Qualifying Exam: Real Analysis

Unofficial solutions by Alex Fu*

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Answer all four questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem, you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Suppose that f is a bounded nonnegative function on a measure space $(X, \mathscr{A}, \mu)$ with $\mu(X) = \infty$. Prove that f is integrable if and only if

$$(*) \qquad \sum_{n=1}^{\infty} \frac{1}{2^n} \mu(\{x \in X : f(x) > 2^{-n}\}) < \infty.$$

Solution. Let M be a finite constant such that $f \le M$ almost everywhere. For convenience, write $E_t = \{x \in X : f(x) > t\}$, and observe that $t \mapsto \mu(E_t)$ is a nonincreasing function.

Suppose that f is integrable. Then, by the tail-sum formula and the monotone convergence theorem,

$$\begin{aligned} \infty &> \int_X f \, \mathrm{d}\mu \\ &= \int_0^M \mu(E_t) \, \mathrm{d}t \\ &\ge \sum_{n=0}^{\infty} \int_{2^{-n}}^{2^{-n+1}} \mu(E_t) \, \mathrm{d}t \\ &\ge \sum_{n=0}^{\infty} 2^{-n} \cdot \mu(E_{2^{-n+1}}), \end{aligned}$$

which implies $(*)$. Conversely, suppose $(*)$. It follows that

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty} 2^{-n} \cdot \mu(E_{2^{-n}}) \\ &\ge \sum_{n=1}^{\infty} \int_{2^{-n}}^{2^{-n+1}} \mu(E_t) \, \mathrm{d}t \\ &= \int_0^1 \mu(E_t) \, \mathrm{d}t. \end{aligned}$$

If $\mu(E_{1/2}) = \infty$, then $(*)$ would not hold, so we must have that $\mu(E_{1/2}) < \infty$, and hence $\mu(E_1) < \infty$. Therefore, $\int_1^M \mu(E_t) \, \mathrm{d}t \le M \cdot \mu(E_1) < \infty$, and we conclude that f is integrable:

$$\int_X f \, \mathrm{d}\mu = \int_0^1 \mu(E_t) \, \mathrm{d}t + \int_1^M \mu(E_t) \, \mathrm{d}t < \infty.$$

*Reach out to me at alexfu.math@usc.edu for any questions, comments, or corrections :)

2. Let f be a real-valued function on $[0,1]$, and let E be the set of points where f is continuous. Prove that E is Lebesgue-measurable.

Solution. For every $\varepsilon > 0$ and every $\delta > 0$, define

$$E_{\varepsilon,\delta} = \{x \in [0,1] : |f(y) - f(z)| < \varepsilon \text{ for all } y, z \in \text{Ball}(x,\delta)\},$$

and observe by the ε-δ definition of continuity that

$$E = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_{1/n,1/m}.$$

Let us show that each $U_n = \bigcup_{m=1}^{\infty} E_{1/n,1/m}$ is open. If $x \in U_n$, then there exists $\delta > 0$ such that $|f(y) - f(z)| < 1/n$ for all $y, z \in \text{Ball}(x,\delta)$. Hence, if $y \in \text{Ball}(x,\delta/2)$, then $\text{Ball}(y,\delta/2) \subseteq \text{Ball}(x,\delta)$, which implies that $y \in E_{1/n,\delta/2} \subseteq U_n$; in other words, $\text{Ball}(x,\delta/2) \subseteq U_n$. Therefore, U_n is open, and we conclude that E is Lebesgue-measurable as the intersection of countably many open sets.

3. Let f be a Lebesgue-integrable function on $\mathbb{R}$, and let $\beta \in (0,1)$. Prove that for almost every $\alpha \in \mathbb{R}$,

$$\int_0^\infty \frac{|f(x)|}{|x-\alpha|^\beta}\,\mathrm{d}x < \infty.$$

Solution. Let $n \geq 1$, and observe that

$$\int_{-n}^{n} |x-\alpha|^{-\beta}\,\mathrm{d}\alpha = \frac{|x+n|^{1-\beta} - |x-n|^{1-\beta}}{1-\beta} = \frac{x^{1-\beta}(|1+n/x|^{1-\beta} - |1-n/x|^{1-\beta})}{1-\beta}.$$

Call this $I_n(x)$. For all $x > n$, we find that up to a constant factor, $I_n(x)$ is asymptotically equivalent to

$$x^{1-\beta}\Big(1 + (1-\beta)\frac{n}{x} - \Big(1 - (1-\beta)\frac{n}{x}\Big)\Big) = 2(1-\beta)nx^{-\beta}$$

as $x \to \infty$; in particular, $\lim_{x\to\infty} I_n(x) = 0$. Because $I_n(x) = \int_{x-n}^{x+n} |\alpha|^{-\beta}\,\mathrm{d}\alpha$ is continuous in x, it follows that I_n is essentially bounded. Now, by Tonelli's theorem, we have that

$$\begin{aligned}\int_{-n}^{n}\int_0^\infty \frac{|f(x)|}{|x-\alpha|^\beta}\,\mathrm{d}x\,\mathrm{d}\alpha &= \int_0^\infty \int_{-n}^{n} \frac{|f(x)|}{|x-\alpha|^\beta}\,\mathrm{d}\alpha\,\mathrm{d}x \\ &= \int_0^\infty |f(x)| \cdot I_n(x)\,\mathrm{d}x \\ &\leq \|f\|_{L^1} \cdot \|I_n\|_{L^\infty} \\ &< \infty.\end{aligned}$$

Hence, the inner integral on the left-hand side is finite for almost every $\alpha \in [-n, n]$. Because $n \geq 1$ is arbitrary, we conclude that $\int_0^\infty |f(x)|/|x-\alpha|^\beta\,\mathrm{d}x < \infty$ for almost every $\alpha \in \mathbb{R}$.

Remark. This problem (and solution) is similar in spirit to question 4 on the spring 2023 exam.

4. Let $(f_n)_{n\geq 1}$ be a sequence of real-valued functions on $[a,b]$ that converges pointwise, let $f = \lim_{n\to\infty} f_n$, and let $V_a^b(f)$ be the total variation of f on $[a,b]$. Show that

$$V_a^b(f) \leq \liminf_{n\to\infty} V_a^b(f_n).$$

Solution. First, let us recall the definition of total variation:

$$V_a^b(f) = \sup\left\{\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| : m \geq 1, a = x_0 < \cdots < x_m = b\right\}.$$

Let $m \geq 1$, and let $a = x_0 < \cdots < x_m = b$. Then, because $f = \lim_{n\to\infty} f_n$,

$$\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| = \liminf_{n\to\infty} \sum_{i=1}^{m} |f_n(x_i) - f_n(x_{i-1})| \leq \liminf_{n\to\infty} V_a^b(f_n).$$

Taking the supremum of the left-hand-side sum over every $m \geq 1$ and every partition $x_0 < \cdots < x_n$ of $[a,b]$, we conclude that

$$V_a^b(f) \leq \liminf_{n\to\infty} V_a^b(f_n).$$