Qualifying Exam: Applied Probability

Unofficial solutions by Alex Fu*

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Answer all three questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem, you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

1. a. A standard deck of 52 cards has 13 cards of each suit: spades, hearts, diamonds, and clubs. A *bridge hand* consists of 13 cards randomly chosen from the deck without replacement, and we say that a bridge hand is *void* in a suit if it contains no cards of that suit.

   Find the probability that a bridge hand is void in at least one suit.

   Solution. Label the suits by 1, 2, 3, and 4, and let A_i be the event that the given bridge hand is void in suit i. Then, by the principle of inclusion-exclusion,

$$\begin{aligned}\mathbb{P}(A_1 \cup A_2 \cup A_3 \cup A_4) &= \sum_{j=1}^{4} (-1)^{j-1} \sum_{n_1 < \cdots < n_j} \mathbb{P}(A_{n_1} \cap \cdots \cap A_{n_j}) \\ &= \sum_{j=1}^{4} (-1)^{j-1} \binom{4}{j} \left(1 - \frac{j}{4}\right)^{13}.\end{aligned}$$

   b. Let A be a set with 10 elements, let B be a set with 4 elements, and let Φ be a random map from A to B chosen uniformly at random from all possible maps. Show that

$$\mathbb{P}(\Phi \text{ is not a surjection}) \le 4 \cdot \left(\frac{3}{4}\right)^{10}.$$

   Solution. Label the elements of B by 1, 2, 3, and 4, and let B_i be the event that i does not belong to the image of Φ. Then, by the union bound,

$$\begin{aligned}\mathbb{P}(\Phi \text{ is not a surjection}) &= \mathbb{P}(B_1 \cup B_2 \cup B_3 \cup B_4) \\ &\le \mathbb{P}(B_1) + \mathbb{P}(B_2) + \mathbb{P}(B_3) + \mathbb{P}(B_4) \\ &= 4 \cdot \left(\frac{3}{4}\right)^{10}.\end{aligned}$$

   c. Find $\mathbb{P}(\Phi \text{ is not a surjection})$.

   Solution. By the principle of inclusion-exclusion, as in part (a),

$$\mathbb{P}(\Phi \text{ is not a surjection}) = \sum_{j=1}^{4} (-1)^{j-1} \binom{4}{j} \left(1 - \frac{j}{4}\right)^{10}.$$

*Reach out to me at alexfu.math@usc.edu for any questions, comments, or corrections :)

2. Let $n \geq 5$, and let $X_1, \dots, X_n$ be i.i.d. random variables that take values uniformly at random in $\{0, \dots, 9\}$. Let N be the number of indices i for which $X_i = X_{i+1} = X_{i+2}$.

Find the mean and variance of N.

Note. For example, if $n = 6$ and $(X_1, \dots, X_n) = (7, 4, 4, 4, 4, 2)$, then $N = 2$: there are two occurrences of $(4, 4, 4)$.

Solution. Let A_i denote the event that $X_i = X_{i+1} = X_{i+2}$, so that

$$N = \sum_{i=1}^{n-2} \mathbb{1}_{A_i}.$$

Then, by the linearity of expectation,

$$\mathbb{E}(N) = \sum_{i=1}^{n-2} \mathbb{P}(A_i) = \frac{n-2}{100}.$$

To compute $\mathrm{var}(N)$ requires a bit more work. Observe that $\mathbb{E}(N^2) = \sum_i \sum_j \mathbb{P}(A_i \cap A_j)$, where:

- If $i = j$, then $\mathbb{P}(A_i \cap A_j) = 1/10^2$. There are $n-2$ such terms.
- If $i+1 = j$ or $i-1 = j$, then $\mathbb{P}(A_i \cap A_j) = 1/10^3$. There are $2(n-3)$ such terms.
- If $i+2 = j$ or $i-2 = j$, then $\mathbb{P}(A_i \cap A_j) = 1/10^4$. There are $2(n-4)$ such terms.
- Otherwise, by independence, $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) \cdot \mathbb{P}(A_j) = 1/10^4$. There are $(n-4)(n-5)$ such terms.

Putting everything together, we find that

$$\begin{aligned}
\mathbb{E}(N^2) &= \frac{n-2}{10^2} + \frac{2(n-3)}{10^3} + \frac{(n-3)(n-4)}{10^4} \\
&= \frac{n^2 + 113n - 248}{10^4}, \\
\mathrm{var}(N) &= \mathbb{E}(N^2) - \mathbb{E}(N)^2 \\
&= \frac{117n - 252}{10^4}.
\end{aligned}$$

3. Let $U_1, U_2, \ldots$ be i.i.d. Uniform$([0,1])$ random variables, and, for each $n \geq 1$, define X_n as the second smallest of the values $U_1, \ldots, U_n$.

   a. For each $t \in [0,1]$, find $\mathbb{P}(X_n > t)$.

   Solution. The cumulative distribution function of the second order statistic of $U_1, \ldots, U_n$ is given by

$$F(t) = \sum_{k=2}^{n} \binom{n}{k} \mathbb{P}(U_1 \leq t)^k \, \mathbb{P}(U_1 > t)^{n-k},$$

   so its complement is given by

$$\begin{aligned} \mathbb{P}(X_n > t) &= \mathbb{P}(U_1 > t)^n + n\mathbb{P}(U_1 \leq t)^1 \, \mathbb{P}(U_1 > t)^{n-1} \\ &= (1-t)^n + nt(1-t)^{n-1}. \end{aligned}$$

   b. Find a sequence of constants $(c_n)_{n\geq 1}$ such that $c_n - \ln X_n$ converges in distribution as $n \to \infty$, and find the cumulative distribution function of the limit.

   Hint: Given $a \in \mathbb{R}$, what is $\lim_{n\to\infty}(1 + a/n)^n$?

   Solution. The cumulative distribution function of $c_n - \ln X_n$, which is continuous, is given by

$$\begin{aligned} \mathbb{P}(c_n - \ln X_n < u) &= \mathbb{P}(X_n > e^{c_n - u}) \\ &= (1 - e^{c_n - u})^n + ne^{c_n - u}(1 - e^{c_n - u})^{n-1}. \end{aligned}$$

   Following the hint, let us fix a constant $a > 0$ and define $c_n = \ln a - \ln n$, so that $e^{c_n} = a/n$. Then,

$$\begin{aligned} \mathbb{P}(c_n - \ln X_n < u) &= \left(1 - \frac{ae^{-u}}{n}\right)^n + ae^{-u}\left(1 - \frac{ae^{-u}}{n}\right)^{n-1} \\ &\to (1 + ae^{-u}) \cdot e^{-ae^{-u}}, \end{aligned}$$

   which is a valid cumulative distribution function, meaning that $c_n - \ln X_n$ does converge in distribution. For simplicity, we can just take $a = 1$, so that

$$\begin{aligned} c_n &= -\ln n, \\ \lim_{n\to\infty} \mathbb{P}(c_n - \ln X_n \leq u) &= (1 + e^{-u}) \cdot e^{-e^{-u}} \qquad \text{for all } u. \end{aligned}$$