Qualifying Exam: Real Analysis

Unofficial solutions by Alex Fu[*]

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Answer all four questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem, you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $f\colon [0,1] \to \mathbb{R}$ be a Lebesgue-measurable function such that $f(x) > 0$ for almost every $x \in [0,1]$, and suppose that $(E_k)_{k\geq 1}$ is a sequence of Lebesgue-measurable subsets of $[0,1]$ with the property that

$$\lim_{k\to\infty} \int_{E_k} f(x)\,\mathrm{d}x = 0.$$

Prove that $\lim_{k\to\infty} m(E_k) = 0$.

Solution. We will prove the contrapositive implication. Let $\varepsilon > 0$, and let $(E_{k_n})_{n\geq 1}$ be a subsequence such that $m(E_{k_n}) \geq \varepsilon$ for every n. Suppose without loss of generality that $E_{k_n} = E_n$. Define $F_j = \{x \in [0,1] : f(x) > 1/j\}$, so that $\{x \in [0,1] : f(x) > 0\} = \bigcup_{j=1}^{\infty} F_j$. By continuity from below, there exists $J \geq 1$ such that $m(F_J) \geq 1 - \varepsilon/2$. Then, for every n, we must have that $m(E_n \cap F_J) \geq \varepsilon/2$; otherwise, we would find the contradiction that $m([0,1]) > 1$. Therefore, for every $n \geq 1$,

$$\int_{E_n} f(x)\,\mathrm{d}x \geq \int_{E_n \cap F_J} f(x)\,\mathrm{d}x \geq \frac{1}{J} \cdot m(E_n \cap F_J) \geq \frac{\varepsilon}{2J},$$

which is positive. It follows that the sequence $(\int_{E_n} f(x)\,\mathrm{d}x)_{n\geq 1}$ cannot converge to 0, and we conclude that

$$m(E_k) \not\to 0 \implies \int_{E_k} f(x)\,\mathrm{d}x \not\to 0.$$

Remark. This is problem 4 on the fall 2021 exam.

[*]Reach out to me at alexfu.math@usc.edu for any questions, comments, or corrections :)

2. Let $(g_n)_{n\geq 1}$ be a sequence of measurable functions on $[0,1]$ with the following properties:

   i. There exists $C < \infty$ such that $|g_n(x)| \leq C$ for every n and almost every $x \in [0,1]$;

   ii. For every $a \in [0,1]$, we have that $\lim_{n\to\infty} \int_0^a g_n(x)\,\mathrm{d}x = 0$.

   Prove that for every $f \in L^1([0,1])$,

$$(*) \qquad \lim_{n\to\infty} \int_0^1 f(x) g_n(x)\,\mathrm{d}x = 0.$$

Solution. Let $\mathscr{F}$ denote the collection of all integrable functions on $[0,1]$ that satisfy $(*)$. We will show that $\mathscr{F} = L^1([0,1])$ via a monotone class argument. Property (ii), together with linearity, implies that $\mathbb{1}_A \in \mathscr{F}$ for every A that is a disjoint union of finitely many subintervals of $[0,1]$. Note that the collection $\mathscr{A}$ of all such A is an algebra. Now, if $A_1, A_2, \ldots \in \mathscr{A}$, and if B is such that $\mathbb{1}_{A_n} \nearrow \mathbb{1}_B$ or $\mathbb{1}_{A_n} \searrow \mathbb{1}_B$, then

$$\lim_{n\to\infty} \left| \int_0^1 \mathbb{1}_{A_n}(x) g_n(x)\,\mathrm{d}x - \int_0^1 \mathbb{1}_B(x) g_n(x)\,\mathrm{d}x \right| \leq \lim_{n\to\infty} \int_0^1 |\mathbb{1}_{A_n}(x) - \mathbb{1}_B(x)| \cdot |g_n(x)|\,\mathrm{d}x = 0$$

by the bounded convergence theorem. Hence, by the monotone class lemma, we find that $\mathbb{1}_B \in \mathscr{F}$ for every Borel subset B of $[0,1]$. By linearity again, we see that every simple integrable function belongs to $\mathscr{F}$. If f is a nonnegative integrable function, then there exists an increasing sequence $(\varphi_n)_{n\geq 1}$ of simple functions that converges pointwise to f; by the dominated convergence theorem with $2C|f|$ as the dominating function,

$$\lim_{n\to\infty} \left| \int_0^1 \varphi_n(x) g_n(x)\,\mathrm{d}x - \int_0^1 f(x) g_n(x)\,\mathrm{d}x \right| \leq \lim_{n\to\infty} \int_0^1 |\varphi_n(x) - f(x)| \cdot |g_n(x)|\,\mathrm{d}x = 0.$$

It follows that every nonnegative integrable function and every nonpositive integrable function belongs to $\mathscr{F}$. Because $\mathscr{F}$ is closed under addition, we conclude that $\mathscr{F} = L^1([0,1])$, as desired.

3. Let E be a measurable subset of $[0,1]$ with Lebesgue measure $m(E) = 99/100$. Show that there exists $x \in [0,1]$ such that for every $r \in (0,1)$,

$$m(E \cap (x-r, x+r)) \geq \frac{r}{4}.$$

Hint: Use the Hardy–Littlewood inequality, which states that

$$m(\{x \in \mathbb{R} : \mathrm{M}f(x) \geq \alpha\}) \leq \frac{3}{\alpha} \cdot \|f\|_{L^1}.$$

Solution. What we want to show is the existence of $x \in [0,1]$ such that

$$\inf_{r>0} \frac{m(E \cap (x-r, x+r))}{2r} \geq \frac{1}{8}.$$

Equivalently, letting $f = \mathbb{1}_{[0,1]\setminus E}$, we want to show that there exists $x \in [0,1]$ such that

$$\mathrm{M}f(x) = \sup_{r>0} \frac{m((x-r, x+r) \setminus E)}{2r} \leq \frac{7}{8}.$$

By the Hardy–Littlewood inequality, we find that

$$m\left(\left\{x \in [0,1] : \mathrm{M}f(x) \geq \frac{7}{8}\right\}\right) \leq \frac{3}{7/8} \cdot m([0,1] \setminus E) = \frac{24}{700} < 1.$$

Hence, the set $\{x \in [0,1] : \mathrm{M}f(x) < 7/8\}$ has positive Lebesgue measure and is in particular nonempty. Therefore, there exists $x \in [0,1]$ such that $m(E \cap (x-r, x+r))/(2r) \geq 1 - \mathrm{M}f(x) > 1/8$ for every $r \in (0,1)$.

4. Let $f\colon [0,1] \to [0,1]$ be Lebesgue-measurable. Prove that for every $M > 0$, there exists $a \in [0,1]$ such that

$$\int_0^1 \frac{1}{|f(x) - a|}\, \mathrm{d}x \geq M.$$

Solution. By Tonelli's theorem, and because the codomain of f is $[0,1]$,

$$\int_0^1 \int_0^1 \frac{1}{|f(x) - a|}\, \mathrm{d}x\, \mathrm{d}a = \int_0^1 \int_0^1 \frac{1}{|f(x) - a|}\, \mathrm{d}a\, \mathrm{d}x = \infty.$$

Let $M > 0$. If $\int_0^1 1/|f(x) - a|\, \mathrm{d}x < M$ for every $a \in [0,1]$, then the integral above would be at most M, which is a contradiction; hence, there exists some $a \in [0,1]$ for which $\int_0^1 1/|f(x) - a|\, \mathrm{d}x \geq M$.

Remark. This problem (and solution) is similar in spirit to question 3 on the spring 2021 exam.