Qualifying Exam: Real Analysis

Unofficial solutions by Alex Fu[*]

Fall 2021

Answer all four questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem, you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let f be a Lebesgue-integrable function on $\mathbb{R}^d$. Prove that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$m(A) < \delta \implies \int_A |f(x)| \, \mathrm{d}m(x).$$

Solution. Write $E_n = \{x \in \mathbb{R}^d : |f(x)| > n\}$. By the dominated convergence theorem, there exists N such that

$$\int_{E_N} |f| \, \mathrm{d}m < \frac{\varepsilon}{2}.$$

Let $\delta = \varepsilon/(2N)$. Then, whenever A is a measurable set such that $m(A) < \delta$, we have that

$$\begin{aligned}
\int_A |f| \, \mathrm{d}m &= \int_{A \cap E_N} |f| \, \mathrm{d}m + \int_{A \setminus E_N} |f| \, \mathrm{d}m \\
&\le \int_{E_N} |f| \, \mathrm{d}m + N \cdot m(A \setminus E_N) \\
&< \frac{\varepsilon}{2} + N \cdot \delta \\
&= \varepsilon.
\end{aligned}$$

Remark. What we have proven is that the measure $A \mapsto \int_A |f| \, \mathrm{d}m$ is absolutely continuous with respect to m.

[*]Reach out to me at alexfu.math@usc.edu for any questions, comments, or corrections :)

2. Let f be a Lebesgue-integrable function on $\mathbb{R}$. Prove that

$$\lim_{h\to 0}\int_{\mathbb{R}} |f(x+h)-f(x)|\,\mathrm{d}x = 0.$$

Solution. If f is continuous, then we are done after applying the dominated convergence theorem with $2|f|$ as the dominating function. Otherwise, let $\varepsilon > 0$, and let g be a continuous function such that $\|f-g\|_{L^1} < \varepsilon/2$. By the triangle inequality and the continuity of g, we find that

$$\limsup_{h\to 0}\int_{\mathbb{R}} |f(x+h)-f(x)|\,\mathrm{d}x \le \limsup_{h\to 0}\int_{\mathbb{R}} |f(x+h)-g(x+h)|\,\mathrm{d}x + \int_{\mathbb{R}} |g(x)-f(x)|\,\mathrm{d}x < \varepsilon.$$

We conclude that $\lim_{h\to 0}\int_{\mathbb{R}} |f(x+h)-f(x)|\,\mathrm{d}x = 0$.

3. Let f and $f_1, f_2, \dots$ be Lebesgue-measurable functions on $\mathbb{R}$, and suppose that there exists a constant $C \geq 0$ such that for every $n \geq 1$,

$$\int_{\mathbb{R}} |f(x) - f_n(x)|\,\mathrm{d}x \leq \frac{C}{n^2}.$$

Prove that $f_n \to f$ almost everywhere as $n \to \infty$.

Solution. Let $g(x) = \limsup_{n\to\infty} |f(x) - f_n(x)|$. For each $k \geq 2$, we find that

$$\begin{aligned}
\int_{\mathbb{R}} g(x)\,\mathrm{d}x &\leq \int_{\mathbb{R}} \sup_{n\geq k} |f(x) - f_n(x)|\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}} \sum_{n=k}^{\infty} |f(x) - f_n(x)|\,\mathrm{d}x \\
&= \sum_{n=k}^{\infty} \int_{\mathbb{R}} |f(x) - f_n(x)|\,\mathrm{d}x \\
&\leq C \sum_{n=k}^{\infty} \frac{1}{n^2} \\
&\leq \frac{C}{k-1}.
\end{aligned}$$

It follows that $\int_{\mathbb{R}} g(x)\,\mathrm{d}x = 0$. Because g is a nonnegative function, we conclude that $g = 0$ almost everywhere, i.e., $f_n \to f$ almost everywhere.

4. Let f be a Lebesgue-measurable function on $\mathbb{R}$ such that $f(x) > 0$ for almost every $x \in \mathbb{R}$, and let $(E_k)_{k\geq 1}$ be a sequence of Lebesgue-measurable subsets of $[0,1]$ such that

$$\lim_{k\to\infty} \int_{E_k} f(x)\,\mathrm{d}x = 0.$$

Prove that $\lim_{k\to\infty} m(E_k) = 0$.

Solution. We will prove the contrapositive implication. Let $\varepsilon > 0$, and let $(E_{k_n})_{n\geq 1}$ be a subsequence such that $m(E_{k_n}) \geq \varepsilon$ for every n. Suppose without loss of generality that $E_{k_n} = E_n$. Define $F_j = \{x \in [0,1] : f(x) > 1/j\}$, so that $\{x \in [0,1] : f(x) > 0\} = \bigcup_{j=1}^{\infty} F_j$. By continuity from below, there exists $J \geq 1$ such that $m(F_J) \geq 1 - \varepsilon/2$. Then, for every n, we must have that $m(E_n \cap F_J) \geq \varepsilon/2$; otherwise, we would find the contradiction that $m([0,1]) > 1$. Therefore, for every $n \geq 1$,

$$\int_{E_n} f(x)\,\mathrm{d}x \geq \int_{E_n \cap F_J} f(x)\,\mathrm{d}x \geq \frac{1}{J} \cdot m(E_n \cap F_J) \geq \frac{\varepsilon}{2J},$$

which is positive. It follows that the sequence $(\int_{E_n} f(x)\,\mathrm{d}x)_{n\geq 1}$ cannot converge to 0, and we conclude that

$$m(E_k) \not\to 0 \implies \int_{E_k} f(x)\,\mathrm{d}x \not\to 0.$$

Remark. This is problem 1 on the spring 2023 exam.