[P \circ_{\xi^+} \hat{q} \right] = \left[P_{\xi'} \hat{q}_{\xi'}^j \right]_{\substack{j \in \mathcal{J} \\ \xi' \in \xi^+}}$ is the $SG \times J$ matrix of present values of the payoffs of the J securities at the immediate successors ξ^+ of ξ. Let us show that with this choice of portfolio the sequential budget constraint is satisfied at each node. We begin with the initial node ξ_0. Premultiplying the SG equations (37) by $\mathbf{1}^\mathsf{T} \in \mathbb{R}^{SG}$ gives

$$\sum_{\xi \in \mathbb{D} \setminus \xi_0} \left(P_\xi p_\xi \left(c_\xi^h - \frac{L_\xi^h}{1 + r_\xi^1} \right) + \gamma^h P_\xi \theta_\xi \right) - \left(\sum_{\xi' \in \xi_0^+} P_{\xi'} \right)$$
$$\times p_{\xi_0} L_{\xi_0}^h = P_{\xi_0} q_{\xi_0} z_{\xi_0}^h \tag{38}$$

and since $\boldsymbol{x}^h \in B^h(\boldsymbol{P}, \boldsymbol{p}, \boldsymbol{r}^1, \Theta)$ and $\sum_{\xi' \in \xi_0^+} P_{\xi'} = \frac{P_{\xi_0}}{1 + r_{\xi_0}^1}$ it follows that

$$-P_{\xi_0} p_{\xi_0} \left(c_{\xi_0}^h - \frac{L_{\xi_0}^h}{1 + r_{\xi_0}^1} \right) - \gamma^h P_{\xi_0} \theta_{\xi_0} + P_{\xi_0} w_0^h - \frac{P_{\xi_0}}{1 + r_{\xi_0}^1} p_{\xi_0} L_{\xi_0}^h$$
$$= P_{\xi_0} q_{\xi_0} z_{\xi_0}^h$$

namely
$$p_{\xi_0} c_{\xi_0}^h + \gamma^h \theta_{\xi_0} + q_{\xi_0} z_{\xi_0}^h = w_0^h.$$

In the same way, for any node $\tilde{\xi}$ with $t(\tilde{\xi}) \geq 1$ premultiplying (37) by $\mathbf{1}^\mathsf{T}$ and using (ii) gives the analogue of (38) with $\mathbb{D}$ replaced by $D(\tilde{\xi})$ and ξ_0 replaced by $\tilde{\xi}$. Using (37) again to express $P_{\tilde{\xi}} \hat{q}_{\tilde{\xi}} z_{\tilde{\xi}^-}^h$ and substituting leads to

$$P_{\tilde{\xi}} \hat{q}_{\tilde{\xi}} z_{\tilde{\xi}^-}^h - P_{\tilde{\xi}} p_{\tilde{\xi}} \left(c_{\tilde{\xi}}^h - \frac{L_{\tilde{\xi}}^h}{1 + r_{\tilde{\xi}}^1} \right) - \gamma^h P_{\tilde{\xi}} \theta_{\tilde{\xi}} + P_{\tilde{\xi}} p_{\tilde{\xi}} L_{\tilde{\xi}^-}^h$$
$$- \frac{P_{\tilde{\xi}}}{1 + r_{\tilde{\xi}}^1} p_{\tilde{\xi}} L_{\tilde{\xi}}^h = P_{\tilde{\xi}} q_{\tilde{\xi}} z_{\tilde{\xi}}^h$$

so that the budget constraint at node $\tilde{\xi}$
$$p_{\tilde{\xi}} c_{\tilde{\xi}}^h + \gamma^h \theta_{\tilde{\xi}} + q_{\tilde{\xi}} z_{\tilde{\xi}}^h = p_{\tilde{\xi}^-} L_{\tilde{\xi}^-}^h + \hat{q}_{\tilde{\xi}} z_{\tilde{\xi}^-}^h$$

is satisfied. It remains to show that the Transversality Condition is satisfied. To this end consider any node $\xi \in \mathbb{D}$ and for any date $T \geq t(\xi)$, consider the nodes of $\mathbb{D}_T(\xi)$ at date T in the subtree $\mathbb{D}(\xi)$. Using (37) and summing over the nodes of $\mathbb{D}_T(\xi)$ gives

$$\sum_{\xi' \in \mathbb{D}_T(\xi)} P_{\xi'} q_{\xi'} z_{\xi'}^h$$
$$= \sum_{\substack{\xi'' \in \mathbb{D}(\xi) \\ t(\xi'') \geq T+1}} \left(P_{\xi''} p_{\xi''} \left(c_{\xi''}^h - \frac{L_{\xi''}^h}{1 + r_{\xi''}^1} \right) + \gamma^h P_{\xi''} \theta_{\xi''} \right)$$
$$- \sum_{\xi' \in \mathbb{D}_T(\xi)} P_{\xi'} p_{\xi'-} L_{\xi'-}^h. \tag{39}$$

By condition (iii) and (iv) all the tails of the series on the right side of (39) converge to zero, and thus $\sum_{\xi' \in \mathbb{D}_T(\xi)} P_{\xi'} p_{\xi'} z_{\xi'}^h \to 0$ when $T \to \infty$, so that $\boldsymbol{x}^h \in \mathbb{B}^h(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta})$.

Step 2. We show that, given price processes $(\boldsymbol{P}, \boldsymbol{p}, \boldsymbol{q})$ satisfying conditions (i)–(iii) of Step 1 and an aggregate consumption process $\boldsymbol{C} = \sum_{h \in \mathcal{H}} \boldsymbol{c}^h \in \ell^\infty(\mathbb{D})$, the government policy $(\boldsymbol{M}, \boldsymbol{\theta}, \boldsymbol{Z})$ is determined by the feasibility conditions and the fiscal rule, and it satisfies $\sum_{\xi' \in \mathbb{D}_T(\xi)} P_{\xi'} W_{\xi'} \to 0$ and $\sum_{\xi' \in \mathbb{D}_T(\xi)} P_{\xi'} \theta_{\xi'} \to 0$ as $T \to \infty$.

Actually without a further assumption there is an indeterminacy in the government's portfolio since by the fiscal rule (7) the taxes θ_ξ at node ξ adjust to accommodate the portfolio Z_{ξ^-} inherited from the previous node. We thus introduce an additional process of parameters $\beta = (\beta_\xi)_{\xi \in \mathbb{D}}$ where $\beta_\xi = (\beta_{\xi'})_{\xi' \in \xi^+}$ determines the composition of the liabilities $(W_{\xi'})_{\xi' \in \xi^+}$ at the successors of node ξ. That is, we assume that the portfolio Z_ξ is chosen so that $(W_{\xi'})_{\xi' \in \xi^+} = d_\xi \beta_\xi$ for some $d_\xi \in \mathbb{R}_{++}$.

Let us show that given $(\boldsymbol{P}, \boldsymbol{p}, \boldsymbol{q})$ satisfying conditions (i)–(iii) of Step 1 and an aggregate consumption process $\boldsymbol{C} \in \ell^\infty_+(\mathbb{D})$, the feasibility and fiscal rule constraints at each node $\xi \in \mathbb{D}$

$$M_\xi = p_\xi C_\xi \tag{40}$$
$$M_\xi + \theta_\xi + q_\xi Z_\xi = W_\xi \tag{41}$$
$$\frac{r_\xi^1 M_\xi}{1 + r_\xi^1} + \theta_\xi = \alpha_\xi W_\xi \tag{42}$$
$$(W_{\xi'})_{\xi' \in \xi^+} = d_\xi \beta_\xi \tag{43}$$

where $W_\xi = M_{\xi^-} + \hat{q}_\xi Z_{\xi^-}$, the liability of the government at the beginning of node ξ, determines the portfolio and tax policy (θ, Z). Start at the initial node ξ_0. (40) determines M_{ξ_0} and since $r_{\xi_0}^1$ is given by $q_{\xi_0}^1 = \frac{1}{1 + r_{\xi_0}^1}$, the seignorage $\frac{r_{\xi_0}^1 M_{\xi_0}}{1 + r_{\xi_0}^1}$ is known. (41) and (42) then determine θ_{ξ_0} and $k_0 = q_{\xi_0} Z_{\xi_0}$. Premultiplying (43) by the vector $P_{\xi_0^+} = (P_{\xi'})_{\xi' \in \xi_0^+}$ gives the equation

$$\left(\sum_{\xi' \in \xi_0^+} P_{\xi'} \right) M_{\xi_0} + P_{\xi_0} q_{\xi_0} Z_{\xi_0}$$
$$= d_{\xi_0} \sum_{\xi' \in \xi_0^+} P_{\xi'} \beta_{\xi_0 \xi'} \iff \frac{P_{\xi_0} M_{\xi_0}}{1 + r_{\xi_0}^1} + P_{\xi_0} k_0 = d_{\xi_0} P_{\xi_0^+} \beta_0$$

which determines d_{ξ_0}. Then

$$(W_{\xi'})_{\xi' \in \xi_0^+} = M_{\xi_0} \mathbf{1} + \left[\hat{q}_{\xi_0^+} \right] Z_{\xi_0}$$

determines Z_{ξ_0} since $\left[\hat{q}_{\xi^+} \right]$ is invertible. Note that

$$\sum_{\xi' \in \xi_0^+} P_{\xi'} W_{\xi'} = \frac{P_{\xi_0} M_{\xi_0}}{1 + r_{\xi_0}^1} + P_{\xi_0} q_{\xi_0} Z_{\xi_0}$$
$$= P_{\xi_0} \left(M_{\xi_0} - \frac{r_{\xi_0}^1 M_{\xi_0}}{1 + r_{\xi_0}^1} + q_{\xi_0} Z_{\xi_0} \right)$$
$$= P_{\xi_0} (1 - \alpha_{\xi_0}) W_0 \tag{44}$$

where the last equality is obtained by substituting the value of θ_{ξ_0} given by (42) into the budget Eq. (41): thus the present value of the government's liabilities has decreased by the factor α_{ξ_0} by virtue of the Ricardian policy (42). Note also that since all the terms $(W_{\xi'})_{\xi' \in \xi_0^+}$ have the same sign ($\beta_{\xi_0} \gg 0$), they are of the same sign as W_0.

By induction we can use (40)–(43) to calculate (θ_ξ, Z_ξ) for all nodes ξ, showing that the relation (44) between W_ξ and $(W_{\xi'})_{\xi' \in \xi^+}$ is satisfied at each node and that the liabilities W_ξ always have

the same sign, which is the sign of W_0. To establish the asymptotic properties, we assume that $W_0 > 0$, so that for all ξ, $W_\xi > 0$: if $W_0 < 0$ it suffices to reverse the inequalities in the analysis that follows and the same asymptotic results hold. It follows from (44) that

$$\sum_{\xi' \in \xi^+} P_{\xi'} W_{\xi'} \leq (1 - \underline{\alpha}) P_\xi W_\xi.$$

Applying this inequality recursively gives

$$\sum_{\substack{\xi'' \in \mathbb{D}(\xi) \\ t(\xi'') = t(\xi) + 2}} P_{\xi''} W_{\xi''} \leq \sum_{\xi' \in \xi^+} (1 - \underline{\alpha}) P_{\xi'} W_{\xi'} \leq (1 - \underline{\alpha})^2 P_\xi W_\xi$$

and moving forward $T - t(\xi)$ periods into the subtree $\mathbb{D}(\xi)$ gives

$$0 < \sum_{\xi'' \in \mathbb{D}_T(\xi)} P_{\xi''} W_{\xi''} \leq (1 - \underline{\alpha})^{T - t(\xi)} P_\xi W_\xi. \tag{45}$$

Thus the Ricardian assumption (RC) implies that the present value of the government's date T liabilities tend to zero when $T \to \infty$ on every subtree of $\mathbb{D}$.

Multiplying (42) by $P_{\xi'}$ for each node $\xi' \in \mathbb{D}_T(\xi)$ and forming the sum of these values gives

$$\sum_{\xi' \in \mathbb{D}_T(\xi)} P_{\xi'} \theta_{\xi'} = \sum_{\xi' \in \mathbb{D}_T(\xi)} \alpha_{\xi'} P_{\xi'} W_{\xi'} - \sum_{\xi' \in \mathbb{D}_T(\xi)} \frac{r_{\xi'}^1}{1 + r_{\xi'}^1} P_{\xi'} M_{\xi'},$$

$$\xi \in \mathbb{D}.$$

By (45) the first term on the right side tends to 0 as $T \to \infty$ and since $P_{\xi'} M_{\xi'} = P_{\xi'} p_{\xi'} C_{\xi'}$ by (iii) of Step 1 and $\boldsymbol{C} \in \ell^\infty(\mathbb{D})$, the second term tends to 0. Thus $\sum_{\xi' \in \mathbb{D}_T(\xi)} P_{\xi'} \theta_{\xi'} \to 0$ as $T \to \infty$.

Step 3. We show that an extensive-form equilibrium is a reduced-form equilibrium. Let

$$\left(\left((\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_g}), (\bar{\boldsymbol{M}}, \bar{\boldsymbol{Z}}, \bar{\boldsymbol{\theta}}) \right), \left((\bar{\boldsymbol{x}}, \widetilde{\bar{\boldsymbol{m}}}, \bar{\boldsymbol{z}}), (\bar{\boldsymbol{y}}, \bar{\boldsymbol{L}}) \right), \right.$$
$$\left. \left(\bar{\boldsymbol{P}}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{\omega}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_p} \right) \right)$$

be an extensive-form equilibrium. The profit maximization in (vi) of Definition 1 implies $\bar{\omega}_\xi = \bar{p}_\xi, \xi \in \mathbb{D}$. Since the government policy is Ricardian the asymptotic properties established in Step 2 hold. Thus if we define $\bar{\Theta} \equiv \sum_{\xi \in \mathbb{D}} \bar{P}_\xi \bar{\theta}_\xi$, all the conditions (i)–(iv) of Step 1 are satisfied, and $\mathbb{B}^h(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, \bar{\boldsymbol{\theta}}) = B^h(\bar{\boldsymbol{P}}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{r}}^1, \bar{\Theta})$. Since $\bar{\boldsymbol{x}}^h$ is optimal in $\mathbb{B}^h$, it is also optimal over B^h so that (2) in Definition 2 is satisfied. It follows that the triple $\left(\left(\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_g}, \bar{\Theta} \right), \bar{\boldsymbol{x}}, \bar{\boldsymbol{P}} \right)$ satisfies all the conditions for a reduced-form equilibrium in Definition 2.

Step 4. We show that from a reduced-form equilibrium we can reconstruct the associated extensive-form equilibrium. Let $\left((\bar{\boldsymbol{B}}, (\bar{\boldsymbol{q}}^j)_{j \in \mathcal{J}_g}, \bar{\Theta}), \bar{\boldsymbol{x}}, \bar{\boldsymbol{P}} \right)$ be a reduced-form equilibrium and let $\bar{\boldsymbol{q}}^j$ be defined by $\bar{P}_\xi \bar{q}_\xi^j = \sum_{\xi' \in \xi^+} \bar{P}_{\xi'} \hat{\bar{q}}_{\xi'}^j, j \in \mathcal{J}_p, \xi \in \mathbb{D}$ with $\hat{\bar{q}}_{\xi'}^j \equiv V_{\xi'}^j$. By assumption $\left[\hat{\bar{q}}_{\xi^+} \right]$ is invertible for all $\xi \in \mathbb{D}$. Given the properties of a reduced-form equilibrium, (i)–(iii) of Step 1 are satisfied and we can apply Step 2 to construct the government policy $(\bar{\boldsymbol{M}}, \bar{\boldsymbol{Z}}, \bar{\boldsymbol{\theta}})$. To show that $\bar{\Theta} = \sum_{\xi \in \mathbb{D}} \bar{P}_\xi \bar{\theta}_\xi$, we sum the agents' budget equations in the reduced-form equilibrium and use the market clearing equations to obtain

$$\sum_{\xi \in \mathbb{D}} \frac{\bar{r}_\xi^1}{1 + \bar{r}_\xi^1} \bar{P}_\xi \bar{M}_\xi + \bar{\Theta} = \sum_{h \in \mathcal{H}} w_0^h = W_0. \tag{46}$$

On the other hand multiplying the government's budget equation (41) at node ξ by $\bar{P}_\xi$ for all $\xi \in \mathbb{D}$ and summing over the whole event-tree leads to

$$\sum_{\xi \in \mathbb{D}} \frac{\bar{r}_\xi^1}{1 + \bar{r}_\xi^1} \bar{P}_\xi \bar{M}_\xi + \sum_{\xi \in \mathbb{D}} \bar{P}_\xi \bar{\theta}_\xi = W_0$$

which combined with (46) implies that $\bar{\Theta} = \sum_{\xi \in \mathbb{D}} \bar{P}_\xi \bar{\theta}_\xi$. Thus all the conditions (i)–(iv) of Step 1 are satisfied and $\mathbb{B}^h(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, \bar{\boldsymbol{\theta}}) = B^h(\bar{\boldsymbol{P}}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{r}}^1, \bar{\Theta})$. Thus for each $h \in \mathcal{H}$, $\bar{\boldsymbol{x}}^h$ is optimal over $\mathbb{B}^h$ and the portfolio strategy $\bar{\boldsymbol{z}}^h$ which finances $\bar{\boldsymbol{x}}^h$ is given by (37). It remains to show that the financial markets clear, i.e. $\sum_{h \in \mathcal{H}} \bar{\boldsymbol{z}}^h = \bar{\boldsymbol{Z}}$. Summing Eq. (37) at node ξ over the agents gives

$$\left[\sum_{\xi'' \in \mathbb{D}(\xi')} \bar{P}_{\xi''} \frac{\bar{r}_{\xi''}^1 \bar{M}_{\xi''}}{1 + \bar{r}_{\xi''}^1} + \sum_{\xi'' \in \mathbb{D}(\xi')} \bar{P}_{\xi''} \bar{\theta}_{\xi''} - \bar{P}_{\xi'} \bar{M}_\xi \right]_{\xi' \in \xi^+}$$
$$= \left[\bar{P} \circ_{\xi^+} \hat{\bar{q}} \right] \sum_{h \in \mathcal{H}} \bar{z}_\xi^h, \quad \forall \xi \in \mathbb{D}. \tag{47}$$

On the other hand multiplying the government budget constraints (41) over the subtree $\mathbb{D}(\xi)$ by the corresponding node prices and, for each $\xi' \in \xi^+$, summing over the subtree $\mathbb{D}(\xi')$ leads to

$$\bar{P}_{\xi'} \hat{\bar{q}}_{\xi'} \bar{Z}_{\xi} + \bar{P}_{\xi'} \bar{M}_\xi = \sum_{\xi'' \in \mathbb{D}(\xi')} \bar{P}_{\xi''} \frac{\bar{r}_{\xi''}^1 \bar{M}_{\xi''}}{1 + \bar{r}_{\xi''}^1} + \sum_{\xi'' \in \mathbb{D}(\xi')} \bar{P}_{\xi''} \bar{\theta}_{\xi''},$$

$$\xi' \in \xi^+$$

which, combined with (47) implies that $\sum_{h \in \mathcal{H}} \bar{z}_\xi^h = \bar{Z}_\xi$ for all $\xi \in \mathbb{D}$. Thus all the properties of Definition 1 are satisfied, and the proof is complete. $\square$

A.2. Uniqueness of equilibrium when $S = 2$

The case where there are two possible inflation rates (π_1, π_2) provides the simplest setting for understanding the difference between conditions (i) and (ii) of Theorem 11. If the inflation rates are positive $0 \leq \pi_1 < \pi_2$ then for any 2×2 Markov matrix $\bar{B}$ there exists an equilibrium with non-negative short-term interest rates, corresponding to gross rates of return on the short-term bond $\bar{R}_1 \geq 1, \bar{R}_2 \geq 1$. If $T = 1$ and

$$\frac{u_c(c(\bar{R}_1), 1 - c(\bar{R}_1))}{1 + \pi_1} \neq \frac{u_c(c(\bar{R}_2), 1 - c(\bar{R}_2))}{1 + \pi_2} \tag{48}$$

then condition (ii) of Theorem 11 is satisfied and the short-term interest rate suffices to anchor the inflation expectations to $\bar{B}$. Condition (i) is then superfluous.

If pricing is risk neutral ($u_c \equiv 1$) then (48) is always satisfied since $\pi_1 \neq \pi_2$ and the two Fisher equations

$$\frac{1}{\bar{R}_1} = \delta \left(\frac{B_{11}}{1 + \pi_1} + \frac{1 - B_{11}}{1 + \pi_2} \right),$$
$$\frac{1}{\bar{R}_2} = \delta \left(\frac{B_{21}}{1 + \pi_1} + \frac{1 - B_{21}}{1 + \pi_2} \right)$$

uniquely determine (B_{11}, B_{21}).

In the case where the representative agent is risk averse, (48) may not hold. In this case it is easy to see from the short-term pricing equation that the short-term bond prices are such that

$$\bar{q}_1 = \frac{1}{\bar{R}_1} = \frac{\delta}{1 + \pi_1} \neq \frac{\delta}{1 + \pi_2} = \frac{1}{\bar{R}_2} = \bar{q}_2. \tag{49}$$

Thus (48) is not satisfied when the characteristics $(u, \delta, \pi_1, \pi_2)$ of the economy are such that

$$\frac{u_c(c(\frac{1 + \pi_1}{\delta}), 1 - c(\frac{1 + \pi_1}{\delta}))}{1 + \pi_1} = \frac{u_c(c(\frac{1 + \pi_2}{\delta}), 1 - c(\frac{1 + \pi_2}{\delta}))}{1 + \pi_2}. \tag{50}$$

In view of Lemma 6(iii), since $\frac{u_c(c(\bar{R}), 1 - c(\bar{R}))}{R}$ is decreasing, the smallest perturbation in π_1 leads (50) to be violated: thus (48) is generically satisfied. When (50) holds so that condition (ii) of Theorem 11 is not satisfied, by (49) the rank condition in (i) is

satisfied, so that expectations can be anchored by using bonds of maturities 1 and 2, (B_{11}, B_{21}) being determined by the two Fisher equations for the two-period bond (recall that (49) and (50) hold)

$$\bar{q}_1^2 = \frac{\delta^2}{1 + \pi_1}\Big(\frac{B_{11}}{1 + \pi_1} + \frac{1 - B_{11}}{1 + \pi_2}\Big),$$

$$\bar{q}_2^2 = \frac{\delta^2}{1 + \pi_2}\Big(\frac{B_{21}}{1 + \pi_1} + \frac{1 - B_{21}}{1 + \pi_2}\Big).$$

Thus when $S = 2$, generically (ii) of Theorem 11 is satisfied. In the exceptional case where (ii) is not satisfied, then (i) is satisfied and the prices of the bonds of maturities 1 and 2 anchor expectations of inflation.

A.3. Proof of Lemma 14

We first assume that $(\boldsymbol{v}, \boldsymbol{a}) \in \Delta_{++}^H \times \mathbb{R}_{++}^H$ are fixed and, to simplify notation, we omit these parameters as arguments of the functions. For $R > 0, \Phi > 0, h \in \mathcal{H}$, and k a large positive number, define the function $\tilde{x}^h : \mathbb{R}_{++}^2 \to \mathbb{R}_{++}^2$ by

$$\tilde{x}^h(\Phi, R) = \operatorname{argmax}\Big\{ v^h u^h(c^h, \ell^h) - \Phi c^h$$

$$- \Phi \frac{\ell^h}{R} \mid 0 \le c^h \le k, \ 0 \le \ell^h \le e^h \Big\}.$$

By Assumption $\mathcal{U}$ and $v^h > 0$, the function which is maximized is strictly concave and has a unique maximum. By Assumption $\mathcal{U}$, for k large enough the maximum cannot occur on the boundary of the constraint set so that the maximum is the solution of the FOCs (a1)–(a2). Let $\big(\tilde{c}^h(\Phi, R), \tilde{\ell}^h(\Phi, R)\big)$ denote the solution of the maximum problem viewed as function of (Φ, R). Since $u_{cc}^h u_{\ell\ell}^h - (u_{c\ell}^h)^2 > 0$, the functions $\tilde{c}^h$ and $\tilde{\ell}^h$ are differentiable. To obtain a solution to (a1)–(a3) Φ must satisfy the market clearing equation

$$\sum_{h \in \mathcal{H}} \tilde{c}^h(\Phi, R) + \sum_{h \in \mathcal{H}} a^h \tilde{\ell}^h(\Phi, R) = \sum_{h \in \mathcal{H}} a^h e^h. \tag{51}$$

By Assumption $\mathcal{U}$, when $\Phi \to 0, \tilde{c}^h(\Phi, R) \to \infty$ and the left side of (51) is greater than the right side. When $\Phi \to \infty$ both $\tilde{c}^h(\Phi, R)$ and $\tilde{\ell}^h(\Phi, R)$ tend to zero so that the left side of (51) is smaller than the right side. Thus it suffices to show that the functions $\tilde{c}^h$ and $\tilde{\ell}^h$ are strictly decreasing functions of Φ to show that Eq. (51) has a unique solution $\Phi(R)$. Differentiating the FOCs (a1)–(a2) gives

$$v^h u_{cc}^h \frac{\partial \tilde{c}^h}{\partial \Phi} + v^h u_{c\ell}^h \frac{\partial \tilde{\ell}^h}{\partial \Phi} = 1$$

$$v^h u_{\ell c}^h \frac{\partial \tilde{c}^h}{\partial \Phi} + v^h u_{\ell\ell}^h \frac{\partial \tilde{\ell}^h}{\partial \Phi} = \frac{a^h}{R}$$

which implies

$$\frac{\partial \tilde{c}^h}{\partial \Phi} = \frac{u_{\ell\ell}^h - \frac{a^h u_{c\ell}^h}{R}}{v^h D^h}, \qquad \frac{\partial \tilde{\ell}^h}{\partial \Phi} = \frac{\frac{a^h u_{cc}^h}{R} - u_{c\ell}^h}{v^h D^h R}$$

where $D^h = u_{cc}^h u_{\ell\ell}^h - (u_{c\ell}^h)^2$. By Assumption $\mathcal{U}$, $D^h > 0$ and the numerators of the fractions are negative, so that both $\tilde{c}^h$ and $\tilde{\ell}^h$ are decreasing functions of Φ: thus (51) has a unique solution $\Phi(R)$.

To sign $\frac{\partial \Phi}{\partial R}$, differentiating (51) gives

$$\sum_{h \in \mathcal{H}}\Big(\frac{\partial \tilde{c}^h}{\partial \Phi} + a^h \frac{\partial \tilde{\ell}^h}{\partial \Phi}\Big)\frac{\partial \Phi}{\partial R} = -\sum_{h \in \mathcal{H}}\Big(\frac{\partial \tilde{c}^h}{\partial R} + a^h \frac{\partial \tilde{\ell}^h}{\partial R}\Big). \tag{52}$$

The derivatives $\partial \tilde{c}^h/\partial R$ and $\partial \tilde{\ell}^h/\partial R$ can be found by differentiating the FOCs (a1)–(a2)

$$v^h u_{cc}^h \frac{\partial \tilde{c}^h}{\partial R} + v^h u_{c\ell}^h \frac{\partial \tilde{\ell}^h}{\partial R} = 0$$

$$v^h u_{\ell c}^h \frac{\partial \tilde{c}^h}{\partial R} + v^h u_{\ell\ell}^h \frac{\partial \tilde{\ell}^h}{\partial R} = \frac{-a^h \Phi}{R^2}$$

which gives

$$\frac{\partial \tilde{c}^h}{\partial R} = \frac{a^h \Phi u_{c\ell}^h}{v^h R^2 D^h} \ge 0, \qquad \frac{\partial \tilde{\ell}^h}{\partial R} = -\frac{a^h \Phi u_{cc}^h}{v^h R^2 D^h} > 0$$

which by (52) implies $\frac{\partial \Phi}{\partial R} > 0$. Solving for $\frac{\partial \Phi}{\partial R}$ using (52) and substituting gives

$$\frac{\partial}{\partial R}\Big(\frac{\Phi}{R}\Big) = \frac{R\frac{\partial \Phi}{\partial R} - \Phi}{R^2}$$

$$= \frac{-R \sum_{h \in \mathcal{H}}\Big(\frac{\partial \tilde{c}^h}{\partial R} + a^h \frac{\partial \tilde{\ell}^h}{\partial R}\Big) - \Phi \sum_{h \in \mathcal{H}}\Big(\frac{\partial \tilde{c}^h}{\partial \Phi} + a^h \frac{\partial \tilde{\ell}^h}{\partial \Phi}\Big)}{R^2 \sum_{h \in \mathcal{H}}\Big(\frac{\partial \tilde{c}^h}{\partial \Phi} + a^h \frac{\partial \tilde{\ell}^h}{\partial \Phi}\Big)}$$

$$\equiv \frac{N}{D}. \tag{53}$$

The denominator D is negative. To sign the numerator we replace the partial derivatives of $\tilde{c}^h$ and $\tilde{\ell}^h$ by their values, which gives

$$N = \Phi \sum_{h \in \mathcal{H}} \frac{a^h u_{c\ell}^h - u_{\ell\ell}^h}{v^h D^h} > 0$$

so that $\frac{\partial}{\partial R}\big(\frac{\Phi}{R}\big) < 0$.

Reverting to the full notation, (51) implicitly the function $\Phi(\boldsymbol{v}, \boldsymbol{a}, R)$, and if the functions c^h and ℓ^h are defined by $c^h(\boldsymbol{v}, \boldsymbol{a}, R) = \tilde{c}^h(\Phi(\boldsymbol{v}, \boldsymbol{a}, R), R), \ell^h(\boldsymbol{v}, \boldsymbol{a}, R) = \tilde{\ell}^h(\Phi(\boldsymbol{v}, \boldsymbol{a}, R), R)$, all the properties of Lemma 14 are satisfied for $(\boldsymbol{v}, \boldsymbol{a}, R) \in \Delta_{++}^H \times \mathbb{R}_{++}^H \times \mathbb{R}_{++}$. To show continuity with respect to $\boldsymbol{v}$ over the whole simplex, suppose that a sequence $(\boldsymbol{v}_n)_{n \ge 0} \in \Delta_{++}^H$ in the interior of the simplex converges to $\bar{\boldsymbol{v}} \in \Delta^H$ with $\bar{v}^h = 0$. Since for some $h', \bar{v}^{h'} \ge 1/H$ and (51) must hold, $\Phi(\boldsymbol{v}_n, \boldsymbol{a}, R)$ stays bounded away from zero and, since $v_n^h \to 0, u_c^h(c^h(\boldsymbol{v}_n, \boldsymbol{a}, R), \ell^h(\boldsymbol{v}_n, \boldsymbol{a}, R))$ and $u_\ell^h(c^h(\boldsymbol{v}_n, \boldsymbol{a}, R), \ell^h(\boldsymbol{v}_n, \boldsymbol{a}, R))$ must tend to ∞, so that $(c^h(\boldsymbol{v}_n, \boldsymbol{a}, R), \ell^h(\boldsymbol{v}_n, \boldsymbol{a}, R))$ tends to 0. $\square$

A.4. Proof of Theorem 15

Since for all $\eta \in \mathscr{S} \times \mathscr{G}$, $\boldsymbol{a}_\eta = (a_\eta^1, \dots, a_\eta^H)$ is fixed, we omit it from the argument of the functions and let $c_\eta^h(\boldsymbol{v}, R), \ell_\eta^h(\boldsymbol{v}, R), \Phi_\eta(\boldsymbol{v}, R)$ denote the functions $c^h(\boldsymbol{v}, \boldsymbol{a}_\eta, R), \ell^h(\boldsymbol{v}, \boldsymbol{a}_\eta, R), \Phi(\boldsymbol{v}, \boldsymbol{a}_\eta, R)$ defined in Lemma 14. Since by (1) of Theorem 15

$$\widetilde{\Phi}_\eta(\boldsymbol{v}, 1) \ge \delta \sum_{\eta' \in \mathscr{S} \times \mathscr{G}} B_{\eta\eta'} \frac{\Phi_{\eta'}(\boldsymbol{v}, 1)}{1 + \pi_{\eta'}} > \lim_{R \to \infty} \widetilde{\Phi}_\eta(\boldsymbol{v}, R)$$

there exists a solution $R_\eta^{\max}(\boldsymbol{v}) \ge 1$ to Eq. (27), so that $R_\eta^{\max} \ge 1$. Thus

$$K = \big\{ \boldsymbol{R} \in \mathbb{R}_+^{S \times G} \mid \boldsymbol{1} \le \boldsymbol{R} \le \boldsymbol{R}^{\max} \big\}$$

is a non-empty compact convex subset of $\mathbb{R}^{S \times G}$. For $\boldsymbol{v} \in \Delta^H$ and $\boldsymbol{R} = (R_1, \dots, R_{S \times G}) \gg 0$, let $\Theta(\boldsymbol{v}, \boldsymbol{R})$ be defined by the equation (obtained by summing the agents' budget equations)

$$\sum_{\eta \in \mathscr{S} \times \mathscr{G}} [I - \delta B]_{\eta_0 \eta}^{-1} \Phi_\eta(\boldsymbol{v}, R_\eta)\Big(\sum_{h \in \mathcal{H}} c_\eta^h(\boldsymbol{v}, R_\eta)\Big)\frac{R_\eta - 1}{R_\eta}$$

$$+ \Phi_{\eta_0}(\boldsymbol{v}, R_{\eta_0})(\Theta - W_0) = 0. \tag{54}$$

$\Theta(\boldsymbol{v}, \boldsymbol{R})$ is the present value of the taxes needed to withdraw the government liabilities $W_0 = \sum_h w_0^h$ from the private sector when the seignorage revenue is given by the first term in (54). For each $h \in \mathcal{H}$ consider the function

$$\zeta^h(\boldsymbol{v}, \boldsymbol{R})$$

$$= \sum_{\eta \in \mathcal{S} \times \mathcal{G}} [I - \delta B]_{\eta_0 \eta}^{-1} \Phi_\eta(\boldsymbol{v}, R_\eta) \left(\frac{a_\eta^h (e^h - \ell_\eta^h(\boldsymbol{v}, R_\eta))}{R_\eta} - c_\eta^h(\boldsymbol{v}, R_\eta) \right)$$

$$+ \Phi_{\eta_0}(\boldsymbol{v}, R_{\eta_0})(w_0^h - \gamma^h \Theta(\boldsymbol{v}, \boldsymbol{R}))$$

which gives the excess of the present value of income over consumption for agent h when the vector of weights is $\boldsymbol{v}$ and the vector of returns is $\boldsymbol{R}$. Given the definition of $\Theta(\boldsymbol{v}, \boldsymbol{R})$, for all $\boldsymbol{v} \in \Delta^H$ and $\boldsymbol{R} \gg 0$, $\sum_{h \in \mathcal{H}} \zeta^h(\boldsymbol{v}, \boldsymbol{R}) = 0$.

Consider the map $\Psi: \Delta^H \times K \to \mathbb{R}^H \times \mathbb{R}^{S \times G}$ defined by

$$\Psi_h(\boldsymbol{v}, \boldsymbol{R}) = \frac{v^h + \max\{\zeta^h(\boldsymbol{v}, \boldsymbol{R}), 0\}}{1 + \sum_{h' \in \mathcal{H}} \max\{\zeta^{h'}(\boldsymbol{v}, \boldsymbol{R}), 0\}}, \quad h \in \mathcal{H}$$

$$\Psi_\eta(\boldsymbol{v}, \boldsymbol{R}) = \widetilde{\Phi}_\eta^{-1} \left(\boldsymbol{v}, \delta \sum_{\eta' \in \mathcal{S} \times \mathcal{G}} \frac{B_{\eta \eta'}}{1 + \pi_{\eta'}} \Phi_{\eta'}(\boldsymbol{v}, R_{\eta'}) \right), \quad (55)$$

$$\eta \in \mathcal{S} \times \mathcal{G}$$

where $\widetilde{\Phi}_\eta^{-1}(\boldsymbol{v}, \cdot)$ denotes the inverse of the decreasing function $R \to \widetilde{\Phi}_\eta(\boldsymbol{v}, R)$. Ψ_h increases the weight of agent h when the present value of his income exceeds the present value of his consumption, and decreases it otherwise. Ψ_η gives the return on the short-term bond in state η which is such that the marginal cost of one unit of the bond is equal to its marginal benefit, when the vector of marginal utilities next period is $(\Phi_{\eta'}(\boldsymbol{v}, R_{\eta'}))_{\eta' \in \mathcal{S} \times \mathcal{G}}$. Since the function $R \to \Phi_\eta(\boldsymbol{v}, R)$ is increasing and $\lim_{R \to 0} \widetilde{\Phi}_\eta(\boldsymbol{v}, R) = \infty$, by (1) of Theorem 15

$$\lim_{R \to \infty} \widetilde{\Phi}_\eta(\boldsymbol{v}, R) < \delta \sum_{\eta' \in \mathcal{S} \times \mathcal{G}} \frac{B_{\eta \eta'}}{1 + \pi_{\eta'}} \Phi_{\eta'}(\boldsymbol{v}, 1)$$

$$\leq \delta \sum_{\eta' \in \mathcal{S} \times \mathcal{G}} \frac{B_{\eta \eta'}}{1 + \pi_{\eta'}} \Phi_{\eta'}(\boldsymbol{v}, R_{\eta'})$$

$$< \lim_{R \to 0} \widetilde{\Phi}_\eta(\boldsymbol{v}, R)$$

so that $\delta \sum_{\eta' \in \mathcal{S} \times \mathcal{G}} \frac{B_{\eta \eta'}}{1 + \pi_{\eta'}} \Phi_{\eta'}(\boldsymbol{v}, R_{\eta'})$ is in the image of the function $R \to \widetilde{\Phi}_\eta(\boldsymbol{v}, R)$ and $\Psi_\eta(\boldsymbol{v}, \boldsymbol{R})$ is well defined. By construction $(\Psi_h(\boldsymbol{v}, \boldsymbol{R}))_{h \in \mathcal{H}}$ is in Δ^H. To show that $(\Psi_\eta(\boldsymbol{v}, \boldsymbol{R}))_{\eta \in \mathcal{S} \times \mathcal{G}}$ is in K, note that

$$\delta \sum_{\eta' \in \mathcal{S} \times \mathcal{G}} \frac{B_{\eta \eta'}}{1 + \pi_{\eta'}} \Phi_{\eta'}(\boldsymbol{v}, R_{\eta'}) \leq \delta \sum_{\eta' \in \mathcal{S} \times \mathcal{G}} \frac{B_{\eta \eta'}}{1 + \pi_{\eta'}} \Phi_{\eta'}(\boldsymbol{v}, R_{\eta'}^{\max})$$

$$= \frac{\Phi_\eta(\boldsymbol{v}, 1)}{R_\eta^{\min}(\boldsymbol{v})} \leq \widetilde{\Phi}_\eta(\boldsymbol{v}, 1) \quad (56)$$

where the equality comes from the definition of $R_\eta^{\min}(\boldsymbol{v})$, and the last inequality comes from (2) of Theorem 15 and the fact that $\Phi_\eta(\boldsymbol{v}, 1) = \widetilde{\Phi}_\eta(\boldsymbol{v}, 1)$ for all $\boldsymbol{v} \in \Delta^H$. Since $\widetilde{\Phi}_\eta^{-1}(\boldsymbol{v}, \cdot)$ is decreasing, (56) implies

$$\Psi_\eta(\boldsymbol{v}, \boldsymbol{R}) = \widetilde{\Phi}_\eta^{-1} \left(\boldsymbol{v}, \delta \sum_{\eta' \in \mathcal{S} \times \mathcal{G}} \frac{B_{\eta \eta'}}{1 + \pi_{\eta'}} \Phi_{\eta'}(\boldsymbol{v}, R_{\eta'}) \right)$$

$$\geq \widetilde{\Phi}_\eta^{-1}(\boldsymbol{v}, \widetilde{\Phi}_\eta(\boldsymbol{v}, 1)) = 1$$

so that $\Psi_\eta(\boldsymbol{v}, \boldsymbol{R}) \geq 1$, for all $\eta \in \mathcal{S} \times \mathcal{G}$. Thus Ψ is a continuous map from $\Delta^H \times K$ into itself and, by Brouwer's Theorem, has a fixed

point $(\bar{\boldsymbol{v}}, \bar{\boldsymbol{R}})$. Let

$$\bar{q}_\eta^1 = \frac{1}{\bar{R}_\eta},$$

$$\bar{x}_\eta^h = (c_\eta^h(\bar{\boldsymbol{v}}, \bar{R}_\eta), \ell_\eta^h(\bar{\boldsymbol{v}}, \bar{R}_\eta)), \quad h \in \mathcal{H},$$

$$\bar{\Phi}_\eta = \Phi_\eta(\bar{\boldsymbol{v}}, \bar{R}_\eta), \quad \eta \in \mathcal{S} \times \mathcal{G},$$

$$\bar{\Theta} = \Theta(\bar{\boldsymbol{v}}, \bar{\boldsymbol{R}})$$

and let the prices of bonds of maturities $\tau = 2, \ldots, T$ be calculated recursively using equations (b1) in Proposition 13. Let us show that $((B, \bar{\boldsymbol{q}}, \bar{\Theta}), (\bar{\boldsymbol{v}}, \bar{\boldsymbol{x}}, \bar{\Phi}))$ is a reduced-form equilibrium. From the construction of the functions (c^h, ℓ^h) in Lemma 14 and by the fixed-point property of $\bar{\boldsymbol{R}}$ in K, the equations (a1)–(a3), (b1)–(b2) of Proposition 13 are satisfied and it suffices to show that the budget equations (a4) hold.

Since $\sum_{h \in \mathcal{H}} \zeta^h(\bar{\boldsymbol{v}}, \bar{\boldsymbol{R}}) = 0$, if $\zeta^h(\bar{\boldsymbol{v}}, \bar{\boldsymbol{R}}) \neq 0$ for some agent, then there is one agent for which the value of excess income is strictly positive and $\sum_{h \in \mathcal{H}} \max\{\zeta^h(\bar{\boldsymbol{v}}, \bar{\boldsymbol{R}}), 0\} > 0$, and there is another agent h' such that $\zeta^{h'}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{R}}) < 0$. For this agent the fixed point property

$$\bar{v}^{h'} \left(1 + \sum_{h \in \mathcal{H}} \max\{\zeta^h(\bar{\boldsymbol{v}}, \bar{\boldsymbol{R}}), 0\} \right) = \bar{v}^{h'}$$

implies $\bar{v}^{h'} = 0$. This in turn implies $(c_\eta^{h'}(\bar{\boldsymbol{v}}, \bar{R}_\eta), \ell_\eta^{h'}(\bar{\boldsymbol{v}}, \bar{R}_\eta)) = 0$ for all $\eta \in \mathcal{S} \times \mathcal{G}$, so that

$$\zeta^{h'}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{R}}) = \sum_{\eta \in \mathcal{S} \times \mathcal{G}} [I - \delta B]_{\eta_0 \eta}^{-1} \bar{\Phi}_\eta \frac{a_\eta^h e^h}{\bar{R}_\eta} - \bar{\Phi}_{\eta_0}(\gamma^{h'} \bar{\Theta} - w_0^{h'}).$$

If $\gamma^{h'} \bar{\Theta} - w_0^{h'} \leq 0$ then it is not possible that $\zeta^{h'} < 0$. If $\gamma^{h'} \bar{\Theta} - w_0^{h'} > 0$, then since $\widetilde{\Phi}$ is decreasing and Φ is increasing

$$\zeta^{h'}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{R}}) \geq \sum_{\eta \in \mathcal{S} \times \mathcal{G}} [I - \delta B]_{\eta_0 \eta}^{-1} \Phi_\eta(\bar{v}, R_\eta^{\max}) \frac{a_\eta^h e^h}{R_\eta^{\max}}$$

$$- \Phi_{\eta_0}(\bar{v}, R_{\eta_0}^{\max})(\gamma^{h'} \bar{\Theta} - w_0^{h'}) > 0$$

by (3) of Theorem 15, contradicting the assumption $\zeta^{h'}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{R}}) < 0$. Thus $\zeta^h(\bar{\boldsymbol{v}}, \bar{\boldsymbol{R}}) = 0$ for all $h \in \mathcal{H}$ and (a4) holds. $\quad\square$

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