Qualifying Exam: Real Analysis

Unofficial solutions by Alex Fu*

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Answer all four questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem, you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $(f_n)_{n\ge 1}$ be a sequence of continuously differentiable functions from $\mathbb{R}^{\ge 0}$ to $\mathbb{R}$ such that $f_n(0) = 0$ for every n and

$$\lim_{n\to\infty} \int_0^\infty |f_n'(x)|^2 \,\mathrm{d}x = 0.$$

Find a proof or counterexample to the following statement:

$$\lim_{n\to\infty} \sup_{x\ge 0} |f_n(x)| = 0.$$

Solution. We will provide the following counterexample. For simplicity, we start by considering continuously differentiable functions with nonnegative derivatives, so that

$$\sup_{x\ge 0} |f_n(x)| = \sup_{x\ge 0} f_n(x) = \sup_{x\ge 0} \int_0^x f_n'(t)\,\mathrm{d}t = \int_0^\infty f_n'(t)\,\mathrm{d}t.$$

Now, it suffices to look for a sequence of nonnegative continuous functions $(g_n)_{n\ge 1} = (f_n')_{n\ge 1}$ that converges to 0 in L^2 but not in L^1. A classic example of a function that belongs to $L^2([1,\infty))$ but not $L^1([1,\infty))$ is $1/x$, and a simple modification leads us straight to the example $g_n(x) = 1/(x+n)$, which satisfies

$$\int_0^\infty g_n(x)^2 \,\mathrm{d}x = \frac{1}{n} \to 0,$$

$$\int_0^\infty g_n(x)\,\mathrm{d}x = \infty \quad \text{for all } n.$$

Therefore, our counterexample is given by

$$f_n(x) = \ln(x+n) - \ln n.$$

*Reach out to me at alexfu.math@usc.edu for any questions, comments, or corrections :)

2. Let $(X, \mathscr{A}, \mu)$ be a complete measure space, let f be a nonnegative integrable function on X, and put $b(t) = \mu(\{x \in X : f(x) \geq t\})$. Show that

$$\int_X f \, \mathrm{d}\mu = \int_0^\infty b(t) \, \mathrm{d}t.$$

Solution. Let $E = \{x \in X : f(x) > 0\}$. Observe that $\int_X f \, \mathrm{d}\mu = \int_E f \, \mathrm{d}\mu$, and observe that $b(t) = \mu(\{x \in E : f(x) \geq t\})$ for every $t > 0$. Thus, we can restrict to E and assume that $X = E$ without loss of generality. It follows that X is σ-finite: for each $n \geq 1$, let $X_n = \{x \in X : f(x) > 1/n\}$, so that $X = \bigcup_{n=1}^\infty X_n$ and

$$\mu(X_n) = \int_X \mathbb{1}_{X_n} \, \mathrm{d}\mu \leq \int_X n \cdot f(x) \, \mathrm{d}\mu < \infty.$$

Now, we can finally apply Tonelli's theorem:

$$\begin{aligned}
\int_X f(x) \, \mathrm{d}\mu &= \int_X \int_0^{f(x)} 1 \, \mathrm{d}t \, \mathrm{d}\mu \\
&= \int_X \int_0^\infty \mathbb{1}\{f(x) \geq t\} \, \mathrm{d}t \, \mathrm{d}\mu \\
&= \int_0^\infty \int_X \mathbb{1}\{f(x) \geq t\} \, \mathrm{d}\mu \, \mathrm{d}t \\
&= \int_0^\infty \mu(\{x \in X : f(x) \geq t\}) \, \mathrm{d}t \\
&= \int_0^\infty b(t) \, \mathrm{d}t.
\end{aligned}$$

3. Let X be a compact metric space, and let μ be a finite Borel measure on X such that $\mu(\{x\}) = 0$ for every $x \in X$. Prove that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mu(E) < \varepsilon$ whenever E is a Borel subset of X with diameter less than δ.

   Solution. Observe that every subset of X with diameter less than δ is contained in an open ball of radius $\delta/2$; hence, it suffices to consider open balls instead of Borel subsets.

   Because μ is a finite measure, we have continuity from above: for each $x \in X$,

$$\lim_{r \to 0^+} \mu(\mathrm{Ball}(x, r)) = \mu\left(\bigcap_{k=1}^{\infty} \mathrm{Ball}\left(x, \frac{1}{k}\right)\right) = \mu(\{x\}) = 0,$$

   so we can choose $r_x > 0$ such that $\mu(\mathrm{Ball}(x, 2r_x)) < \varepsilon$. Then, because $\{\mathrm{Ball}(x, r_x) : x \in X\}$ is an open cover of the compact metric space X, there exist finitely many points $x_1, \ldots, x_n$ such that $X = \bigcup_{i=1}^{n} \mathrm{Ball}(x_i, r_{x_i})$. Let

$$\delta = \min\left\{r_{x_1}, \ldots, r_{x_n}\right\}.$$

   Note that $\delta > 0$. With this value of δ, for every $x \in X$, there exists $i \in \{1, \ldots, n\}$ such that $x \in \mathrm{Ball}(x_i, r_{x_i})$, and it follows that $\mu(\mathrm{Ball}(x, \delta)) \le \mu(\mathrm{Ball}(x_i, 2r_{x_i})) < \varepsilon$. Thus, by monotonicity, $\mu(E) < \varepsilon$ whenever E is a Borel subset of X with diameter less than 2δ.

4. a. Let $f \in L^1([0,\infty))$. Prove that

$$g(x) = \int_0^\infty \frac{f(y)}{x+y}\,\mathrm{d}y$$

is differentiable at every $x > 0$.

Solution. First, observe that $g(x)$ is finite for every $x > 0$:

$$|g(x)| \le \int_0^\infty \frac{|f(y)|}{x+y}\,\mathrm{d}y \le \frac{1}{x}\int_0^\infty |f(y)|\,\mathrm{d}y < \infty.$$

Then, let us verify the definition of differentiability directly. For every $x > 0$ and $0 < |h| < x/2$,

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{1}{h}\int_0^\infty \frac{f(y)}{(x+h)+y} - \frac{f(y)}{x+y}\,\mathrm{d}y \\ &= -\int_0^\infty \frac{f(y)}{(x+y+h)(x+y)}\,\mathrm{d}y,\end{aligned}$$

which we again observe to be finite; it follows that

$$\begin{aligned}\left|\frac{g(x+h)-g(x)}{h} - \left(-\int_0^\infty \frac{f(y)}{(x+y)^2}\,\mathrm{d}y\right)\right| &= \left|\int_0^\infty \frac{f(y)}{x+y}\left(\frac{1}{x+y} - \frac{1}{x+y+h}\right)\mathrm{d}y\right| \\ &= |h|\cdot\left|\int_0^\infty \frac{f(y)}{(x+y)^2(x+y+h)}\,\mathrm{d}y\right| \\ &\le |h|\cdot\frac{2}{x^3}\int_0^\infty |f(y)|\,\mathrm{d}y,\end{aligned}$$

which tends to 0 as $h \to 0$. Hence, at every $x > 0$, the derivative of g exists and is equal to

$$-\int_0^\infty \frac{f(y)}{(x+y)^2}\,\mathrm{d}y.$$

b. Find an example of $f \in L^1([0,\infty))$ such that g is not differentiable at $x = 0$.

Solution. Let $f(y) = 1/(y+1)^2$, and note that $\int_0^\infty f(y)\,\mathrm{d}y = 1 < \infty$. Then, for each $n \ge 1$, define

$$a_n = \frac{g(1/n)-g(0)}{1/n} = -\int_0^\infty \frac{f(y)}{y(y+1/n)}\,\mathrm{d}y.$$

Because $(f(y)/[y(y+1/n)])_{n\ge1}$ is a nondecreasing sequence of nonnegative functions, by the monotone convergence theorem, we have that

$$\lim_{n\to\infty} a_n = -\int_0^\infty \frac{1}{y^2(y+1)^2}\,\mathrm{d}y,$$

which we can evaluate using partial fraction decomposition:

$$\begin{aligned}-\int_0^\infty \frac{1}{y^2(y+1)^2}\,\mathrm{d}y &= -\int_0^\infty \frac{-2y+1}{y^2} + \frac{2y+3}{(y+1)^2}\,\mathrm{d}y \\ &= \int_0^\infty \frac{2}{y} - \frac{1}{y^2} - \frac{2}{y+1} - \frac{1}{(y+1)^2}\,\mathrm{d}y \\ &= \left[2\ln y + \frac{1}{y} - 2\ln(y+1) + \frac{1}{y+1}\right]_{0^+}^\infty \\ &= -\infty.\end{aligned}$$

If g were differentiable at $x = 0$, then $g'(0) = \lim_{n\to\infty} a_n > -\infty$, a contradiction; hence, g is not differentiable at $x = 0$.