Qualifying Exam: Real Analysis

Unofficial solutions by Alex Fu[*]

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Answer all four questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem, you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a Lebesgue-measurable function, and suppose that E is a measurable subset of $\mathbb{R}$ such that $0 < \int_E f(x)\,\mathrm{d}x < \infty$. Show that for every $t \in (0,1)$, there exists a measurable set $E_t \subseteq E$ for which

$$\int_{E_t} f(x)\,\mathrm{d}x = t \int_E f(x)\,\mathrm{d}x.$$

Solution. Write $E_\alpha = E \cap (-\infty, \alpha]$, and define $F\colon \mathbb{R} \to \mathbb{R}$ by

$$F(\alpha) = \int_{E_\alpha} f(x)\,\mathrm{d}x.$$

Let us show that F is continuous at every $\alpha \in \mathbb{R}$. Let $\varepsilon > 0$. Applying the dominated convergence theorem with $|f| \cdot \mathbb{1}_E$ as the dominating function, we obtain $N > 0$ such that

$$\int_{\mathbb{R}} |f(x)| \cdot \mathbb{1}_{E \cap [\alpha - 1/N, \alpha + 1/N]}\,\mathrm{d}x < \varepsilon;$$

hence, whenever $|\delta| < 1/N$, we have by monotonicity that $|F(\alpha+\delta) - F(\alpha)| < \varepsilon$. Now, F is continuous on all of $\mathbb{R}$, and we know that $\lim_{\alpha \to -\infty} F(\alpha) = 0$ and $\lim_{\alpha \to \infty} F(\alpha) = \int_E f(x)\,\mathrm{d}x$. Given any $t \in (0,1)$, by the intermediate value theorem, we can find some $\beta_t \in \mathbb{R}$ such that

$$F(\beta_t) = \int_{E_{\beta_t}} f(x)\,\mathrm{d}x = t \int_E f(x)\,\mathrm{d}x.$$

Taking $E_t = E_{\beta_t}$, we are done.

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2. Let $f \in L^1(\mathbb{R})$, and let $F(t) = \int_{\mathbb{R}} f(x) e^{itx}\,\mathrm{d}x$. Prove that $F\colon \mathbb{R} \to \mathbb{C}$ is continuous, and prove that

$$\lim_{t\to-\infty} F(t) = \lim_{t\to\infty} F(t) = 0.$$

Note: There is a typo in the original problem statement. The codomain of F is $\mathbb{C}$, not $\mathbb{R}$.

Solution. Let $t \in \mathbb{R}$. For every sequence $(\delta_n)_{n\geq 1}$ that converges to 0, by the dominated convergence theorem with $2|f|$ as the dominating function,

$$\lim_{n\to\infty} |F(t+\delta_n) - F(t)| \leq \lim_{n\to\infty} \int_{\mathbb{R}} |f(x)| \cdot |e^{i\delta_n x} - 1|\,\mathrm{d}x = 0.$$

This shows that F is continuous at t.

Now, because $x \mapsto f(-x)$ also belongs to $L^1(\mathbb{R})$ and because $\lim_{t\to-\infty} F(t) = \lim_{t\to\infty} F(-t)$, it suffices to prove that $\lim_{t\to\infty} F(t) = 0$. Observe that for every $u > 0$,

$$\begin{aligned} F(u) &= \int_{\mathbb{R}} f\left(x + \frac{\pi}{u}\right) e^{iu(x+\pi/u)}\,\mathrm{d}x \\ &= -\int_{\mathbb{R}} f\left(x + \frac{\pi}{u}\right) e^{iux}\,\mathrm{d}x. \end{aligned}$$

If f is continuous, then it follows from the dominated convergence theorem that

$$\begin{aligned} 2|F(u)| &= \left| \int_{\mathbb{R}} \left(f(x) - f\left(x + \frac{\pi}{u}\right)\right) e^{iux}\,\mathrm{d}x \right| \\ &\leq \int_{\mathbb{R}} \left| f(x) + f\left(x - \frac{\pi}{u}\right)\right| \mathrm{d}x \\ &\xrightarrow{u\to\infty} 0. \end{aligned}$$

Otherwise, let $\varepsilon > 0$, and let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\|f - g\|_{L^1} < \varepsilon$, so that

$$\limsup_{t\to\infty} \left| \int_{\mathbb{R}} f(x) e^{itx}\,\mathrm{d}x \right| \leq \|f - g\|_{L^1} + \lim_{t\to\infty} \left| \int_{\mathbb{R}} g(x) e^{itx}\,\mathrm{d}x \right| < \varepsilon.$$

This proves that $\lim_{t\to\infty} F(t) = 0$. We remark that this result is the *Riemann–Lebesgue lemma*.

3. Compute the following limit:

$$\lim_{n\to\infty}\int_0^n \left(1+\frac{x^2}{n}\right)^{-(n+1)} \mathrm{d}x.$$

Solution. By the observation that

$$0 \le \int_n^\infty \frac{1}{(1+x^2/n)^{n+1}}\,\mathrm{d}x \le \int_n^\infty \frac{1}{(1+x)^{n+1}}\,\mathrm{d}x = \frac{1}{n(n+1)^n},$$

we find that the given limit is equal to

$$\lim_{n\to\infty}\int_0^\infty \frac{1}{(1+x^2/n)^{n+1}}\,\mathrm{d}x.$$

Then, by the bounded convergence theorem, we have that

$$\lim_{n\to\infty}\int_0^\infty \frac{1}{(1+x^2/n)^{n+1}}\,\mathrm{d}x = \int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

4. Let $f \in L^1(\mathbb{R})$, and consider the maximal function

$$\mathrm{M}f(x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(t)|\,\mathrm{d}t.$$

Prove that there exists a constant $A > 0$ such that for every $\alpha > 0$,

$$m(\{x \in \mathbb{R} : \mathrm{M}f(x) > \alpha\}) \le \frac{A}{\alpha} \|f\|_{L^1}.$$

Solution. In other words, we are asked to prove the Hardy–Littlewood maximal inequality. Write $E_\alpha = \{x \in \mathbb{R} : \mathrm{M}f(x) > \alpha\}$, and write $R_h(x) = (1/(2h)) \int_{x-h}^{x+h} |f(t)|\,\mathrm{d}t$. For each $x \in E_\alpha$, there exists $h_x > 0$ such that $R_{h_x}(x) > \alpha$. Now, let $c < m(E_\alpha)$, and, by the inner regularity of the Lebesgue measure, let K be a compact subset of E_α such that $c < m(K)$. Then, the collection $\{\mathrm{Ball}(x, h_x) : x \in E_\alpha\}$ is an open cover of K, so it includes a finite subcover $\mathscr{B} = \{B_1, \ldots, B_n\}$ of K. Let us choose a subcollection of $\mathscr{B}$ consisting of disjoint open balls as follows:

1. Let A_1 be one of the B_i with the greatest measure;

j. For each $j \ge 2$, let A_j be one of the B_i with the greatest measure among all B_i disjoint from $A_1 \cup \cdots \cup A_{j-1}$, until there is no such B_i.

As a result, we have that $c < m(K) \le \sum_{i=1}^n m(B_i) \le 3 \sum_{j\ge1} m(A_j)$. Writing $A_j = \mathrm{Ball}(x_j, h_{x_j})$, it follows that

$$\begin{aligned}
c &\le 3 \sum_{j\ge1} 2h_{x_j} \\
&\le 3 \sum_{j\ge1} \frac{1}{\alpha} \int_{x_j-h_j}^{x_j+h_j} |f(t)|\,\mathrm{d}t \\
&\le \frac{3}{\alpha} \int_{\mathbb{R}} |f(t)|\,\mathrm{d}t \\
&= \frac{3}{\alpha} \|f\|_{L^1}.
\end{aligned}$$

Because $c < m(E_\alpha)$ is arbitrary, we conclude that $m(E_\alpha) \le (3/\alpha)\|f\|_{L^1}$.