Qualifying Exam: Real Analysis

Unofficial solutions by Alex Fu[*]

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Answer all four questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem, you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Prove that for almost every $x \in [0,1]$, there exist at most finitely many rational numbers $r = p/q$, where $q \geq 2$ and p/q is in reduced form, such that
$$\left| x - \frac{p}{q} \right| < \frac{1}{(q \log q)^2}.$$

   Hint: Consider intervals of length $2/(q \log q)^2$ centered at rational points p/q.

   Solution. Let E denote the set of all $x \in [0,1]$ with the property given in the problem statement; we want to show that $m(E) = 1$. For each $q \geq 2$, define
$$E_q = \bigcup_{p:\gcd(p,q)=1} \left(\frac{p}{q} - \frac{1}{(q \log q)^2}, \frac{p}{q} + \frac{1}{(q \log q)^2} \right),$$
   and observe that E_q is a disjoint union of at most q intervals of length $2/(q \log q)^2$ each. Hence,
$$m(E_q) < \frac{2}{q(\log q)^2}.$$
   Because $\sum_{q \geq 2} m(E_q) < \infty$, by the Borel–Cantelli lemma, we conclude that
$$m(E) = 1 - m\left(\limsup_{q \to \infty} E_q \right) = 1.$$

   Remark. The final step requires the observation that
$$[0,1] \setminus \limsup_{q \to \infty} E_q = \{x \in [0,1] : \text{there exist at most finitely many } q \text{ such that } x \in E_q\}.$$

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2. Let S be a closed subset of $\mathbb{R}$, and let $f \in L^1([0,1])$. Suppose that for every measurable subset E of $[0,1]$ with $m(E) > 0$, we have that
$$\frac{1}{m(E)} \int_E f(x)\,\mathrm{d}x \in S.$$
Prove that $f(x) \in S$ for almost every $x \in [0,1]$.

Solution. Let $U = \mathbb{R} \setminus S$, and suppose for the sake of contradiction that $m(f^{-1}(U)) > 0$. In particular, $f^{-1}(U)$ is a nonempty open set, so there exists an open interval $(a,b) \subseteq U$ such that $m(f^{-1}((a,b))) > 0$. Using continuity from below, choose $\delta > 0$ such that $[a+\delta, b-\delta] \subseteq (a,b)$ and $m(f^{-1}([a+\delta, b-\delta])) > 0$; then, for the measurable set $E = f^{-1}([a+\delta, b-\delta])$, we have that
$$a+\delta \leq \frac{1}{m(E)} \int_E f(x)\,\mathrm{d}x \leq b-\delta.$$
However, $[a+\delta, b-\delta]$ is disjoint from S, so we have obtained a contradiction. We conclude that $m(f^{-1}(S)) = 1$.

3. Compute the following limit:

$$\lim_{n\to\infty}\int_0^1 \frac{1+nx}{(1+x)^n}\,\mathrm{d}x.$$

Solution. For all $x>0$ and for every $n\geq 2$, observe that

$$\frac{1+nx}{(1+x)^n} \leq \frac{1+nx}{\binom{n}{2}x^2} = \frac{2}{x^2}\cdot\frac{1}{n(n-1)} + \frac{2}{x}\cdot\frac{1}{n-1},$$

which converges to 0 as $n\to\infty$, and observe that $(1+nx)/(1+x)^n \leq 1$ by the binomial theorem. Hence, by the bounded convergence theorem,

$$\lim_{n\to\infty}\int_0^1 \frac{1+nx}{(1+x)^n}\,\mathrm{d}x = \int_0^1 \lim_{n\to\infty}\frac{1+nx}{(1+x)^n}\,\mathrm{d}x = 0.$$

4. Let $(X, \mathscr{A}, \mu)$ be a finite measure space, and let $(f_n)_{n\geq 1}$ be a sequence of nonnegative measurable functions on X. Prove that $f_n \to 0$ in measure if and only if

$$\lim_{n\to\infty} \int_X \frac{f_n(2+f_n)}{(1+f_n)^2}\, \mathrm{d}\mu = 0.$$

Solution. Write $\varphi(y) = (y(2+y))/(1+y)^2$. The key observation is that for all $y \geq 0$,

$$3\min\{1,y\} \geq \varphi(y) \geq \frac{1}{2}\min\{1,y\},$$

of which a relevant corollary is that

$$\int_X \varphi(f_n)\, \mathrm{d}\mu \to 0 \quad \text{if and only if} \quad \int_X \min\{1, f_n\}\, \mathrm{d}\mu \to 0.$$

Let $g_n = \min\{1, f_n\}$. If $f_n \to 0$ in measure, then, because $\mu(\{x \in X : f_n(x) \geq \varepsilon\}) \geq \mu(\{x \in X : g_n(x) \geq \varepsilon\})$ for every $\varepsilon > 0$, we find that $g_n \to 0$ in measure as well; by the bounded convergence theorem, $\int_X g_n\, \mathrm{d}\mu \to 0$.

Conversely, if $\int_X g_n\, \mathrm{d}\mu \to 0$, then $g_n \to 0$ in measure; because $\mu(\{x : g_n(x) \geq \varepsilon\}) = \mu(\{x : f_n(x) \geq \varepsilon\})$ for $0 < \varepsilon \leq 1$, and because $\lim_{n\to\infty} \mu(\{x : f_n(x) \geq \varepsilon\})$ is a decreasing function of ε, we find that $f_n \to 0$ in measure as well.

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Remark. The motivation for considering $g_n = \min\{1, f_n\}$, a "truncated" version of f_n, is the bounded convergence theorem for finite measure spaces: $g_n \to 0$ in measure if and only if $\int_X g_n\, \mathrm{d}\mu \to 0$. More generally, if φ is any function such that the ratios $\varphi/\min\{1,\cdot\}$ and $\min\{1,\cdot\}/\varphi$ are bounded, then $f_n \to 0$ in measure if and only if $\int_X \varphi(f_n)\, \mathrm{d}\mu = 0$.

For example, let k be any positive integer, and define

$$\varphi(y) = 1 - \frac{1}{(1+y)^k}.$$

Then, we find the following inequalities:

$$\varphi(y) \geq \frac{ky}{(1+y)^k} \geq \frac{k}{2^k}y \qquad \text{whenever } 0 \leq y \leq 1,$$

$$\varphi(y) \leq \frac{\sum_{i=1}^k \binom{k}{i} y}{(1+y)^k} \leq (2^k - 1)y \qquad \text{whenever } 0 \leq y \leq 1,$$

and of course $k/2^k \leq 1/2 \leq \varphi(y) \leq 1$ whenever $y \geq 1$. Putting everything together, we have that

$$(2^k - 1)\min\{1,y\} \geq \varphi(y) \geq \frac{k}{2^k}\min\{1,y\}.$$

When $k = 2$, we obtain precisely the function φ we defined in the solution above.