Qualifying Exam: Applied Probability

Unofficial solutions by Alex Fu*

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Answer all three questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem, you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

1. Let X_1 and X_2 be independent random variables with distributions Poisson(λ_1) and Poisson(λ_2) respectively.

   a. Find $\mathbb{P}(X_1 = k \mid X_1 + X_2 = n)$ for $0 \le k \le n$.

   Solution. By direct computation,

$$\begin{aligned}
\mathbb{P}(X_1 = k \mid X_1 + X_2 = n) &= \frac{\mathbb{P}(X_1 = k, X_1 + X_2 = n)}{\mathbb{P}(X_1 + X_2 = n)} \\
&= \frac{\mathbb{P}(X_1 = k, X_2 = n-k)}{\sum_{\ell=0}^{n} \mathbb{P}(X_1 = \ell, X_2 = n-\ell)} \\
&= \frac{\mathbb{P}(X_1 = k) \cdot \mathbb{P}(X_2 = n-k)}{\sum_{\ell=0}^{n} \mathbb{P}(X_1 = \ell) \cdot \mathbb{P}(X_2 = n-\ell)} \\
&= \frac{\lambda_1^k \lambda_2^{n-k} \cancel{e^{-\lambda_1} e^{-\lambda_2}} / (k!(n-k)!)}{\sum_{\ell=0}^{n} \lambda_1^{\ell} \lambda_2^{n-\ell} \cancel{e^{-\lambda_1} e^{-\lambda_2}} / (\ell!(n-\ell)!)} \\
&= \frac{\binom{n}{k} \lambda_1^k \lambda_2^{n-k}}{\sum_{\ell=0}^{n} \binom{n}{\ell} \lambda_1^{\ell} \lambda_2^{n-\ell}} \\
&= \frac{\binom{n}{k} \lambda_1^k \lambda_2^{n-k}}{(\lambda_1 + \lambda_2)^n} \\
&= \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k}.
\end{aligned}$$

   Remark. What we have found, in other words, is that the conditional distribution of X_1 given that $X_1 + X_2 = n$ is Binomial(n, p) with $p = \lambda_1 / (\lambda_1 + \lambda_2)$. By symmetry, it follows that the conditional distribution of X_2 given that $X_1 + X_2 = n$ is Binomial($n, 1-p$).

   b. Find $\mathbb{E}(X_1^2 + X_2^2 \mid X_1 + X_2 = n)$.

   Solution. Let us first compute $\mathbb{E}(X_1^2 \mid X_1 + X_2 = n)$. Write $p = \lambda_1 / (\lambda_1 + \lambda_2)$. Because direct computation

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is difficult, we will apply a common trick used when the probability mass function involves factorials:

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\begin{aligned}
\mathbb{E}(X_1(X_1-1) \mid X_1+X_2=n) &= \sum_{k=0}^{n} k(k-1)\cdot \mathbb{P}(X_1=k \mid X_1+X_2=n) \\
&= \sum_{k=2}^{n} k(k-1)\cdot \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= n(n-1) \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} p^k (1-p)^{n-k} \\
&= n(n-1)p^2 \sum_{k=2}^{n} \binom{n-2}{k-2} p^{k-2} (1-p)^{(n-2)-(k-2)} \\
&= n(n-1)p^2, \\
\mathbb{E}(X_1 \mid X_1+X_2=n) &= \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} \\
&= np, \\
\mathbb{E}(X_1^2 \mid X_1+X_2=n) &= \mathbb{E}(X_1(X_1-1) \mid X_1+X_2=n) + \mathbb{E}(X_1 \mid X_1+X_2=n) \\
&= n(n-1)p^2 + np.
\end{aligned}
$$

By symmetry, we have that

$$
\mathbb{E}(X_2^2 \mid X_1+X_2=n) = n(n-1)(1-p)^2 + n(1-p).
$$

By the linearity of expectation, we conclude that

$$
\begin{aligned}
\mathbb{E}(X_1^2+X_2^2 \mid X_1+X_2=n) &= [n(n-1)p^2+np] + [n(n-1)(1-p)^2 + n(1-p)] \\
&= 2n^2p^2 - 2n^2p - 2np^2 + n^2 + 2np.
\end{aligned}
$$

Remark. Of course, if you remember the formula for the second moment of a binomial random variable, then you can save yourself most of the work.

2. Let X and Y be independent random variables with distributions Exponential(μ) and Exponential(λ) respectively. Let $U = \max\{X, Y\}$ and $V = \min\{X, Y\}$.

   a. Find $\mathbb{E}(U)$ and $\mathbb{E}(V)$.

      Solution. By the tail-sum formula,

$$\begin{aligned}
\mathbb{E}(V) &= \int_0^\infty \mathbb{P}(V > v)\,\mathrm{d}v \\
&= \int_0^\infty \mathbb{P}(X > v, Y > v)\,\mathrm{d}v \\
&= \int_0^\infty \mathbb{P}(X > v)\cdot\mathbb{P}(Y > v)\,\mathrm{d}v \\
&= \int_0^\infty e^{-\mu v}\cdot e^{-\lambda v}\,\mathrm{d}v \\
&= \frac{1}{\mu+\lambda}.
\end{aligned}$$

      By the identity $X + Y = U + V$ and by the linearity of expectation, it follows that

$$\begin{aligned}
\mathbb{E}(U) &= \mathbb{E}(X) + \mathbb{E}(Y) - \mathbb{E}(V) \\
&= \frac{1}{\mu} + \frac{1}{\lambda} - \frac{1}{\mu+\lambda}.
\end{aligned}$$

      Remark. In the solution above, we actually found that $\mathbb{P}(V > v) = e^{-(\mu+\lambda)v}$ for every $v \geq 0$; in other words, V has distribution Exponential($\mu+\lambda$).

   b. Find $\operatorname{cov}(U, V)$.

      Hint: This requires no integration.

      Solution. By the observation that $UV = XY$, we find that

$$\begin{aligned}
\mathbb{E}(UV) &= \mathbb{E}(XY) \\
&= \mathbb{E}(X)\cdot\mathbb{E}(Y) \\
&= \frac{1}{\mu\lambda}, \\
\operatorname{cov}(U,V) &= \mathbb{E}(UV) - \mathbb{E}(U)\cdot\mathbb{E}(V) \\
&= \frac{1}{\mu\lambda} - \frac{1}{\mu(\mu+\lambda)} - \frac{1}{\lambda(\mu+\lambda)} + \frac{1}{(\mu+\lambda)^2}.
\end{aligned}$$

   c. Find the probability density function, $f_Z(z)$, of $Z = V/U$.

      Solution. For $0 < z \leq 1$, let us compute

$$\begin{aligned}
f_Z(z) &= \int_0^\infty f_{Z,U}(z,u)\,\mathrm{d}u \\
&= \int_0^\infty |u|\cdot f_{V,U}(uz,u)\,\mathrm{d}u,
\end{aligned}$$

      where, as a reminder, u is the Jacobian determinant of the linear transformation $(v,u) \mapsto (z,u)$:

$$\left|\det\begin{bmatrix} \partial v/\partial z & \partial v/\partial u \\ \partial u/\partial z & \partial u/\partial u \end{bmatrix}\right| = \left|\det\begin{bmatrix} u & z \\ 0 & 1 \end{bmatrix}\right| = |u|.$$

Now, the joint distribution of V and U is given by

$$\begin{aligned}
f_{V,U}(v,u) &= f_{X,Y}(v,u) + f_{X,Y}(u,v)\\
&= f_X(v)\cdot f_Y(u) + f_X(u)\cdot f_Y(v)\\
&= \mu\lambda(e^{-(\mu v+\lambda u)} + e^{-(\mu u+\lambda v)})
\end{aligned}$$

whenever $v < u$. Thus, for $0 < z < 1$, we find that

$$\begin{aligned}
f_Z(z) &= \int_0^\infty |u|\cdot\mu\lambda(e^{-(\mu uz+\lambda u)} + e^{-(\mu u+\lambda uz)})\,\mathrm{d}u\\
&= \mu\lambda\int_0^\infty ue^{-(\mu z+\lambda)u} + ue^{-(\mu+\lambda z)u}\,\mathrm{d}u\\
&= \frac{\mu\lambda}{(\mu z+\lambda)^2} + \frac{\mu\lambda}{(\mu+\lambda z)^2}.
\end{aligned}$$

For $z = 1$, we find that the probability density is halved:

$$\begin{aligned}
f_Z(1) &= \int_0^\infty u\cdot\mu\lambda e^{-(\mu+\lambda)u}\,\mathrm{d}u\\
&= \frac{\mu\lambda}{(\mu+\lambda)^2}.
\end{aligned}$$

3. Let $n \geq 2$, and let $X_1, \ldots, X_n$ be i.i.d. Uniform($[0,1]$) random variables. Let A denote the number of *ascents* in the sequence $(X_1, \ldots, X_n)$:

$$A = \#\{1 \leq i \leq n-1 : X_i < X_{i+1}\}.$$

Similarly, let D denote the number of *descents* in $(X_1, \ldots, X_n)$:

$$D = \#\{1 \leq j \leq n-1 : X_j > X_{j+1}\}.$$

Note: Your answers to the following questions should be functions of n.

Hint: This problem requires no integration at all.

a. Find $\mathbb{P}(A = 0)$, and find $\mathbb{E}(A)$.

Solution. By the symmetry of all $n!$ possible orderings of $X_1, \ldots, X_n$, we have that

$$\mathbb{P}(A = 0) = \mathbb{P}(X_1 > \cdots > X_n) = \frac{1}{n!}.$$

Now, because the given random variables are i.i.d. and continuous, we know that with probability one, $X_1, \ldots, X_n$ take on n distinct values. Consequently, with probability one, $A + D = n - 1$. Thus, by symmetry again, we find that

$$\mathbb{E}(A) = \frac{n-1}{2}.$$

Remark. One way to see the second instance of symmetry is to consider $1 - X_1, \ldots, 1 - X_n$.

b. Find $\mathbb{P}(A = 1 \mid X_1 < X_2)$.

Solution. If $n = 2$, then $\mathbb{P}(A = 1 \mid X_1 < X_2) = 1$. Otherwise, $n \geq 3$, and

$$\begin{aligned}
\mathbb{P}(A = 1 \mid X_1 < X_2) &= \mathbb{P}(X_2 > \cdots > X_n \mid X_1 < X_2) \\
&= \frac{\mathbb{P}(X_1 < X_2, X_2 > \cdots > X_n)}{\mathbb{P}(X_1 < X_2)} \\
&= \frac{(n-1)/n!}{1/2} \\
&= \frac{2(n-1)}{n!}.
\end{aligned}$$

(It turns out that this formula holds for all $n \geq 2$.)

c. Find $\mathbb{P}(X_i < X_{i+1}, X_j > X_{j+1})$ for all i and j.

Solution. If $i = j$, then $\mathbb{P}(X_i < X_{i+1}, X_j > X_{j+1}) = 0$. Otherwise, take $i < j$ without loss of generality. In the particular case where $i + 1 = j$,

$$\mathbb{P}(X_i < X_{i+1}, X_j > X_{j+1}) = \mathbb{P}(X_{j-1} < X_j, X_j > X_{j+1}) = \frac{2}{3!} = \frac{1}{3}.$$

In the remaining case where $i + 1 < j$, we have by independence that

$$\mathbb{P}(X_i < X_{i+1}, X_j > X_{j+1}) = \mathbb{P}(X_i < X_{i+1}) \cdot \mathbb{P}(X_j > X_{j+1}) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

To summarize, for $1 \leq i, j \leq n-1$,

$$\mathbb{P}(X_i < X_{i+1}, X_j > X_{j+1}) = \begin{cases} 0 & \text{if } i = j, \\ 1/3 & \text{if } i + 1 = j \text{ or } i - 1 = j, \\ 1/4 & \text{otherwise.} \end{cases}$$

d. Find $\operatorname{cov}(A, D)$.

Solution. Observe that we can write A and D as sums of $n-1$ indicator functions. By the bilinearity of covariance, we have that

$$
\begin{aligned}
\operatorname{cov}(A, D) &= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \operatorname{cov}(\mathbb{1}\{X_i < X_{i+1}\}, \mathbb{1}\{X_j > X_{j+1}\}) \\
&= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \mathbb{P}(X_i < X_{i+1}, X_j > X_{j+1}) - \mathbb{P}(X_i < X_{i+1}) \cdot \mathbb{P}(X_j > X_{j+1}),
\end{aligned}
$$

where $n-1$ terms correspond to the case of $i = j$ and $2(n-2)$ terms correspond to the case of $i+1 = j$ or $i-1 = j$. Therefore, by part (c), we find that

$$
\begin{aligned}
\operatorname{cov}(A, D) &= (n-1)\left(0 - \frac{1}{4}\right) + 2(n-2)\left(\frac{1}{3} - \frac{1}{4}\right) \\
&= -\frac{n}{12} - \frac{1}{12}.
\end{aligned}
$$