Qualifying Exam: Applied Probability

Unofficial solutions by Alex Fu*

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Answer all three questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem, you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

1. Suppose that each of k jobs is assigned at random to one of servers 1, 2, 3, and 4. Here, "at random" means that there are 4^k equally likely outcomes.

   Let E be the event that every server gets at least one job. Find $\mathbb{P}(E)$.

   Solution. For each $i \in \{1,2,3,4\}$, let A_i be the event that server i gets no jobs. Then E is the complement of the event $A_1 \cup A_2 \cup A_3 \cup A_4$, and by the principle of inclusion-exclusion,

$$\begin{aligned}\mathbb{P}(E) &= 1 - \mathbb{P}(A_1 \cup A_2 \cup A_3 \cup A_4) \\ &= 1 - \sum_{j=1}^{4} (-1)^{j-1} \binom{4}{j} \mathbb{P}(A_1 \cap \cdots \cap A_j) \\ &= 1 - \sum_{j=1}^{4} (-1)^{j-1} \binom{4}{j} \left(1 - \frac{j}{4}\right)^k \\ &= \sum_{j=0}^{4} (-1)^{j} \binom{4}{j} \left(1 - \frac{j}{4}\right)^k .\end{aligned}$$

   Remark. This is essentially the same question as Problem 1(a) on the Fall 2022 exam.

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2. For each $n \geq 1$, let X_n be a random variable with distribution Poisson(n). Find a sequence of constants $(a_n)_{n\geq 1}$ such that $\sqrt{X_n} - a_n$ converges in distribution as $n \to \infty$, and find the limiting distribution.

   Hint: What is the limit in distribution of $(X_n - n)/\sqrt{n}$?

   Solution. Let us follow the hint. Let $\xi_1, \xi_2, \ldots$ be i.i.d. Poisson(1) random variables, and write $W_n = \sum_{i=1}^n \xi_i$, so that W_n has distribution Poisson(n) for all n. Then, by the central limit theorem,

$$\frac{W_n - n}{\sqrt{n}} \xrightarrow{\mathsf{d}} \text{Normal}(0, 1).$$

   It follows that the same is true for $(X_n - n)/\sqrt{n}$. Now, write $Y_n = X_n/n$, so that

$$\sqrt{n}(Y_n - 1) = \frac{X_n - n}{\sqrt{n}} \xrightarrow{\mathsf{d}} \text{Normal}(0, 1).$$

   Applying the delta method using the square root function, which has a nonzero derivative of $1/2$ at the input 1, we find that we can take $a_n = n$:

$$\sqrt{n}(\sqrt{Y_n} - \sqrt{1}) = \sqrt{X_n} - \sqrt{n} \xrightarrow{\mathsf{d}} \text{Normal}\left(0, \frac{1}{4}\right).$$

3. Let U and V be independent random variables with distribution Normal$(0,1)$.

   a. Given constants $\mu_X, \mu_Y \in \mathbb{R}$, $\sigma_X, \sigma_Y > 0$, and $\rho \in [-1,1]$, find constants a, b, c such that if

$$X = \mu_X + aU,$$
$$Y = \mu_Y + bU + cV,$$

   then (X,Y) is multivariate normal with covariance matrix

$$\begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}.$$

   Solution. Since X and Y are linear combinations of the independent normal random variables U and V, we know that (X,Y) is multivariate normal. The covariance matrix of (X,Y) is

$$\begin{bmatrix} \operatorname{var}(X) & \operatorname{cov}(X,Y) \\ \operatorname{cov}(X,Y) & \operatorname{var}(Y) \end{bmatrix},$$

   where, by the given assumption of independence,

$$\begin{aligned}
\operatorname{var}(X) &= \operatorname{var}(aU) \\
&= a^2 \operatorname{var}(U) \\
&= a^2, \\
\operatorname{cov}(X,Y) &= \operatorname{cov}(aU, bU + cV) \\
&= \operatorname{cov}(aU, bU) + \operatorname{cov}(aU, cV) \\
&= ab \operatorname{cov}(U,U) + ac \operatorname{cov}(U,V) \\
&= ab, \\
\operatorname{var}(Y) &= \operatorname{var}(bU) + \operatorname{var}(cV) \\
&= b^2 + c^2.
\end{aligned}$$

   Thus, to obtain the desired covariance matrix, we can take the constants to be

$$\begin{cases} a = \sigma_X, \\ b = \rho\sigma_Y, \\ c = \sqrt{1-\rho^2}\,\sigma_Y. \end{cases}$$

   b. Find the conditional density of Y given $X = x$.

   Hint: Use the definition of X and Y directly.

   Solution. We know that the conditional distribution of Y given $X = x$ is normal. By direct computation,

$$\begin{aligned}
\mathbb{E}(Y \mid X = x) &= \mu_Y + \mathbb{E}(bU \mid X = x) + \mathbb{E}(cV \mid X = x) \\
&= \mu_Y + \mathbb{E}(bU \mid \mu_X + aU = x) + \mathbb{E}(cV) \\
&= \mu_Y + \frac{b}{a}(x - \mu_X), \\
\operatorname{var}(Y \mid X = x) &= \operatorname{var}\left(\mu_Y + \frac{b}{a}(x - \mu_X) + cV \mid X = x\right) \\
&= \operatorname{var}(cV \mid X = x) \\
&= \operatorname{var}(cV) \\
&= c^2.
\end{aligned}$$

Therefore, we have that $Y \mid \{X = x\} \sim \text{Normal}(\mu_Y + b(x-\mu_X)/a, c^2)$, i.e.,

$$f_{Y|X}(y \mid x) = \frac{1}{\sqrt{2\pi}c} \exp\left\{-\frac{(y-\mu_Y - b(x-\mu_X)/a)^2}{2c^2}\right\}$$
$$= \frac{1}{\sqrt{2\pi(1-\rho^2)}\sigma_Y} \exp\left\{-\frac{(y-\mu_Y - \rho\sigma_Y(x-\mu_X)/\sigma_X)^2}{2(1-\rho^2)\sigma_Y^2}\right\}.$$

c. Let $\xi_1, \xi_2, \ldots$ be i.i.d. Normal(0,1) random variables, and write $S_n = \sum_{i=1}^n \xi_i$. For $m < n$ and for all s, find the conditional density of S_m given $S_n = s$.

Solution. Similarly to part (b), we know that the conditional distribution of S_m given $S_n = s$ is normal, and we find its parameters as follows. Because $n \cdot \mathbb{E}(\xi_1 \mid S_n = s) = s$, we find that

$$\mathbb{E}(S_m \mid S_n = s) = \frac{m}{n} s.$$

Now, write $\Delta = S_m - (m/n)S_n$; observe that (Δ, S_n) is multivariate normal and

$$\text{cov}(\Delta, S_n) = \text{cov}(S_m, S_n) - \frac{m}{n}\text{var}(S_n) = 0,$$

so Δ and S_n are independent. It follows that

$$\begin{aligned}
\text{var}(S_m \mid S_n = s) &= \text{var}(\Delta \mid S_n = s) \\
&= \text{var}(\Delta) \\
&= \text{var}\left(\frac{n-m}{n} S_m\right) + \text{var}\left(\frac{m}{n}(S_n - S_m)\right) \\
&= \frac{(n-m)^2 m}{n^2} + \frac{m^2(n-m)}{n^2} \\
&= \frac{m(n-m)}{n}.
\end{aligned}$$

Therefore, we conclude that

$$f_{S_m|S_n}(t \mid s) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{n}{m(n-m)}} \exp\left\{-\frac{1}{2}\frac{n}{m(n-m)}\left(t - \frac{m}{n}s\right)^2\right\}.$$