Qualifying Exam: Real Analysis

Unofficial solutions by Alex Fu*

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Answer all four questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem, you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let μ be a measure on $\mathbb{R}$, and let $f: \mathbb{R} \to \mathbb{R}$ be a nonnegative function that is integrable with respect to μ. Prove that for every $\varepsilon > 0$, there exists a μ-measurable subset A of $\mathbb{R}$ such that $\mu(A) < \infty$ and

$$\int_A f \, \mathrm{d}\mu \geq \int_{\mathbb{R}} f \, \mathrm{d}\mu - \varepsilon.$$

Solution. Let $E_n = \{x \in \mathbb{R} : f(x) \leq n\}$, and let $E_{m,n} = \{x \in \mathbb{R} : m < f(x) \leq n\}$. For each $n \geq 1$, by the monotone convergence theorem, there exists $M_n \geq 1$ such that

$$\int_{E_{M_n,n}} f \, \mathrm{d}\mu \geq \int_{E_n} f \, \mathrm{d}\mu - \frac{\varepsilon}{2}.$$

Again by the monotone convergence theorem, there exists N such that $\int_{E_N} f \, \mathrm{d}\mu \geq \int_{\mathbb{R}} f \, \mathrm{d}\mu - \varepsilon/2$, for which

$$\int_{E_{M_N,N}} f \, \mathrm{d}\mu \geq \int_{\mathbb{R}} f \, \mathrm{d}\mu - \varepsilon.$$

Therefore, let $A = E_{M_N,N}$, and observe by Markov's inequality that indeed

$$\mu(A) \leq \mu(\{x \in \mathbb{R} : f(x) > M_N\}) \leq \frac{1}{M_N} \int_{\mathbb{R}} f \, \mathrm{d}\mu < \infty.$$

*Reach out to me at alexfu.math@usc.edu for any questions, comments, or corrections :)

2. Prove that the following limit exists, or prove that it does not exist:

$$\lim_{n\to\infty}\int_0^1 \frac{(3x-1)^{2n}}{1+(3x-1)^{2n}}\,\mathrm{d}x.$$

Solution. Write $f_n(x) = 1/(1+(3x-1)^{2n})$, and observe that

$$\lim_{n\to\infty}\int_0^1 \frac{(3x-1)^{2n}}{1+(3x-1)^{2n}}\,\mathrm{d}x = 1 - \lim_{n\to\infty}\int_0^1 f_n(x)\,\mathrm{d}x.$$

Also observe that the sequence $(f_n(x))_{n\geq 1}$ is nondecreasing when $0 \leq x \leq 1/3$ and decreasing when $1/3 < x \leq 1$. Because every f_n is nonnegative, by the monotone convergence theorem,

$$\lim_{n\to\infty}\int_0^{1/3} f_n(x)\,\mathrm{d}x = \int_0^{1/3} \frac{1}{1+\mathbb{1}\{x=0\}}\,\mathrm{d}x = \frac{1}{3};$$

by the dominated convergence theorem, with 1 as the dominating function,

$$\lim_{n\to\infty}\int_{1/3}^1 f_n(x)\,\mathrm{d}x = \int_{1/3}^1 0\,\mathrm{d}x = 0.$$

We conclude that the given limit exists and is equal to $2/3$.

3. Let m denote the Lebesgue measure on $\mathbb{R}^n$, and suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is such that $\int |f|\,\mathrm{d}m > 0$. Prove that the Hardy–Littlewood maximal function

$$\mathrm{M}f(x) = \sup_{r>0} \frac{1}{m(\mathrm{Ball}(x,r))} \int_{\mathrm{Ball}(x,r)} |f(y)|\,\mathrm{d}y$$

does not belong to $L^1(\mathbb{R}^n)$.

Solution. Since $\int |f|\,\mathrm{d}m > 0$, by the monotone convergence theorem, we can choose $r > 0$ such that

$$0 < \int_{\mathrm{Ball}(0,r)} |f(y)|\,\mathrm{d}y < \infty.$$

Let I be the value of this integral. For $|x| > r$, by the observation that $\mathrm{Ball}(x, |x|+r) \supseteq \mathrm{Ball}(0,r)$, we find that

$$\begin{aligned}
\mathrm{M}f(x) &\geq \frac{1}{m(\mathrm{Ball}(x,|x|+r))} \int_{\mathrm{Ball}(x,|x|+r)} |f(y)|\,\mathrm{d}y \\
&\geq \frac{1}{m(\mathrm{Ball}(x,|x|+r))} \int_{\mathrm{Ball}(0,r)} |f(y)|\,\mathrm{d}y \\
&= \frac{c}{(|x|+r)^n} \cdot I,
\end{aligned}$$

where c is a constant (for the volume of an n-dimensional ball). Consequently,

$$\int_{\mathbb{R}^n} \mathrm{M}f(x)\,\mathrm{d}m(x) \geq cI \int_{\{|x|>r\}} \frac{1}{(|x|+r)^n}\,\mathrm{d}x = \infty,$$

so $\mathrm{M}f$ does not belong to $L^1(\mathbb{R}^n)$.

4. Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is Lebesgue-measurable, then there exists a Borel-measurable function $g\colon \mathbb{R} \to \mathbb{R}$ such that $f = g$ almost everywhere with respect to the Lebesgue measure.

Solution. Let $(\varphi_n)_{n\geq 1}$ be a sequence of simple Lebesgue-measurable functions that converges pointwise to f, and, for each n, write

$$\varphi_n = \sum_{i=1}^{m_n} c_i \cdot \mathbb{1}_{E_i}.$$

Then, because the Lebesgue σ-algebra on $\mathbb{R}$ is the completion of the Borel σ-algebra, each E_i contains a Borel subset B_i such that $E_i \setminus B_i$ is a Lebesgue-null set (i.e., a subset of a Borel-null set). Let

$$\psi_n = \sum_{i=1}^{m_n} c_i \cdot \mathbb{1}_{B_i},$$

so that $\varphi_n = \psi_n$ almost everywhere for each n; it follows that

$$f = \lim_{n\to\infty} \varphi_n = \lim_{n\to\infty} \psi_n \quad \text{almost everywhere.}$$

Now, every ψ_n is Borel-measurable, and the pointwise limit of a sequence of Borel-measurable functions is Borel-measurable, so we are done after taking $g = \lim_{n\to\infty} \psi_n$.