Fall 2021

1a) $\ell(\mu \mid x) = \det(2\pi\Sigma)^{-1/2} \exp\left(\frac{-(x-\mu)^T \Sigma^{-1} (x-\mu)}{2} \right)$

$$\lambda(x) = \frac{\ell(\mu_0 \mid x)}{\ell(\mu_1 \mid x)} = \exp\left(-\frac{1}{2}(x-\mu_0)^T \Sigma^{-1}(x-\mu) + \frac{1}{2}(x-\mu_1)^T \Sigma^{-1}(x-\mu_1)\right)$$

$$= \exp\left(x^T \Sigma^{-1}\mu_0 - x^T\Sigma^{-1}\mu_1 - \frac{1}{2}\left[\mu_0^T\Sigma^{-1}\mu_0 - \mu_1\Sigma^{-1}\mu_1\right]\right)$$

(Assume Σ, Σ^{-1} symmetric)

$$= C \exp\left((\mu_0 - \mu_1)^T \Sigma^{-1} x\right), \quad C > 0$$

reject H_0 when $\lambda(x)$ small

$\Rightarrow$ reject H_0 when $\log(\lambda(x))$ small

$$\log(\lambda(x)) = C + (\mu_0 - \mu_1)^T \Sigma^{-1} x$$

$\Rightarrow$ reject H_0 when $(\mu_0 - \mu_1)^T \Sigma^{-1} x$ small

Note $\Sigma^{1/2} Z + \mu_0 \overset{Dist}{=} X$, where $Z \sim N(0, I_n)$

so $\Sigma^{-1/2}(X - \mu_0) \overset{Dist}{=} Z$

$\Rightarrow$ reject H_0 when $(\mu_0 - \mu_1)^T \Sigma^{-1/2}\left(\Sigma^{-1/2}(X - \mu_0)\right)$ is small

$$(\mu_0 - \mu_1)^T \Sigma^{-1/2}(Z) \sim N\left(0, \|(\mu_0 - \mu_1)^T \Sigma^{-1/2}\|_2^2\right)$$

1a) cont.

$\Rightarrow$ Reject H_0 when $\dfrac{(\mu_0-\mu_1)^T \Sigma^{-1}(X-\mu_0)}{\|(\mu_0-\mu_1)\Sigma^{-1/2}\|_2}$ small,

with $T(x) := \dfrac{(\mu_0-\mu_1)^T \Sigma^{-1}(X-\mu_0)}{\|(\mu_0-\mu_1)\Sigma^{-1/2}\|_2} \sim N(0,1)$

$\mathbb{P}_{H_0}(T(x) \le z_\alpha) = \alpha$,

Reject when $T(x) \le z_\alpha$

1b) $\beta(\mu_1) = \mathbb{P}_{H_1}(\text{Reject})$

$= \mathbb{P}_{H_1}(T(x) \le z_\alpha)$

$= \mathbb{P}_{H_1}\left(\dfrac{(\mu_0-\mu_1)^T \Sigma^{-1}(X-\mu_0)}{\|\cdots\|_2} \le z_\alpha\right)$

$= \mathbb{P}_{H_1}\left(\dfrac{(\mu_0-\mu_1)^T \Sigma^{-1}(X-\mu_1+\mu_1-\mu_0)}{\|\cdots\|} \le z_\alpha\right)$

$= \mathbb{P}_{H_1}\left(\underbrace{\dfrac{(\mu_0-\mu_1)^T \Sigma^{-1}(X-\mu_1)}{\|\cdots\|_2}}_{\sim N(0,1)} \le z_\alpha + \underbrace{\dfrac{(\mu_0-\mu_1)^T \Sigma^{-1}(\mu_0-\mu_1)}{\|\cdots\|^2}}_{=\|(\mu_0-\mu_1)^T\Sigma^{-1/2}\|_2}\right)$

(1b) $\cdots = \mathbb{P}_{\mathcal{N}(0,1)}\left(Z \leq Z_\alpha + \|(\mu_0 - \mu_1)\Sigma^{-1/2}\|_2 \right)$

Assume $X = \mathcal{N}_n(\mu \mathbb{1}_n, \sigma^2 I)$, $\mathbb{1}_n$ is ones vector.

$$\beta(\mu_1) = \mathbb{P}_{\mathcal{N}(0,1)}\left(Z \leq Z_\alpha + \|(\mu_0 - \mu_1)\tfrac{1}{\sigma} I\|\right)$$

$$= \mathbb{P}_{\mathcal{N}(0,1)}\left(Z \leq Z_\alpha + \tfrac{n}{\sigma}\|\mu_0 - \mu_1\|\right)$$

1c) It is true that there exists symmetric $\Sigma^{1/2}$

s.t. $(\Sigma^{1/2})^2 = \Sigma$, with $\Sigma^{1/2}$ singular.

Let $Z \sim \mathcal{N}_n(0, I)$.

$\Sigma^{1/2} Z + \mu$ is multivariate normal.

$\mathbb{E}\, \Sigma^{1/2} Z + \mu = \mu = \mathbb{E} X$

$\operatorname{Cov}\left((\Sigma^{1/2}Z+\mu)_i, (\Sigma^{1/2}Z+\mu)_j\right) = \Sigma_{ij} = \operatorname{Cov}(X_i, X_j)$

so $\Sigma^{1/2} Z + \mu \overset{Dist}{=} X$

Now, support $(Z) = \mathbb{R}^n$

$\Rightarrow$ support $X = \text{colspace}(\Sigma^{1/2}) + \mu$

so $\text{support}_{H_0}(X) = \text{colspace}(\Sigma^{1/2}) + \mu_0 =: S_{H_0}$

$\text{support}_{H_1}(X) = \text{colspace}(\Sigma^{1/2}) + \mu_1 =: S_{H_1}$

1c) cont. Let Cond 1 Be

$$S_{H_0} \neq S_{H_1}$$

or, equivalently,

$$\text{Proj}_{\text{Col}(\Sigma^{1/2})^\perp}(\mu_0) \neq \text{Proj}_{\text{Col}(\Sigma^{1/2})^\perp}(\mu_1)$$

or, equivalently,

$$(I - \Sigma^{1/2+}\Sigma^{1/2})\mu_0 \neq (I - \Sigma^{1/2+}\Sigma^{1/2})\mu_1$$

where A^+ is the Moore-Penrose Pseudoinverse.

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suppose Condition 1

Consider the test:

test form [Reject H_0 if $X \notin S_{H_0}$

$$\alpha = \mathbb{P}_{H_0}(X \notin \text{colspace}(\Sigma^{1/2}) + \mu_0) = 0$$

$$\beta = \mathbb{P}_{H_1}(X \notin \text{colspace}(\Sigma^{1/2}) + \mu_0) = 1$$

Since $\text{colspace}(\Sigma^{1/2}) + \mu_0$ has no points in common with $\text{colspace}(\Sigma^{1/2}) + \mu_1$ under condition 1.

So a $\alpha = 0$, $\beta = 1$ test exists.

1(c) cont. suppose Condition 1 is false

A general test has the form: Reject H_0 if $X \in R$.

For an $\alpha = 0$ test:

$$\alpha = \mathbb{P}_{H_0}(X \in R) = 0$$

so $S_{H_0} \cap R$ is a null set w.r.t

the lebesgue measure on S_{H_0}.

we assume $S_{H_0} = S_{H_1}$

so $\beta = \mathbb{P}_{H_1}(X \in R) = 0$

since $S_{H_1} \cap R$ is a null set.

Thus any $\alpha = 0$ test is also $\beta = 0$

so no $\alpha = 0$, $\beta = 1$ test exists.

2a) $l(p \mid x_1, \ldots, x_n) = p^{\sum x_i} (1-p)^{n-\sum x_i} = \left(\frac{p}{1-p}\right)^{\sum x_i} (1-p)^n$

$(p \neq 1)$

$l \in C^1([0,1])$. If $\sum x_i \neq 0, n$; then set:

$$\frac{dl}{dp} = \sum x_i \, p^{\sum x_i - 1} (1-p)^{n-\sum x_i} - (n - \sum x_i)) \, p^{\sum x_i} (1-p)^{n-\sum x_i - 1} = 0$$

$$\Downarrow$$

$$\sum x_i (1-\hat{p}) - (n - \sum x_i)) \hat{p} = 0$$

$$\sum x_i - \cancel{p \sum x_i} - np + \cancel{p \sum x_i} = 0$$

$$\hat{p} = \frac{1}{n} \sum x_i, \text{ if } \sum x_i \neq 0, n$$

if $\sum x_i = 0$, $\hat{p} = 0 = \frac{1}{n} \sum x_i$

if $\sum x_i = n$, $\hat{p} = 1 = \frac{1}{n} \sum x_i$

$$\hat{p} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

2b) $\sup_{p \in [0,1]} \ell_n(p) = \frac{1}{n} \sum_{i=1}^{n} x_i$

$$\Lambda_n = \log \left(\left(\frac{\bar{X}}{p_0} \right)^{\sum x_i} \cdot \left(\frac{1-\bar{X}}{1-p} \right)^{n - \sum x_i} \right)$$

$$= \sum x_i \log \left(\frac{\bar{X}}{p_0} \right) + (n - \sum x_i) \log \left(\frac{1-\bar{X}}{1-p} \right)$$

$$= n \left(\bar{X} \log \left(\frac{\bar{X}}{p_0} \right) + (1 - \bar{X}) \log \left(\frac{1-\bar{X}}{1-p} \right) \right)$$

2c) $$2\Lambda_n = 2n\Big[(\bar{X} - p_0 + p_0)\log\Big(1 + \frac{\bar{X} - p_0}{p_0}\Big) + (1 - p_0 + p_0 - \bar{x})\log\Big(1 + \frac{p_0 - \bar{x}}{1 - p_0}\Big)\Big]$$

$$= 2n\Big[((\bar{X} - p_0) + p_0)\Big(\frac{\bar{X} - p_0}{p_0} - \frac{1}{2}\Big(\frac{\bar{X} - p_0}{p_0}\Big)^2\Big) + ((1 - p_0) + (p_0 - \bar{x}))\Big(\frac{p_0 - \bar{X}}{1 - p_0} - \frac{1}{2}\Big(\frac{p_0 - \bar{X}}{1 - p_0}\Big)^2\Big)\Big] + n\,O((\bar{X} - p_0)^3)$$

$$= 2n\Big[\frac{(\bar{X} - p_0)^2}{p_0} + \bar{X} - p_0 - \frac{1}{2}\frac{(\bar{X} - p_0)^2}{p_0} + \frac{(p_0 - \bar{x})^2}{1 - p_0} + p_0 - \bar{X} - \frac{1}{2}\frac{(\bar{X} - p_0)^2}{1 - p_0}\Big] + n\,O((\bar{X} - p_0)^3)$$

$$= n\Big(\frac{p_0(\bar{X} - p_0)^2}{p_0(1 - p_0)} + \frac{(1 - p_0)(\bar{X} - p_0)^2}{p_0(1 - p_0)}\Big) + o(1)$$

$$= n\frac{(\bar{X} - p_0)^2}{p_0(1 - p_0)} + o(1)$$

2c)
cont.

$$\mathbb{E}\, X_i = p_0$$

$$\operatorname{var} X_i = p_0(1-p_0)$$

Note $$\frac{n(\bar{X}-p_0)^2}{p_0(1-p_0)} = \left(\frac{\sqrt{n}(\bar{X}-p_0)}{\sqrt{p_0(1-p_0)}}\right)^2$$

By CLT: $$\frac{\sqrt{n}(\bar{X}-p_0)}{\sqrt{p_0(1-p_0)}} \xrightarrow{D} N(0,1)$$

so $$\frac{n(\bar{X}-p_0)^2}{p_0(1-p_0)} \xrightarrow{D} \chi^2(1)$$

Spring 2021

1.1) $$G_n(x) = \frac{\sup_{\lambda=\lambda_0} L(\lambda|X)}{\sup L(\lambda|X)}$$

$$L(\lambda|x) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\Sigma x_i}}{\prod_{i=1}^{n} x_i!}$$

$$\sup L(\lambda|x): \frac{d}{d\lambda}\frac{e^{-n\lambda}\lambda^{\Sigma x_i}}{\prod_{i=1}^{n} x_i!} = \frac{-n e^{-n\lambda}\lambda^{\Sigma x_i} + e^{-n\lambda}(\Sigma x_i \lambda^{\Sigma x_i - 1})}{\prod_{i=1}^{n} x_i!}$$

$$= 0$$

$$\Rightarrow -n\lambda_1 + \Sigma x_i = 0$$

$$\lambda_{n_1} = \frac{1}{n}\Sigma x_i$$

$$G_n(x) = \frac{e^{-n\lambda_0}\lambda_0^{\Sigma x_i}}{\prod_{i=1}^{n} x_i!} \cdot \frac{\prod_{i=1}^{n} x_i!}{e^{-n\bar{X}}\bar{X}^{\Sigma x_i}} = \boxed{e^{-n(\lambda_0 - \bar{X})}\left(\frac{\lambda_0}{\bar{X}}\right)^{\Sigma x_i}}$$

1.2) $-2\ln G_n(X) = 2n(\lambda_0-\bar{X}) - 2\ln\left(\frac{\lambda_0}{\bar{X}}\right)n\bar{X}$

$$= 2n(\lambda_0-\bar{X}) + 2n\bar{X}\ln\left(1+\frac{\bar{X}-\lambda_0}{\lambda_0}\right)$$

$$\overset{\text{taylor}}{\simeq} 2n(\lambda_0-\bar{X}) + 2n\bar{X}\left[\frac{\bar{X}-\lambda_0}{\lambda_0} - \frac{1}{2}\left(\frac{\bar{X}-\lambda_0}{\lambda_0}\right)^2\right]$$

$$= 2n(\lambda_0-\bar{X})\left(1-\frac{\bar{X}}{\lambda_0}\right) + \frac{\bar{X}}{\lambda_0}\left(\frac{\sqrt{n}(\bar{X}-\lambda_0)}{\sqrt{\lambda_0}}\right)^2$$

$$= \underbrace{\left(2-\frac{\bar{X}}{\lambda_0}\right)}_{\to 1}\underbrace{\left(\frac{\sqrt{n}(\bar{X}-\lambda_0)}{\sqrt{\lambda_0}}\right)^2}_{\to \chi^2(1)} \xrightarrow[D]{C.L.T} \chi^2(1)$$

1.3) Proceed as Before:

$$-2\ln(G_n(X)) \simeq \left(\frac{\bar{X}-\lambda_0}{\sqrt{\frac{\bar{X}}{n}}}\right)^2$$

$$= \left(\frac{\bar{X}+\delta/\sqrt{n}-\lambda_0-\frac{\delta}{\sqrt{n}}}{\sqrt{\frac{\bar{X}}{n}}}\right)^2$$

$$= \left(\underbrace{\frac{\bar{X}-\lambda_0-\frac{\delta}{\sqrt{n}}}{\sqrt{\frac{\bar{X}}{n}}}}_{\xrightarrow{D} N(0,1)} + \frac{\delta}{\sqrt{\bar{X}}}\right)^2$$

$$\xrightarrow{D} \text{Non-Central } \chi^2\left(1, \frac{\delta^2}{\lambda_0}\right)$$

2.1) $\mathbb{E} Z_n = n\,\mathbb{E} W_n - \frac{n-1}{n} \sum_{i=1}^{n} \mathbb{E}\left(t_{n-1}(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n\right)$

$$= n\left[\theta + \frac{a}{n}\right] - \frac{n-1}{n}[n]\left[\theta + \frac{a}{n-1}\right]$$

$$= n\theta + a - (n-1)\theta - a$$

$$= \theta$$

2.2) $\mathbb{E} Z_n = n \mathbb{E} W_n - \left(\frac{n-1}{n}\right) \sum_{i=1}^{n} \mathbb{E}\left[t_{n-1}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)\right]$

$$= n\left(\theta + \frac{a}{n} + \frac{b}{n^2} + O\left(\frac{1}{n^3}\right)\right) - \frac{n-1}{n} \cdot n\left[\theta + \frac{a}{n-1} + \frac{b}{(n-1)^2} + O\left(\frac{1}{(n-1)^3}\right)\right]$$

$$= \theta + 0 \cdot a + \left[\frac{b}{n} - \frac{b}{n-1}\right] + O\left(\frac{1}{n^3}\right)$$

$$= \theta + \frac{b(n-1)}{n(n-1)} - \frac{bn}{n(n-1)} + O\left(\frac{1}{n^3}\right)$$

$$= \theta + \frac{-1}{n(n-1)} + O\left(\frac{1}{n^3}\right)$$

$$= \theta + O\left(\frac{1}{n^2}\right)$$

2.3) $\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2 = \frac{1}{n^2}\mathbb{E}\sum_{i=1}^{n}(X_i^2) + \frac{1}{n^2}\sum_{i\neq j}\mathbb{E}X_i X_j$

$$= \frac{\theta}{n} + \frac{n(n-1)}{n^2}\theta^2$$

$$= \frac{\theta + (n-1)\theta^2}{n}$$

$$= \theta^2 + \frac{\theta - \theta^2}{n}$$

By 2.1), Z_n is unbiased.

Just in case:

$$\mathbb{E}Z_n = n\left[\theta^2 + \frac{\theta-\theta^2}{n}\right] - \frac{n-1}{n}[n]\left[\theta^2 + \frac{\theta-\theta^2}{n-1}\right]$$

$$= \theta^2 \checkmark$$

Fall 2020

1a) $\mathbb{P}(X \le x) = \mathbb{P}(x_1, \ldots, x_n \le x)$

$= \prod_{i=1}^{n} \mathbb{P}(X_i \le x)$

$= \prod_{i=1}^{n} F(x_i)$

$\frac{d}{dx} \mathbb{P}(X \le x) = \sum_{i=1}^{n} f(x_i) \prod_{j \ne i}^{n-1} F(x_j)$

$= n\, f(x)\, F(x)^{n-1} = f_n(x)$

1b) Let $m < n$.

$$\frac{f_n(x)}{f_m(x)} = \frac{n f(x) F(x)^{n-1}}{m f(x) F(x)^{m-1}}$$

$= \frac{n}{m} F(x)^{n-m}$, which is nondecreasing in $\underline{F(x)}$

1c) $\mathbb{P}(X \leq x \mid X) \Rightarrow \mathbb{1}\{X \leq x\}$, independent of n, so X sufficient stat for n.

F strictly increasing, so one-to-one, &

$\mathbb{P}(X \leq x \mid F(X))$ is also independent of n.

Thus $F(X)$ is a sufficient statistic for n.

By Karlin-Rubin, the UMP test of level α

has form: Reject if $F(x) \geq t_0$, we assume $t_0 \in$ Range $F(x)$.

$\alpha = \mathbb{P}_{\theta_0}(F(X) > t_0)$ $\qquad \underline{\theta_0 : n = 5}$

$$\begin{aligned}
1-\alpha &= \mathbb{P}_{\theta_0}(F(X) \leq t_0) \\
&= \mathbb{P}_{\theta_0}(X \leq F^{-1}(t_0)) \\
&= \prod_{i=1}^{5} F(F^{-1}(t_0)) \\
&= \prod_{i=1}^{5} t_0 = t_0^5
\end{aligned}$$

$\sqrt[5]{1-\alpha} = t_0$ $\qquad \Rightarrow$ Reject if $F(X) \geq \sqrt[5]{1-\alpha}$

d a) $g(x) = g(\mu) + g'(\mu)(x-\mu) + \frac{g''(\mu)}{2}(x-\mu)^2 + o(|\bar{X}-\mu|^3)$

since $\bar{X} \approx \mu$ for large n by LLN.

$b = \mathbb{E}\, g(\bar{X}_n) - g(\mu)$

$= \mathbb{E}\left[g'(\mu)(\bar{X}-\mu) + \frac{g''(\mu)}{2}(\bar{X}-\mu)^2 + o(|\bar{X}-\mu|^3)\right]$

Leading term: $\frac{g''(\mu)}{2}\mathbb{E}(\bar{X}-\mu)^2$

$= \frac{g''(\mu)}{2}\frac{\sigma^2}{n}$ since $\mathbb{E}\bar{X} = 0$, $\operatorname{var}\bar{X} = \frac{1}{n}\sigma^2$

2 b) $\widehat{Bias}_{Boot}(g(\bar{X}), g(M)) = \overline{g(\bar{X}^*)} - g(\bar{X})$

Suppose we have n data points

Let $\{X_i^*\}_{i=1}^{n^n}$ be the enumeration of all

n^n possible resamplings of size n, with replacements of

$\{X_1, \ldots, X_n\}$, our original sample.

$\overline{X_i^*}$ is the mean of the resample X_i^*

So $\widehat{Bias}_{Boot}(g(\bar{X}), g(M)) = \frac{1}{n^n} \sum_{i=1}^{n^n} g(\overline{X_i^*}) - g(\bar{X})$

One can also resample $B < n^n$ times instead of the full n^n, for computational feasibility.

Then $Bias_{Boot}(g(\bar{X}), g(M)) = \frac{1}{B} \sum_{i=1}^{B} g(\overline{X_i^*}) - g(\bar{X})$

2c) $\bar{X} = \frac{1}{2}(1+3) = 2$, $n = 2$.

$$X_i^* = \underset{i=1}{(1,1)},\ \underset{i=2}{(1,3)},\ \underset{i=3}{(3,1)},\ \underset{i=4}{(3,3)}$$

$$\widehat{\text{Bias}}_{\text{Boot}}(g(\bar{X}), g(\mu)) = \frac{1}{2^2}\left(\bar{X}_1^{*2} + \bar{X}_2^{*2} + \bar{X}_3^{*2} + \bar{X}_4^{*2}\right) - \bar{X}^2$$

$$= \frac{1}{4}\left(1^2 + 2^2 + 2^2 + 3^2\right) - 2^2$$

$$= \frac{18}{4} - 4$$

$$= \frac{1}{2}$$

By 2a), we know $b = \mathbb{E}\left[\frac{g''(\mu)}{2}(\bar{X}-\mu)^2\right] = \mathbb{E}(\bar{X}-\mu)^2$

$= \text{var}(\bar{X})$

$$\text{var}(\bar{X}) = \frac{1}{2}\text{var}(X) = \frac{1}{2}\left(\frac{1}{2-1}\left[(1-2)^2 + (3-2)^2\right]\right) = \underline{1}$$

2c) cont.

using Bootstrap, $g(\mu) \simeq g(\bar{x}) - b_{boot}$

$$= 4 - \tfrac{1}{2} = 3\tfrac{1}{2}$$

using 2a) $g(\mu) \simeq g(\bar{x}) - b_{2a)}$

$$= 4 - 1 = 3$$

I'd use $g(\mu) = 3$ as an estimate?

we don't have to depend on bootstrap being unbiased.

But then, we don't know if the sample variance is unbiased.

why not split the difference? $g(\mu) = 3\tfrac{1}{4}$

Spring 2020

1) a)

$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu_0)^2}{2\sigma^2}}$$

$$\ell(\mu \mid x_1, \dots, x_n) = \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$\lambda(x_1, \dots, x_n) = \frac{e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2}}{e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_1)^2}}$$

$$= e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[(x_i-\mu_0)^2-(x_i-\mu_1)^2\right]}$$

$$= e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[-2x_i\mu_0+\mu_0^2+2x_i\mu_1-\mu_1^2\right]}$$

Reject when λ small, $\lambda \le C$

$\rightarrow$ ~~when~~ $-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[-2x_i\mu_0+\mu_0^2+2x_i\mu_1-\mu_1^2\right] \le C$

$\rightarrow$ when $\sum_{i=1}^{n}\left[2x_i(\mu_1-\mu_0)\right] + n(\mu_0^2-\mu_1^2) \ge C$

$\rightarrow$ when $\begin{cases} \bar{X} \ge c_1, & \text{iff } \mu_1 > \mu_0 \\ \bar{X} \le c_1, & \text{iff } \mu_1 < \mu_0 \end{cases}$

(a)

Assuming $\mu_1 > \mu_0$

$$\bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$$

$$\mathbb{P}_{H_0}(\bar{X} \geq C_\alpha) = \alpha$$

$$\mathbb{P}_{H_0}\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \geq \frac{C_\alpha - \mu_0}{\sigma/\sqrt{n}}\right) = \alpha$$

$$Z \sim N(0,1)$$

$$\mathbb{P}_{H_0}\left(Z \geq \frac{C_\alpha - \mu_0}{\sigma/\sqrt{n}}\right) = \alpha$$

$$\frac{C_\alpha - \mu_0}{\sigma/\sqrt{n}} = Z_{1-\alpha} \quad \text{where } \mathbb{P}(Z \leq Z_\alpha) = \alpha$$

$$C_\alpha = Z_{1-\alpha}\frac{\sigma}{\sqrt{n}} + \mu_0$$

Reject if $\bar{X} \geq Z_{1-\alpha}\frac{\sigma}{\sqrt{n}} + \mu_0$

Assuming $\mu_1 < \mu_0$

$$\mathbb{P}_{H_0}(\bar{X} \leq C_\alpha) = \alpha$$

$$1 - \mathbb{P}_{H_0}(\bar{X} \geq C_\alpha) = \alpha$$

$$1 - \alpha = \mathbb{P}_{H_0}(\bar{X} \geq C_\alpha)$$

$\vdots$

$$C_\alpha = Z_\alpha \frac{\sigma}{\sqrt{n}} + \mu_0$$

Reject if $\bar{X} \leq Z_\alpha \frac{\sigma}{\sqrt{n}} + \mu_0$

Assume $\mu_1 > \mu_0$

1b) $\beta(\mu) = \mathbb{P}_{H_1}(\text{Reject } H_0)$

$= \mathbb{P}_{H_1}\left(\bar{X} \geq z_\alpha \frac{\sigma}{\sqrt{n}} + \mu_0\right)$

$= \mathbb{P}_{H_0}\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \geq z_\alpha + \frac{\sqrt{n}}{\sigma}(\mu_0 - \mu)\right)$

$= \mathbb{P}_{H_0}\left(Z \geq z_\alpha + \frac{\sqrt{n}}{\sigma}(\mu_0 - \mu)\right)$

$= 1 - F_Z\left(z_\alpha + \frac{\sqrt{n}}{\sigma}(\mu_0 - \mu)\right)$ F_Z CDF of $\mathcal{N}(0,1)$

Assume $\mu_1 < \mu_0$

$\beta(\mu) = \mathbb{P}_{H_1}(\text{Reject } H_0)$

$= \mathbb{P}_{H_1}\left(\bar{X} \leq z_{1-\alpha} \frac{\sigma}{\sqrt{n}} + \mu_0\right)$

$= \mathbb{P}_{H_1}\left(Z \leq z_{1-\alpha} + \frac{\sqrt{n}}{\sigma}(\mu_0 - \mu)\right)$

$= F_Z\left(z_{1-\alpha} + \frac{\sqrt{n}}{\sigma}(\mu_0 - \mu)\right)$

(c) $H_0: r = p$, $H_1: r = q$

$$P_n(x_1, \dots, x_n) = \prod_{i=1}^{n} p(x_i) := P_n(x)$$

$$Q_n(x_1, \dots, x_n) = \prod_{i=1}^{n} q(x_i) := Q_n(x)$$

$$\lambda(x) = \prod_{i=1}^{n} \frac{p(x_i)}{q(x_i)}$$

NP. Reject if $\lambda(x)$ small

$$\Rightarrow \log \lambda(x) = \sum \log \frac{p}{q}(x_i) \quad \text{small}$$

$$\frac{\left(\frac{1}{n}\sum \log \frac{p}{q}(x_i) - D(p\|q)\right)}{\tau_{p\|q}/\sqrt{n}} \quad \text{small}$$

$\leftarrow \sim N(0,1)$ By C.L.T.

For α-level

$$\mathbb{P}_{H_0}\left(\frac{\frac{1}{n}\sum \log \frac{p}{q} - D(p\|q)}{\tau_{p\|q}/\sqrt{n}} < z_\alpha\right) \simeq \alpha$$

$$\mathbb{P}_{H_0}\left(\log \lambda < \sqrt{n}\, z_\alpha\, \tau_{p\|q} + n D(p\|q)\right) \simeq \alpha$$

1(c) cont

$$\mathbb{P}_{H_0}\left(\lambda < \exp\left(\sqrt{n}\, z_\alpha \tau_{p\|q} + nD(p\|q)\right)\right) \approx \alpha$$

Reject when $\lambda(x) < \exp\left(\sqrt{n}\, z_\alpha \tau_{p\|q} + nD(p\|q)\right)$

for $\approx \alpha$-level test

(d) $\beta = \mathbb{P}_{H_a}\left(\log \lambda(x) < \sqrt{n}\, \tau^2_{p\|a}\, z_\alpha + n D(p\|a) \right)$

$$= \mathbb{P}_{H_a}\left(-\log \lambda(x) > -\sqrt{n}\, \tau_{p\|a}\, z_\alpha + n D(p\|a) \right)$$

$$= \mathbb{P}_{H_a}\left(\frac{-\frac{1}{n}\log \lambda(x) - D(a\|p)}{\tau_{a\|p}/\sqrt{n}} > \frac{\tau_{p\|a}}{\tau_{a\|p}} z_\alpha - \frac{D(p\|a) + D(a\|p)}{\tau_{a\|p}/\sqrt{n}} \right)$$

$$= \mathbb{P}_{H_a}\left(Z > \frac{\tau^2_{p\|a}}{\tau^2_{a\|p}} z_\alpha - \frac{D(p\|a) + D(a\|p)}{\tau^2_{a\|p}/\sqrt{n}} \right)$$

(1 b) $\log\left(\frac{p}{a}\right) = -\frac{1}{2\sigma^2}\left[-2x\mu_0 + \mu_0^2 + 2x\mu_1 - \mu_1^2 \right]$

$$D(p\|a) = \mathbb{E}\log\left(\frac{p}{a}\right) = -\frac{1}{2\sigma^2}\left[-2\mu_0^2 + \mu_0^2 + 2\mu_0\mu_1 - \mu_1^2 \right]$$

$$= \frac{(\mu_0 - \mu_1)^2}{2\sigma^2}$$

$D(a\|p) = D(p\|a)$ by symmetry

d) $\overset{\#}{\text{cont}}$

$$\tau^2_{pkn} = \text{Var}\left(-\frac{1}{2\sigma^2}\left[2X(\mu_0-\mu_1)\right]\right)$$

$$= \frac{1}{\sigma^4}(\mu_0-\mu_1)^2\,\sigma^2$$

$$= \frac{(\mu_0-\mu_1)^2}{\sigma^2}$$

$\tau^2_{alt} = \tau^2_{pkn}$ by symmetry

So: $$\beta = \mathbb{P}_{\mu=\mu_1}\left(Z > 1\,z_\alpha - \frac{(\mu_0-\mu_1)^2/\sigma^2}{(\mu_0-\mu_1)/\sigma/\sqrt{n}}\right)$$

$$= \mathbb{P}_{\mu=\mu_1}\left(Z > z_\alpha - \frac{\sqrt{n}(\mu_0-\mu_1)}{\sigma}\right)$$

\# not quite a match

which is the result from b

2)

$$f(x) = k(a-x)\,\mathbb{1}\{0 \le x \le a\}$$

$$1 = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_0^a k(a-x)\,dx$$

$$= k\left[ax - \frac{1}{2}x^2\right]\Big|_0^a$$

$$= k\left[a^2 - \frac{1}{2}a^2\right]$$

$$= k\,\frac{a^2}{2}$$

$$k = \frac{2}{a^2}$$

$$F(t) = \mathbb{P}(X \le t)$$

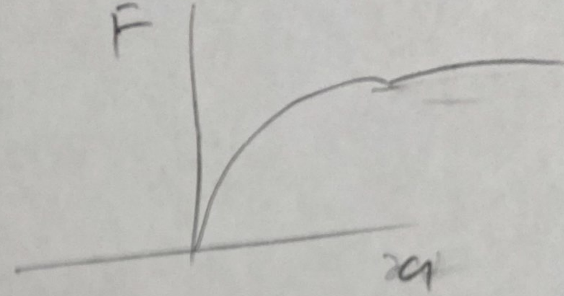
$$= \int_0^t f(x)\,dx$$

$$= \frac{2}{a^2}\left[at - \frac{1}{2}t^2\right]$$

2) cont.

Find F^{-1}

Assume $0 \le t \le a$



$$F(t) = \frac{2}{a^2}\left[-\frac{1}{2}t^2 + at\right]$$

$$= -\frac{1}{a^2}\left[t^2 - 2at + a^2 - a^2\right].$$

$$= -\frac{1}{a^2}(t-a)^2 + 1$$

$$-a^2\left[F(t) - 1\right] = (t-a)^2$$

$$a\sqrt{1-F(t)} = t - a$$

$$a\sqrt{1-F(t)} + a = t$$

$$\underline{F^{-1} = a\left(\sqrt{1-F(t)} + 1\right)}$$

our variable $\overset{D}{\sim} F^{-1}(U)$

$$\mathbb{P}(F^{-1}(U) \le t) = \mathbb{P}(U \le F(t)) = F(t) \quad \checkmark$$

1a) $\mathcal{L}(\theta|x) = \prod_{i=1}^{n} \theta e^{-\theta x_i}$ <u>Fall 2019</u>

$= \theta^n e^{-\theta \Sigma x_i}$

$\mathcal{L}$ is C^∞, supported on $(0,\infty)$

$\lim_{\theta\to\infty} \mathcal{L}(\theta, x) = 0$, $\lim_{\theta^+\to 0} \mathcal{L}(0, x) = 0$

so $\mathcal{L}$ takes max when $\frac{d\mathcal{L}}{d\theta} = 0$

$\frac{d\mathcal{L}}{d\theta} = n\theta^{n-1} e^{-\theta \Sigma x_i} + \theta^n(-\Sigma x_i) e^{-\theta \Sigma x_i} = 0$

$n - \hat{\theta} \Sigma x_i = 0$

$\boxed{\hat{\theta} = \frac{1}{\bar{x}}}$

(b)

$$\lambda(x) = \frac{\ell(\theta_0 \mid x)}{\sup_{\theta > 0} \ell(\theta, x)}$$

$$= \frac{\theta_0^n e^{-\theta_0 \Sigma x_i}}{\left(\frac{1}{\bar{x}}\right)^n e^{-\frac{\Sigma x_i}{\bar{x}}}}$$

$$= (\bar{x}\theta_0)^n e^{-(\theta_0 \Sigma x_i) + n}$$

$$= (\bar{x}\theta_0)^n e^{-n(\theta_0 \bar{x} - 1)}$$

N.P. test indicates: reject H_0 when $\lambda(x)$ small

$$-2\log(\lambda x) = -2\left[n\log(\bar{x}\theta_0) - n(\theta_0\bar{x} - 1)\right]$$

$$= -2n\left[\log(1 + (\theta_0\bar{x} - 1)) - o(\theta_0\bar{x} - 1)\right]$$

$$= -2n\left[\cancel{(\theta_0\bar{x} - 1)} - \frac{1}{2}(\theta_0\bar{x} - 1)^2 + o((\theta_0\bar{x} - 1)^2) - \cancel{(\theta_0\bar{x} - 1)}\right]$$

$$= n(\theta_0\bar{x} - 1)^n + n\,o((\theta_0\bar{x} - 1)^2)$$

$$= \left(\frac{\bar{x} - 1/\theta_0}{1/\sqrt{n\theta_0^2}}\right)^2 + n \cdot o(\cdots)$$

N.P. test $\Rightarrow$ reject when $-2\log(\lambda x)$ large.

1b) $\mathbb{E}X = \frac{1}{\theta_0}$

$\text{var}\, X = \frac{1}{\theta_0^2}$ Using CLT,

$$\left(\frac{\bar{X} - 1/\theta_0}{1/\sqrt{n\theta_0^2}}\right)^2 \xrightarrow{D} Z^2(1)$$

and $n \cdot o\left((\theta_0 \bar{X} - 1)^2\right) \xrightarrow{P} 0$.

For α Level test, Reject when

$-2\log(\lambda(x))$ is high, and

$\mathbb{P}_{H_0}(-2\log(\lambda(x)) > c) = \alpha$, $c > 0$

$\simeq \mathbb{P}_{N(0,1)}(Z^2 \geq c) = \alpha$

$= \mathbb{P}_{N(0,1)}(Z \leq -\sqrt{c}$ or $Z \geq \sqrt{c}) = \alpha$

$\Rightarrow \mathbb{P}_{N(0,1)}(Z \leq -\sqrt{c}) = \frac{\alpha}{2}$

$-\sqrt{c} = Z_{\alpha/2}$ where $\mathbb{P}(Z \leq Z_\alpha) = \alpha$.

$c = Z_{\alpha/2}^2$

1b)
cont. Reject H_0 when:

$$\left(\frac{\bar{X} - 1/\theta_0}{1/\sqrt{n\theta_0^2}}\right)^2 \geq Z_{\alpha/2}^2$$

$$\Rightarrow \quad \frac{\bar{X} - 1/\theta_0}{1/\sqrt{n\theta_0^2}} \leq Z_{\alpha/2}, \quad \frac{\bar{X} - 1/\theta_0}{1/\sqrt{n\theta_0^2}} \geq -Z_{\alpha/2}$$

Reject when:

$$\bar{X} \notin \left(1/\theta_0 + \frac{Z_{\alpha/2}}{\sqrt{n\theta_0^2}}, \ 1/\theta_0 - \frac{Z_{\alpha/2}}{\sqrt{n\theta_0^2}}\right)$$

1c) Let $\theta_1 \leq \theta_0$, $\theta_1 > \theta_0$

$$\Lambda(x) = \left(\frac{\theta_1}{\theta_2}\right)^n e^{-n\bar{x}(\theta_1 - \theta_2)}$$

Reject the $\Lambda(x)$ small

$\Rightarrow$ $\underset{<0}{\underline{-n\bar{x}(\theta_1 - \theta_2)}}$ small

$\Rightarrow$ $\bar{x}$ small

$\bar{x} \leq c$, where c depends on α.

2a)

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\hat{X}_1} \\ & & 1 & 3 \\ \hline \hat{X}_2 & 1 & 1/4 & 1/4 \\ \hline & 3 & 1/4 & 1/4 \\ \hline \end{array}$$

b) $$\hat{\theta}^* = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad \hat{\theta} = 2$$

$$\hat{\theta}^* - \hat{\theta} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda(-0.5) \approx P_{BS}(\hat{\theta}^* - \hat{\theta} \leq -0.5)$$

$$= 1/4$$

2c) Find the MLE given X_1, X_2:

$$\ell(\theta \mid X_1, X_2) = \frac{1}{4} e^{-(|x-1| + |x-3|)}$$

MLE is $[1, 3]$

take $\theta = 2$, the middle value.

Let B be a large #.

Draw B Sample Pairs from $f_{\theta=2}(x)$, and

Calculate $\hat{\theta}_1, \ldots, \hat{\theta}_n$ from them,

where $\hat{\theta}_i = \dfrac{X_1^* + X_2^*}{2}$, the average of

a sample pair (X_1^*, X_2^*).

Let $\hat{\theta}^*$ be emperically distributed across $\hat{\theta}_1, \ldots, \hat{\theta}_n$.

Then $\mathbb{P}_F(\hat{\theta} - \theta \le t) \approx \mathbb{P}_{Emp}(\hat{\theta}^* - 2 \le t)$

$$= \frac{1}{B} \sum_{i=1}^{B} \mathbb{1}\{\hat{\theta}_i - 2 \le t\}$$

Spring 2019

1a) $\sum X_i \sim \mathcal{N}(n\theta, n\theta)$

$\sum X_i - n\theta \sim \mathcal{N}(0, n\theta)$

$\frac{1}{\sqrt{n\theta}}\left(\sum X_i - n\theta\right) \sim \mathcal{N}(0,1)$

$\frac{1}{\sqrt{n\theta}}\left(\sum X_i - n\theta\right)$ is a pivot)

1b) Let us create an α-level confidence interval,

i.e $\mathbb{P}_\theta(\theta \in I(X)) = 1-\alpha$

$\mathbb{P}\left(Z_{\frac{\alpha}{2}} \le \frac{1}{\sqrt{n\theta}}\left(\sum X_i - n\theta\right) \le Z_{1-\frac{\alpha}{2}}\right) = \alpha$

where Z_α is defined as: $\underset{\mathcal{N}(0,1)}{\mathbb{P}}(Z \le Z_\alpha) = \alpha$.

we shall assume here $\sum X_i \ge 0$,

witch is almost sure as $n \to \infty$.

(b) cont.

$$z_{\alpha/2} \leq \frac{1}{\sqrt{n\theta}} \left(\sum x_i - n\theta \right)$$

(may not be true for any θ if $\sum x_i < 0$)

$$\Rightarrow \quad z_{\alpha/2} \leq \frac{\sqrt{n}}{\sqrt{\theta}} \left(\bar{X} - \theta \right)$$

$$\frac{\sqrt{\theta}}{\sqrt{n}} z_{\alpha/2} + \sqrt{n\theta} \leq \frac{\sqrt{n}}{\sqrt{\theta}} \bar{X}$$

$$\theta + \frac{\sqrt{\theta}}{\sqrt{n}} z_{\alpha/2} + \frac{z_{\alpha/2}^2}{4n} \leq \bar{X} + \frac{z_{\alpha/2}^2}{4n}$$

$$\left(\sqrt{\theta} + \frac{z_{\alpha/2}}{2\sqrt{n}} \right)^2 \leq \bar{X} + \frac{z_{\alpha/2}^2}{4n}$$

$$\sqrt{\theta} \leq \sqrt{\bar{X} + \frac{z_{\alpha/2}^2}{4n}} - \frac{z_{\alpha/2}}{2\sqrt{n}}$$

$$\theta \leq \left(\sqrt{\bar{X} + \frac{z_{\alpha/2}^2}{4n}} - \frac{z_{\alpha/2}}{2\sqrt{n}} \right)^2$$

(b) cont.

Likewise $\quad Z_{1-\alpha/2} \geq \frac{1}{\sqrt{n\theta}} \left(\sum x_i - n\theta \right)$ $\left(\begin{array}{c} \text{May always} \\ \text{be true if} \\ \sum x_i < 0 \end{array} \right)$

$$\Rightarrow \theta \geq \left(\sqrt{\bar{x} + \frac{z^2_{1-\alpha/2}}{4n}} - \frac{z_{1-\alpha/2}}{2\sqrt{n}} \right)^2$$

Observing $\quad z_{1-\alpha/2} = -z_{\alpha/2}$

$$\theta \geq \left(\sqrt{\bar{x} + \frac{z^2_{\alpha/2}}{4n}} + \frac{z_{\alpha/2}}{2\sqrt{n}} \right)^2$$

$$I(x) = \left(\left(\sqrt{\bar{x} + \frac{z^2_{\alpha/2}}{4n}} + \frac{z_{\alpha/2}}{2\sqrt{n}} \right)^2, \left(\sqrt{\bar{x} + \frac{z^2_{\alpha/2}}{4n}} - \frac{z_{\alpha/2}}{2\sqrt{n}} \right)^2 \right)$$

2a) MLR in X for $f_\theta(x)$ means that:

If $\theta_2 > \theta_1$, $\dfrac{f_{\theta_2}(x)}{f_{\theta_1}(x)}$ is monotone in in x

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} = \frac{\binom{\theta_2}{x}\binom{N-\theta_2}{n-x}}{\binom{\theta_1}{x}\binom{N-\theta_1}{n-x}}$$

$$= \frac{\dfrac{\theta_2!}{x!(x-\theta_2)!} \dfrac{N-\theta_2!}{(n-x)!(N-\theta_2-n+x)!}}{\dfrac{(\theta_1)!}{x!(x-\theta_1)!} \dfrac{(N-\theta_1)!}{(n-x)!(N-\theta_1-n+x)!}}$$

$$\propto \frac{(N-\theta_2-n+x)!}{(N-\theta_1-n+x)!} \frac{(x-\theta_1)!}{(x-\theta_2)!}$$

$$= \frac{1}{(N-\theta_2-n+x+1)\cdots(N-\theta_1-n+x)} \frac{1}{(x-\theta_1+1)\cdots(x-\theta_2)}$$

decreasing in X.

So $f_\theta(x)$ has an MLR in X.

2b) $\mathbb{P}_{H_0}(X=n) = \mathbb{1}_{\{X=n\}}$, as in, if all balls are white, then the probability of drawing all white balls is 1.

If $\gamma(x)$ is an α-level test, then

$$\sum_{k=0}^{n} \gamma(x)\,\mathbb{P}_{H_0}(X=k) = 1-\alpha$$

but, $$\sum_{k=0}^{n} \gamma(x)\,\mathbb{P}_{H_0}(X=k) = \gamma(n)$$

So all α-level test have $\gamma(n) = 1-\alpha$

The power function of such tests is:

$$\beta(\gamma,\theta) = \sum_{k=0}^{n} \gamma(x)\,\mathbb{P}_{\theta}(X=k)$$

$$= (1-\alpha)\,\mathbb{P}_{\theta}(X=n) + \sum_{k=0}^{n} \gamma(x)\,\mathbb{P}_{\theta}(X=k)$$

which is minimized if $\gamma(x)=0$ for $0 \le x \le n$

So the UMP test is

$$\gamma(x) = \begin{cases} 1-\alpha, & x=n \\ 0, & x \neq n \end{cases}$$

2c) $\mathbb{P}_{\theta=N-1}(\text{Reject } H_0 \text{ w/ } \gamma) \geq 1-\beta$

$\mathbb{P}_{\theta=N-1}(\text{Accept } H_0 \text{ w/ } \gamma) < \beta$

$$\beta > \sum_{k=0}^{n} \gamma(k) f_{\theta=N-1}(k)$$

$$= \gamma(n) \binom{N-1}{n} \binom{1}{0} \Big/ \binom{N}{n}$$

$$= (1-\alpha) \left(\frac{(N-1)!}{(n)!(N-n-1)!} \right) \Big/ \frac{(N!)}{(N-n)!\,(n)!}$$

$$= (1-\alpha) \frac{N-n}{N}$$

$$N \frac{\beta}{1-\alpha} > N-n$$

$$n > N\left(1-\frac{\beta}{1-\alpha}\right)$$

$$n = \max\left(0, N\left(1-\frac{\beta}{1-\alpha}\right)\right)$$

Fall 2018

1a) $w_i \sim \mathcal{N}(x_i'\beta + \varepsilon_i)$

$$\mathbb{P}(\mathbb{1}(w_i > 0) = 1) = \mathbb{P}(w_i > 0)$$
$$= \mathbb{P}(w_i - x_i'\beta > -x_i'\beta)$$
$$= \mathbb{P}(\varepsilon_i > -x_i'\beta)$$
$$= \mathbb{P}(\varepsilon_i < x_i'\beta)$$
$$= \Phi(x_i'\beta)$$

$$\mathbb{P}(\mathbb{1}(w_i > 0) = 0) = 1 - \Phi(x_i'\beta)$$

ε_i are iid, so $\mathbb{1}(w_i > 0)$ are independent.

So Y_i has same dist. as $\mathbb{1}(w_i > 0)$.

1(b) $L(\beta, w_1, \ldots, w_n)$

$= \prod_{i=1}^{n} L(\beta, w_i)$

$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(w_i - x_i'\beta)^2}{2}\right)$

$l_c(\beta, w_1, \ldots, w_n) = \log L(\beta, w_1, \ldots, w_n)$

$= \sum_{i=1}^{n} -\frac{(w_i - x_i'\beta)^2}{2} + n \log\left(\frac{1}{\sqrt{2\pi}}\right)$

1(c) $\mathbb{E}\left[(w_i - x_i'\tilde{\beta})^2 \mid y\right]$

$= \mathbb{E}\left[(w_i - x_i'\tilde{\beta})^2 \mid y_1, \ldots, y_n\right]$

$= \mathbb{E}\left[(w_i - x_i'\tilde{\beta})^2 \mid y_i\right]$ $\quad (w_i \perp\!\!\!\perp y_j,\ j \neq i)$

1c) $\mathbb{E}_{\beta}\left[(w_i - x_i'\beta)^2 \mid y_i = 1\right]$
cont.

$= \mathbb{E}_{\beta}\left[(\varepsilon_i + x_i'\beta - x_i'\tilde{\beta})^2 \mid \varepsilon_i + x_i'\beta > 0\right]$

$= (x_i'\beta - x_i'\tilde{\beta})^2 + 2(x_i'\beta - x_i'\tilde{\beta})\,\mathbb{E}(\varepsilon_i \mid \varepsilon_i > -x_i'\beta)$

$\qquad + \mathbb{E}(\varepsilon_i^2 \mid \varepsilon_i > -x_i'\beta)$

$$= (x_i'\tilde{\beta})^2 - 2x_i'\beta\, x_i'\tilde{\beta} - 2x_i'\tilde{\beta}\,\frac{1}{\mathbb{P}(\varepsilon_i > x_i'\beta)}\int_{-x_i'\beta}^{\infty} \frac{t}{\sqrt{2\pi}} e^{-t^2/2}\,dt + C$$

$$= (x_i'\tilde{\beta})^2 - 2x_i'\tilde{\beta}\left[x_i'\beta - \frac{1}{\Phi(x_i'\beta)} \left.\frac{e^{-\frac{x_i'\beta}{2}}}{\sqrt{2\pi}}\right|_{-x_i'\beta}^{\infty}\right] + C$$

$$= (x_i'\tilde{\beta})^2 - 2x_i'\tilde{\beta}\left[x_i'\beta + \frac{\phi(x_i'\beta)}{\Phi(x_i'\beta)}\right] + C$$

$\mathbb{E}_{\beta}\left[(w_i - x_i'\beta)^2 \mid y_i = 0\right]$

$= \mathbb{E}_{\beta}\left[\;\text{- - - -}\; \mid \varepsilon_i + x_i'\beta < 0\right]$

$\vdots$

$$= (x_i'\tilde{\beta})^2 - 2x_i'\tilde{\beta}\left[x_i'\beta - \frac{1}{\mathbb{P}(\varepsilon_i < -x_i'\beta)}\phi(x_i'\beta)\right] \cdots + C''$$

...

1d) $Q(\tilde{\beta}, \beta^{(n)}) = \mathbb{E}_{\beta^{(n)}}\left[l_c(\tilde{\beta}) \mid Y \right]$

$$= -\frac{1}{2}\left[\sum_{i=1}^{n} \mathbb{E}\left[(w_i - x_i'\beta^{(n)})^2 \mid Y_i = 1 \right] \mathbb{I}(Y_i = 1) + \mathbb{E}\left[(w_i - x_i'\beta^{(n)})^2 \mid Y_i = 0 \right] \mathbb{I}(Y_i = 0) \right] + C$$

$$= -\frac{1}{2}\left[\left((x_i\tilde{\beta})^2 - 2x_i'\tilde{\beta}\left(x_i'\beta^{(n)} + \frac{\phi(x_i'\beta)}{\Phi(x_i'\beta)} \right) \right) \mathbb{I}(Y_i = 1) + \left((x_i\tilde{\beta})^2 - 2x_i'\tilde{\beta}\left(x_i'\beta^{(n)} - \frac{\phi(x_i'\beta)}{\Phi(x_i'\beta)} \right) \right) \mathbb{I}(Y_i = 0) \right] + C$$

$$= -\frac{1}{2}\left[(X\tilde{\beta})^T(X\tilde{\beta}) - 2(X\tilde{\beta})(X\beta^{(n)}) - 2(X\tilde{\beta})V \right] + C$$

$$\frac{d\,Q(\beta, \beta^{(n)})}{d\tilde{\beta}} = -\frac{1}{2}\left[2(X^TX)\tilde{\beta} - 2X^TX\beta^{(k)} - X^TV \right] = 0$$

$$\tilde{\beta} = \beta^{(n)} + (X^TX)^{-1}X^TV(\beta^{n})$$

$\beta^{(n+1)}$, maximizes $Q(\tilde{\beta}, \beta^{(n)})$

2a) $\bar{Y} \sim \mathcal{N}(\Theta, \frac{1}{\sqrt{n}})$, conditioned on Θ.

$$f_{\bar{Y},\Theta}(y,t) = f_{\bar{Y}|\Theta}(y|t)\, f_\Theta(t)$$

$$= \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{n(y-t)^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

$$= \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{ny^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2+2nyt-nt^2}{2}}$$

$$= \frac{\sqrt{n}}{\sqrt{2\pi}} e^{-\frac{ny^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{-(n+1)\left[t^2-\frac{2n}{n+1}yt+\left(\frac{n}{n+1}\right)^2y^2\right]+\frac{n^2}{n+1}y^2}{2}}$$

$$= \frac{\sqrt{n}}{\sqrt{n+1}} \frac{1}{\sqrt{2\pi}} e^{\frac{-ny^2+\frac{n^2y^2}{n+1}}{2}} \frac{\sqrt{n+1}}{\sqrt{2\pi}} e^{-\frac{\left[t-\frac{n}{n+1}y\right]^2}{2/n+1}}$$

$$= \sqrt{\frac{n}{n+1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{n}{n+1}y^2/2} \frac{\sqrt{n+1}}{\sqrt{2\pi}} e^{-\frac{\left[t-\frac{n}{n+1}y\right]^2}{2/n+1}}$$

$$f_{\bar{Y}}(y) = \int_{-\infty}^{\infty} f_{\bar{Y},\Theta}(y,t)\,dt = \sqrt{\frac{n}{n+1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{n}{n+1}y^2/2}$$

2b)

$$f_{\theta|\bar{Y}}(t,y) = \frac{f_{\theta,\bar{Y}}(t,y)}{f_{\bar{Y}}(y)}$$

$$= \frac{\sqrt{n+1}}{\sqrt{2\pi}} e^{-\frac{\left[t - \frac{n}{n+1} y\right]^2}{2/n+1}}$$

2c) The density of θ conditioned on $\bar{Y}$ indicates

$$\theta \sim \mathcal{N}\left(\frac{n}{n+1} y, \frac{1}{n+1}\right)$$

$$\mathbb{E}(\theta | \bar{Y} = y) = \frac{n}{n+1} y$$

$$\text{Var}(\theta | \bar{Y} = y) = \frac{1}{n+1} \xrightarrow{n \to \infty} 0$$

so $\theta \xrightarrow{d} \bar{Y}$ as $n \to \infty$

but $\bar{Y} \xrightarrow{d} \theta$ as $n \to \infty$

so $\theta | \bar{Y} \xrightarrow{n \to \infty} \theta$

Spring 2018

1) Let $y = (x_{11}, x_{12}, x_{21}, x_{22}, x_3, x_4)$

$$L(\theta \mid x) = \frac{200!}{x_1! x_2! x_3! x_4!} \left(\frac{1}{2}+\frac{\theta}{r}\right)^{x_1} \left(\frac{1}{4}-\frac{\theta}{4}\right)^{x_2} \left(\frac{1}{4}-\frac{\theta}{3}\right)^{x_3} \left(\frac{\theta}{12}\right)^{x_4}$$

Let $y \sim M\left(200; \frac{1}{2}, \frac{\theta}{2}, \frac{1}{4}-\frac{\theta}{3}, \frac{\theta}{12}, \frac{1}{4}-\frac{\theta}{3}, \frac{\theta}{12}\right)$

$\downarrow$

$$L(\theta \mid x, y) = \frac{200!}{x_{11}! x_{12}! x_{21}! x_{22}! x_3! x_4!} \left(\frac{1}{2}\right)^{x_{11}} \left(\frac{\theta}{2}\right)^{x_{12}} \left(\frac{1}{4}-\frac{\theta}{3}\right)^{x_{21}+x_3} \left(\frac{\theta}{12}\right)^{x_{22}+x_4}$$

E-step: what is $\mathbb{E}\left(\log L(\theta \mid x, y) \mid \theta^k, x\right)$?

$$\log L(\theta \mid x, y) \propto x_{12} \log \theta/2 + (x_{21}+x_3) \log\left(\frac{1}{4}-\frac{\theta}{3}\right) + (x_{22}+x_4) \log\left(\frac{\theta}{12}\right)$$

1) Cont $\mathbb{E}(\log L(\theta|x,y)|x,\theta^n)$

$$= \sum_{n=0}^{x_1}\sum_{m=0}^{x_2} \mathbb{P}_\theta\left(X_{11}=n, X_{12}=x_1-n, X_{21}=m, X_{22}=X_2-m \mid X_1, X_2, X_3, X_4, \theta^{(n)}\right)$$

$$\cdot \log L(\theta|X,y)$$

(Denote $\mathbb{P}(\cdot\cdot|\cdot\cdot)$ above by $P_{\theta^n}(n,m|X)$)

$$\propto \sum_{n=0}^{x_1}\sum_{m=0}^{x_2} P_{\theta^n}(n,m|X)\left[(x_1-n)\log\left(\frac{\theta}{4}\right) + (m+x_3)\log\left(\frac{1}{4}-\frac{\theta}{3}\right) + (x_2+x_4-m)\log\frac{\theta}{12}\right.$$

<u>M-step</u> $\frac{d}{dx}\mathbb{E}(\log L(\theta|x,y)|x,\theta^n) = 0$

$$= \sum_{n=0}^{x_1}\sum_{m=0}^{x_2} P_{\theta^n}(n,m|X)\left[\frac{x_1-n+x_2-m}{\theta} - \frac{m+x_3}{\frac{3}{4}-\theta}\right]$$

$$= \frac{A^n}{\theta} - \frac{B^n}{\frac{3}{4}-\theta} \quad \text{where } A^n, B^n \text{ above.}$$

$$\left(\frac{3}{4}-\theta\right)A^n = B^n\theta \Rightarrow \theta^{n+1} = \frac{3}{4}A^n \Big/ (A^n+B^n)$$

a) $\bar{X} \sim \mathcal{N}(\theta, \frac{\sigma^2}{n})$, given θ.

$$f_{\theta,\bar{x}}(t,x) = f_{\bar{x}|\theta}(x,t)\, f_\theta(t)$$

$$= \frac{\sqrt{n}}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-n(x-t)^2}{2\sigma^2}\right) \frac{1}{\sqrt{2\pi}\,\tau} \exp\left(\frac{-(t-\mu)^2}{2\tau^2}\right)$$

$$= \frac{\sqrt{n}}{2\pi\sigma\tau} \exp\left(\frac{-n(x-t)^2}{2\sigma^2} - \frac{(t-\mu)^2}{2\tau^2}\right)$$

b) By the form of $f_{\theta,\bar{x}}$, $f_{\theta,\bar{x}}$ is distributed as a bivariate normal,

$$f_{\theta,\bar{x}}(t,x) = \frac{\sqrt{n}}{2\pi\sigma\tau} \exp\left(\frac{-x^2}{2\sigma^2/n} - \frac{t^2}{\frac{2\sigma^2/n\,\tau^2}{\tau^2+\sigma^2/n}} + \frac{2xt\tau^2 + 2\mu t\sigma^2/n}{2\sigma^2/n\,\tau^2} - \frac{\mu^2}{2\tau^2}\right)$$

where $\sigma_1, \mu_1 \to X$

$\sigma_2, \mu_2 \to \theta$

$$(1-\rho^2)\sigma_1^2 = \sigma^2/n,$$

$$(1-\rho^2)\sigma_2^2 = \frac{\sigma^2/n\,\tau^2}{\tau^2+\sigma^2/n}$$

$$\frac{(1-\rho^2)\sigma_1\sigma_2}{\rho} = \sigma^2/n$$

2(b) cont.

$$\rho^2 = \left(\frac{\rho}{(1-\rho^2)\sigma_1\sigma_2}\right)^2 \left[(1-\rho^2)\sigma_1^2\right]\left[(1-\rho^2)\sigma_2^2\right]$$

$$= \left(\frac{1}{\sigma^2/n}\right)^2 \left(\sigma^2/n\right)\left(\frac{\sigma^2/n\,\tau^2}{\tau^2+\sigma^2/n}\right)$$

$$\boxed{\rho = \frac{\tau}{\sqrt{\tau^2+\sigma^2/n}}}$$

$$1-\rho^2 = \frac{\sigma^2/n}{\tau^2+\sigma^2/n}$$

$$\boxed{\sigma_1^2 = \frac{\sigma^2/n}{1-\rho^2} = \tau^2+\sigma^2/n}$$

$$\boxed{\sigma_2^2 = \frac{\frac{\sigma^2/n\,\tau^2}{\tau^2+\sigma^2/n}}{1-\rho^2} = \tau^2}$$

$$\mathbb{E}(\theta) = \mu_1 = \mu$$

$$\mathbb{E}(\bar{X}) = \mu_2 = \mu$$

2b) cont. So $\bar{X} \sim \mathcal{N}(\mu, \tau^2 + \sigma^2/n)$

2c) Let z_p be s.t. $\underset{\mathcal{N}(0,1)}{\mathbb{P}}(Z \leq z_p) = p$

$$1-\alpha = \mathbb{P}\left(z_{\alpha/2} \leq \frac{\bar{X} - \mu}{\sqrt{\tau^2 + \sigma^2/n}} \leq z_{1-\frac{\alpha}{2}}\right)$$

$$z_{\alpha/2} \leq \frac{\bar{X} - \mu}{\sqrt{\tau^2 + \sigma^2/n}} \Rightarrow \mu \leq \bar{X} + z_{\alpha/2}\sqrt{\tau^2 + \sigma^2/n}$$

$$\Rightarrow \mu \geq \bar{X} - z_{1/2}\sqrt{\tau^2 + \sigma^2/n}$$

$$I_{\bar{X}}^{(1-\alpha)} = \left(\bar{X} + z_{\alpha/2}\sqrt{\tau^2 + \sigma^2/n}, \bar{X} - z_{1/2}\sqrt{\tau^2 + \sigma^2/n}\right)$$

$$I_{\theta}^{(1-\alpha)} = \left(\bar{X} + z_{1/2}\tau, \bar{X} - z_{1/2}\tau\right)$$

$$I_{\bar{X}}(1-\alpha) \xrightarrow{n \to \infty} I_{\theta}(1-\alpha)$$

Fall 2017

(1a) Let $X^n = (x_1, \dots, x_n)$ observed

$X^m = (x_{n+1}, \dots, x_{n+m})$ unobserved

$$L(\theta \mid X^n, X^m) = \prod_{i=1}^{n+m} \left(\frac{1}{\theta}\right) \exp(-x_i/\theta)$$

$$= \frac{1}{\theta^{n+m}} \exp\left(-\frac{1}{\theta} \sum_{i=1}^{n+m} x_i\right)$$

$$\ell(\theta \mid X^n, X^m) = -(n+m)\log(\theta) - \frac{1}{\theta} \sum_{i=1}^{n+m} x_i$$

(1b) $$\frac{d}{d\theta} \ell(\theta \mid X^n, X^m) = -\frac{(n+m)}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n+m} x_i = 0$$

$$\theta (n+m) = \sum_{i=1}^{n+m} x_i$$

$$\theta = \overline{X}$$

1c) $\mathbb{E}[X \mid X > t] = \int_0^\infty X \cdot f_\theta(X \mid X > t)\, dt$

$$= \int_0^\infty X \frac{f_\theta(X)\, \mathbb{I}_{X>t}}{\mathbb{P}(X>t)}$$

$$= \frac{\int_0^\infty X \frac{1}{\theta} \exp\left(-\frac{X}{\theta}\right) \mathbb{I}_{X>t}}{\int_t^\infty \frac{1}{\theta} \exp\left(-\frac{X}{\theta}\right)}$$

$$= \frac{\int_t^\infty X \frac{1}{\theta} \exp\left(-\frac{X}{\theta}\right)}{-\exp\left(-\frac{X}{\theta}\right) \Big|_t^\infty}$$

$$= e^{\frac{t}{\theta}} \left[-X \exp\left(-\frac{X}{\theta}\right) - \theta \exp\left(-\frac{X}{\theta}\right) \right] \Big|_t^\infty$$

$$= e^{t/\theta} \left[t \exp\left(-\frac{t}{\theta}\right) + \theta \exp\left(-\frac{t}{\theta}\right) \right]$$

1c) $\mathbb{E}[X \mid X < t] = \dfrac{\int_0^\infty x \, f_\theta(x) \, \mathbb{1}_{x<t}}{\mathbb{P}(X < t)}$

cont.

$$= \frac{\int_0^t x \frac{1}{\theta} \exp(-\frac{x}{\theta})}{1 - e^{-\frac{t}{\theta}}}$$

$$= \frac{\left[-x\exp(-\frac{x}{\theta}) - \theta \exp(-\frac{x}{\theta})\right]\Big|_0^t}{1 - e^{-t/\theta}}$$

$$= \frac{1 - t\exp(-\frac{t}{\theta}) - \theta \exp(-\frac{t}{\theta})}{1 - e^{-t/\theta}}$$

1d) E-step:

a vector

$$\mathbb{E}\left(\ell(\theta \mid X^n, X^m) \mid \theta^n, X^n, \mathbb{I}(X^m > t)\right)$$

$$= \mathbb{E}\left(-(n+m)\log\theta - \frac{1}{\theta}\sum_{i=1}^{n+m} X_i \mid \theta^n, X^n, \mathbb{I}(X^m > t)\right)$$

$$= -(n+m)\log\theta - \frac{1}{\theta}\sum_{i=1}^{n} \mathbb{E}(X_i \mid \theta^n, X_i) - \frac{1}{\theta}\sum_{i=n+1}^{n+m} \mathbb{E}(X_i \mid \theta^n, \mathbb{I}_{X_i > t})$$

$$= -(n+m)\log\theta - \frac{1}{\theta}\sum_{i=1}^{m} X_i$$

$$- \frac{1}{\theta}\sum_{i=n+1}^{n+m} \underbrace{\left[e^{t/\theta^n}\mathbb{E}_{\theta^n}(X_i \mid X_i > t) + (1 - e^{t/\theta^n})\mathbb{E}_{\theta^n}(X_i \mid X_i < t)\right]}_{P_{i,\theta^n}}$$

$$= -(n+m)\log\theta - \frac{1}{\theta}\sum_{i=1}^{m} X_i - \frac{1}{\theta}\sum_{i=n+1}^{n+m} P_{i,\theta^n}$$

1d) M-Step:
Cont

$$\frac{d}{d\theta} \mathbb{E}\left(\ell(\theta \mid x^n, x^m) \mid \theta^n, x^n, \mathbb{I}_{x^m > t}\right)$$

$$= -\frac{n+m}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i + \frac{1}{\theta^2}\sum_{i=n+1}^{m} P_i = 0$$

$$\theta^{(n+1)} = \frac{1}{n+m}\left(\sum_{i=1}^{n} x_i + \sum_{i=n+1}^{m} P_{i,\theta^n}\right)$$

2a) $L(\mu \mid X) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x_i-\mu)^2}{2}\right)$

$$= \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{\sum (x_i-\mu)^2}{2}\right)$$

$$\log L(\mu \mid X) \propto -\sum_{i=1}^{n} \frac{(x_i+\mu)^2}{2}$$

$$\Lambda(x) = \frac{L(\theta_0 \mid x)}{\sup L(\theta \mid x)} = \frac{\exp\left(-\frac{\sum x_i^2}{2}\right)}{\exp\left(-\frac{\sum (x_i-\bar{x})^2}{2}\right)}$$

$$= \exp\left(-\frac{\sum x_i^2}{2} + \frac{\sum (x_i-\bar{x})^2}{2}\right)$$

Reject when $-\frac{\sum x_i^2}{2} + \frac{\sum (x_i-\bar{x})^2}{2}$ small

2a)
Cont.

$-\Sigma x_i^2 + \Sigma x_i^2 - 2\bar{X}\Sigma x_i + n\bar{X}^2$ small

$\Rightarrow -n\bar{X}^2$ small

$\Rightarrow \bar{X}^2$ large

$\Rightarrow |\bar{X}|$ large

$\bar{X} \sim \mathcal{N}(0, \frac{1}{n})$

$\sqrt{n}\bar{X} \sim \mathcal{N}(0,1)$

Reject if $\sqrt{n}\bar{X} \notin [z_{\alpha/2}, z_{1-\frac{\alpha}{2}}]$

$\Rightarrow \bar{X} \notin [\sqrt{n}\, z_{\alpha/2}, \sqrt{n}\, z_{1-\alpha/2}]$

2b) No. Considering $H_0 = 0$, $H_1 = -1$,

The UMP test is one tailed, but our test is two tailed.

That is $\beta_{\substack{\text{Two}\\\text{Tailed}}}(-1) < \beta_{\substack{\text{left}\\\text{tailed}}}(-1)$

when a one tailed test Rejects if

$$\overline{X} \leq \sqrt{n}\, Z_{\alpha/2}$$

2c) The UMP test for $H_0 = 0$, $H_1 = -1$

is: Reject if $\bar{X} \le \sqrt{n} Z_{\alpha}$ (Test 1)

The UMP test for $H_0 = 0$, $H_1 = 1$

is: Reject if $\bar{X} \ge \sqrt{n} Z_{1-\alpha}$ (test 2)

obviously $\beta_1(-1) \ne \beta_2(-1)$

so no UMP test exists.

Spring 2017

1a) $\bar{X} = \frac{1}{n}\sum X_i \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$, $S^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim T_{n-1}$$

Let $\mathbb{P}(T_{n-1} \leq T_{\alpha, n-1}) \leq \alpha$

$$\mathbb{P}(T_{\frac{\alpha}{2}, n-1} \leq T_{n-1} \leq T_{1-\frac{\alpha}{2}, n-1}) = 1-\alpha$$

$$\mathbb{P}\left(T_{\frac{\alpha}{2}, n-1} \leq \frac{\bar{X} - \mu}{S/\sqrt{n}} \leq T_{1-\frac{\alpha}{2}, n-1}\right) = 1-\alpha$$

$$\mathbb{P}\left(\bar{X} - T_{1-\frac{\alpha}{2}, n-1}\frac{S}{\sqrt{n}} \leq \mu \leq \bar{X} - T_{\frac{\alpha}{2}, n-1}\frac{S}{\sqrt{n}}\right) = 1-\alpha$$

1b) $(n-1)\frac{S^2}{\sigma^2} \sim \chi^2_{(n-1)}$

Let $\mathbb{P}(\chi^2_{(n-1)} \leq \chi^2_{\alpha, n-1}) = \alpha$

$$\mathbb{P}\left(\chi^2_{\frac{\alpha_2}{2}, n-1} \leq (n-1)\left(\frac{S^2}{\sigma^2}\right) \leq \chi^2_{\frac{\alpha_2}{2}, n-1}\right)$$

$$= \mathbb{P}\left(\frac{\sigma^2}{n-1}\chi^2_{\frac{\alpha_2}{2}, n-1} \leq S^2 \leq \frac{\sigma^2}{n-1}\chi^2_{1-\frac{\alpha_2}{2}, n-1}\right) = 1-\alpha_2$$

$$\mathbb{P}\left(\frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha_2}{2}, n-1}} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{\frac{\alpha_2}{2}, n-1}}\right) = 1-\alpha_2$$

1c) Choose $\alpha_1 = \frac{\alpha}{2}$, $\alpha_0 = \frac{\alpha}{2}$

$$\mathbb{P}(\mu \in I_1(x) \;\&\; \sigma^2 \in I_2(x))$$

$$= 1 - \mathbb{P}(\mu \notin I_1(x) \text{ or } \sigma^2 \notin I_2(x))$$

$$\geq 1 - \left[\mathbb{P}(\mu \notin I_1(x)) + \mathbb{P}(\sigma^2 \notin I_2(x))\right]$$

$$= 1 - \left[\frac{\alpha}{2} + \frac{\alpha}{2}\right]$$

$$= 1 - \alpha$$

2) a) Let $\gamma(x)$ be the MP test, $X = (X_1, \dots, X_n)$

so $\alpha = \int_{\mathbb{R}^n} \gamma(x) P_0^n(x) dx$, where $P_0^n(x)$ is the n-dimentional density of $(X_1, \dots, X_n)$ under H_0.

& if $\gamma'(x)$ is also a test w/ $\int_{\mathbb{R}^n} \gamma'(x) P_0(x) dx = \alpha$

then $\int_{\mathbb{R}^n} \gamma'(x) P_a^n(x) \leq \int_{\mathbb{R}^n} \gamma'(x) P_a^n(x) dx$

Let $\gamma'(x) = \alpha$ for all x.

$\alpha = \int \gamma'(x) P_0(x) dx = \int \gamma'(x) P_a(x) dx$

$$\begin{aligned} \beta - \alpha &= \int \gamma(x) P_a(x) dx - \alpha \\ &\geq \int \gamma'(x) P_a(x) dx - \alpha \\ &= 0 \implies \text{so } \beta \geq \alpha \end{aligned}$$

2a) cont B_3 Neyman Pearson sufficiency:

Rejection region takes form $P_a \geq h P_0$

$$\beta - \alpha = \int \gamma(x)\,(P_a - P_0)\,dx$$
$$= \underbrace{\int_{\text{Accept}} \gamma(x)(P_a - P_0)\,dx}_{=0} + \int_{\text{Reject}} (P_a - P_0)\,dx$$

$$= \int_{\text{Reject}} \gamma(x)\,(P_a - P_0)$$

$$= \int_{P_a > h P_0} (P_a - P_0)\,dx$$

$$\neq 0 \qquad (\text{if } h \neq 0)$$

However, if $P_a = P_0$, $\beta - \alpha = \int \gamma(x)\,(0)\,dx = 0 \Rightarrow \alpha = \beta$.

So $\beta > \alpha$ unless $P_a = P_0$

NP: Reject when

$\mathbb{1}_{[0,1]} > 3\,\mathbb{1}_{[1/3, 2/3]}$

Reject on

$[0, 1/3] \cup [2/3, 1]$

2b) Suppose $\alpha = \left(\frac{1}{3}\right)^n$

$$\gamma := \begin{cases} 1, & x_1, \ldots, x_n \in [1/3, 2/3] \\ 0, & \text{else} \end{cases}$$

$$\alpha_\gamma = \int_{\mathbb{R}^n} \gamma(x)\, p_0\, dx = \left(\frac{1}{3}\right)^n = \alpha \checkmark$$

Suppose $\alpha > \left(\frac{1}{3}\right)^n$

$$\gamma = \begin{cases} 1, & x_1, \ldots, x_n \in [1/3, 2/3] \\ \frac{\alpha - \left(\frac{1}{3}\right)^n}{1 - \left(\frac{1}{3}\right)^n}, & \text{else} \end{cases}$$

then
$$\alpha_\gamma = \int_{\mathbb{R}^n} \gamma(x)\, p_0\, dx = \int_{\text{Accept}} \gamma(x)\, p_0\, dx + \int_{\text{Reject}} \gamma(x)\, p_0(x)\, dx$$

$$= \left(\frac{1}{3}\right)^n + \left(\frac{\alpha - \left(\frac{1}{3}\right)^n}{1 - \left(\frac{1}{3}\right)^n}\right) \cdot \left(1 - \left(\frac{1}{3}\right)^n\right)$$

$$= \alpha \checkmark$$

2b) cont Suppose $\alpha < (\frac{1}{3})^n$

$$\gamma(x) = \begin{cases} \frac{\alpha}{(1/3)^n}, & x_1, \dots, x_n \in [1/3, 2/3) \\ 0, & \text{else} \end{cases}$$

$$\alpha_\gamma = \int \gamma(x) \, P_0(x) \, dx$$

$$= \frac{\alpha}{(\frac{1}{3})^n} \left(\frac{1}{3}\right)^n$$

$$= \alpha \checkmark$$

Fall 2016

1a) let $X_1, \dots, X_n$ be bernoullis,

$T(x) = \sum x_i \sim B(n, p)$

For fixed t, let $\alpha_1 = \alpha$ $\alpha_2 = 0$, so

$\alpha_1 + \alpha_2 = \alpha$.

Note $\mathbb{P}(T \le t \mid p) = \sum_{k=0}^{t} \binom{n}{k} p^k (1-p)^{n-k}$ is

decreasing in p for fixed t.

So we need $\Theta_U(t)$, $\Theta_L(t)$ s.t.

$\mathbb{P}(T \le t \mid p = \Theta_U(t)) = \alpha_1$

$\mathbb{P}(T \ge t \mid p = \Theta_L(t)) = \alpha_2$

then $[\Theta_L(t), \Theta_U(t)]$ is a $1-\alpha$ confidence

interval for p.

In fact, we only need the above for $t = 0$.

(17) cont

$$\mathbb{P}(T \leq 0 \mid P = \theta_U(0)) = \alpha$$

$$(1-P)^n = \alpha$$

$$1 - \sqrt[n]{\alpha} = P$$

$$\theta_U(1) = 1 - \sqrt[n]{\alpha}$$

$$\mathbb{P}(T \geq 0 \mid P = \theta_L(0)) = 0$$

No restrictions

$\left[0, 1 - \sqrt[n]{\alpha}\right]$ is a $1-\alpha$ confidence interval for P.

(5) $\sqrt[n]{\alpha} = e^{\frac{1}{n}\log\alpha}$

$$e^{\frac{1}{n}\log\alpha} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{n}\log\alpha\right)^k}{k!}$$

first order, small α, large n $\approx 1 + \frac{1}{n}\log\alpha$

$$1 - \sqrt[n]{\alpha} \approx \left(1 - \left(1 + \frac{1}{n}\log\alpha\right)\right)$$

$$= -\frac{1}{n}\log\alpha$$

for $\alpha = 0.05$

$$1 - \sqrt[n]{\alpha} \approx -\frac{1}{n}\log 0.05$$

$$\approx \frac{3}{n}$$

so $CI \approx \left[0, \frac{3}{n}\right]$

2a) $L(\Psi|x) = \prod_{j=1}^{n} \left(\sum_{i=1}^{g} \pi_i f_i(x_j) \right)$, $0 \le \pi_i \le 1 \ \forall i$, $\sum \pi_i = 1$

$$\ell(\Psi|x) = \sum \log \left(\sum_{i=1}^{g} \pi_i f_i(x_j) \right)$$

$$\max_{\substack{\|\Psi\|_1 = 1 \\ 0 \le \pi_i \le 1 \\ \forall i}} \sum_{j=1}^{n} \log \left(\sum_{i=1}^{g} \pi_i f_i(x_j) \right)$$

2b) $L(\Psi | w_j = w, Z_{ij} = 1) = \pi_i f_i(w_j)$ (for single data point)

$$L(\Psi | w, Z_{ij}) = \sum_{i=1}^{g} \pi_i Z_{ij} f_i(w)$$

$$L(\Psi | w, Z) = \prod_{j=1}^{n} \left(\sum_{i=1}^{g} \pi_i Z_{ij} f_i(w) \right)$$

2c) $$L(\psi \mid w, z) = \prod_{j=1}^{n} \left[\left(\prod_{i=1}^{g-1} \left(\pi_i f_i(w_j) \right)^{z_{ij}} \right) \left[\left(1 - \sum_{h=1}^{g-1} \pi_h\right) f_g(w_j) \right]^{z_{gj}} \right]$$

$$\ell(\psi \mid w, z) = \sum_{j=1}^{n} \sum_{i=1}^{g-1} z_{ij} \log(\pi_i f_i(w_j))$$

$$+ \sum_{j=1}^{n} z_{gj} \log\left(\left(1 - \sum_{h=1}^{g-1} \pi_h\right) f_g(w_j)\right)$$

$$\frac{d\ell}{d\pi_h} = \sum_{j=1}^{n} \frac{z_{mj}}{\pi_m} - \sum_{j=1}^{n} \frac{z_{gj}}{1 - \sum_{\ell=1}^{g-1} \pi_\ell} = 0$$

$$\frac{1}{\pi_k} \sum_{j=1}^{n} z_{kj} = \frac{1}{\pi_g} \sum_{j=1}^{g} z_{gj} \quad \text{for any } k,\ 1 \le k \le g-1$$

$$\frac{1}{\pi_k} \sum_j z_{kj} = \frac{1}{\pi_\ell} \sum_j z_{\ell j} \quad \text{for any } k, \ell$$

$$1 = \sum_h \pi_h = \sum_h \frac{\pi_m \sum_j z_{hj}}{\sum_j z_{mj}} \Rightarrow 1 = \pi_m \frac{\sum_{ij} z_{ij}}{\sum_j z_{mj}} \Rightarrow \boxed{\pi_m = \frac{\sum_j z_{mj}}{\sum_{ij} z_{ij}}}$$

E-step

2d) $\mathbb{E}\, \ell(\Psi \mid z, w) \mid \Psi^{(r)}, w$

$$= \mathbb{E} \sum_j \ell(\Psi \mid z_j, w_j) \mid \Psi^{r}\, w$$

$$= \sum_j \mathbb{E}\, \ell(\Psi \mid z_j, w_j) \mid \Psi^{(r)}, w_j)$$

$$= \sum_j \left[\sum_i \ell(\Psi \mid z_j, w_j) f(z_j \mid w_j, \Psi^{r}) \right]$$

$$= \sum_j \sum_i^{g} \log(\pi_i f_i(w_j)) \frac{f(z_j, w_j, \Psi^{(r)})}{f(w_j, \Psi^{(r)})}$$

$$= \sum_j \left[\sum_{i=1}^{g-1} \log \pi_i f_i(w_j) \frac{\pi_i^{(r)} f_i(w_j)}{\sum_{\ell=1}^{g} \pi_\ell^{(r)} f_\ell(w_j)} \right. \quad \leftarrow P_{r,i,j}^{(r)}$$

$$\left. + \log \pi_g f_g(w_j) \frac{\pi_g^{(r)} f_g(w_j)}{\sum_{\ell=1}^{g} \pi_\ell^{(r)} f_\ell(w_j)} \right]$$

2d) M-step

$$\frac{\partial \mathbb{E}}{\partial \pi_n} = \sum_{j=1}^{n} \frac{1}{\pi_n} P_{n,j}^{(r)} - \sum_{j=1}^{n} \frac{1}{\pi_g} P_{g,j}^{(r)}$$

$$\Rightarrow \frac{1}{\pi_n} \sum P_{n,j}^{(r)} = \frac{1}{\pi_g} P_{g,j}^{(r)}$$

$$\Rightarrow \boxed{\pi_n^{(r+1)} = \frac{\sum_j P_{n,j}^{(r)}}{\sum_{j,n}^{n\,g} P_{n,j}^{(r)}}}$$

Fall 2015

1a) Let $X^* = (X_{(1)}^*, X_{(2)}^*, X_{(3)}^*)$

Let $P_{ijk} = \mathbb{P}(X_{(1)}^*, X_{(2)}^{(*)}, X_{(3)}^{(*)}$ has i $X_{(1)}$, j $X_{(2)}$, k $X_{(3)})$

$$P_{111} = \frac{3!}{3^3} = \frac{2}{9}$$

$$P_{210} = P_{120} = P_{201} = P_{102} = P_{021} = P_{012} = \frac{3}{3^3} = \frac{1}{9}$$

$$P_{300} = P_{030} = P_{003} = \frac{1}{27}$$

1(b)

$$\widehat{\text{bias}}_{\text{Boot}}(\theta) = \mathbb{E}\,\theta^* - \hat{\theta}$$

$$\hat{\theta} = X_{(2)}$$

$$\mathbb{E}\,\theta^* = X_{(2)}P_{111} + X_{(1)}P_{210} + X_{(2)}P_{120} + X_{(1)}P_{201}$$

$$+ X_{(3)}P_{102} + X_{(2)}P_{021} + X_{(3)}P_{012}$$

$$+ X_{(1)}P_{300} + X_{(2)}P_{030} + X_{(3)}P_{003}$$

$$= \frac{7}{27}X_{(1)} + \frac{13}{27}X_{(2)} + \frac{7}{27}X_{(3)}$$

$$\widehat{\text{bias}}_{\text{Boot}}(\theta) = \frac{7}{27}(X_{(1)}) - \frac{14}{27}X_{(2)} + \frac{7}{27}X_{(3)}$$

1(c)

$$\mathrm{var}(\tilde{\theta}) = \mathbb{E}_{\rightarrow}(\theta^{*} - \hat{\theta})^2$$

$$= (X_{(2)} - X_{(2)})^2 P_{111} + (X_{(2)} - X_{(1)})^2 P_{210}$$

$$+ 0^2 P_{120} + (X_{(1)} - X_{(1)})^2 P_{201}$$

$$+ (X_{(3)} - X_{(2)})^2 P_{102} + (0)^2 P_{021}$$

$$+ (X_{(3)} - X_{(1)})^2 P_{012} + (X_{(1)} - X_{(1)})^2 P_{300}$$

$$+ 0 \quad P_{030} + (X_{(2)} - X_{(1)})^2 P_{003}$$

$$= \frac{7}{14}(X_{(2)} - X_{(1)})^2 + \frac{7}{14}(X_{(3)} - X_{2})^2$$

$$= \frac{7}{14} X_{(1)}^2 - X_{(1)} X_{(2)} + X_{(2)}^2 - X_{(2)} X_{(3)} + \frac{7}{14} X_{(3)}^2$$

2 a)

$$f_Z(z) = \frac{1}{2\pi |\Sigma|^{1/2}} \exp\left(-\frac{z^T \Sigma^{-1} z}{2}\right)$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\left(\frac{x^2 - 2xy\rho + y^2}{2(1-\rho^2)}\right)\right)$$

$$\text{where } \Sigma^{-1} = \frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}$$

$$L(\rho) = \frac{1}{2\pi(1-\rho^2)^{n/2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\sum x_i^2 - 2\rho\sum x_i y_i + \sum y_i^2\right]\right)$$

$$= \frac{1}{(2\pi)^{n/2}|\Sigma|^{n/2}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} z_i^T \Sigma^{-1} z_i\right)$$

2b) N.P Procedure: Reject when:

$$\frac{L(0)}{L(\rho)} \quad \text{small}$$

$$\Rightarrow (1-\rho_0)^{n/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left(Z_i^T Z_i - Z_i^T \Sigma_{\rho_0}^{-1} Z_i\right)\right) \quad \text{small}$$

$$\Rightarrow \sum x_i^2 + \sum y_i^2 - \frac{\sum x_i^2}{1-\rho^2} - \frac{\sum y_i^2}{1-\rho^2} + \frac{2\rho \sum x_i y_i}{1-\rho^2} \quad \text{large}$$

$$\Rightarrow \frac{-\rho^2 \sum x_i^2 - \rho^2 \sum y_i^2 + 2\rho \sum x_i y_i}{1-\rho^2} \quad \text{large}$$

$$\Rightarrow \underline{-\rho\left(\sum x_i^2 + \sum y_i^2\right)} + \sum x_i y_i \quad \text{large}$$

$$=: T(Z)$$

Need to know distribution of $T(Z)$, then test

is: Reject when $T(Z) > T_{1-\alpha}^{(n,\rho)}$

where $\mathbb{P}(T(Z) < T_{1-\alpha}^{(n,\rho)}) = 1-\alpha$

2c) NP says α-level UMP test for $H_0 = 0$ v.s. $H_a = \rho_0$ is

$$R_1 = \{ X : -\rho_0 (\sum x_i^2 + \sum y_i^2) + \sum x_i y_i > T_{1-\alpha}^{n,\rho_0} \}.$$

for $H_0 = 0$ v.s. $H_a = \rho_0'$, each UMP test looks like (up to null set):

$$R_2(k) = \{ X : -\rho_0' (\sum x_i^2 + \sum y_i^2) + \sum x_i y_i \geq k \}$$

If R_1 is a UMP test for $H_0 = 0$ v.s. $H_a = \rho_0' \neq \rho_0$, then $\exists k$ s.t. $R_1 = R_2(k)$

suppose so:

$$-\rho_0 (\sum x_i^2 + \sum y_i^2) + \sum x_i y_i > T_{1-\alpha}^{n,\rho_0}$$

$$\text{iff } -\rho_0' (\sum x_i^2 + \sum y_i^2) + \sum x_i y_i > \underbrace{T_{1-\alpha}^{n,\rho} + (\rho_0 - \rho_0')(\sum x_i^2 + \sum y_i^2)}_{k :=}$$

k is a function of x, y $\times$

R_1 is not UMP for other tests.

Fall 2011

(I)

a) $P(Y_1, \ldots, Y_n | \theta) = \frac{1}{\theta^n} \prod_{i=1}^{n} \mathbb{1}\{Y_i \in [0, \theta]\}$

b) If $Y_i < 0$ for some i, $P(Y_1, \ldots, Y_n | \theta) = 0$, no max exists

(assume $Y_i \geq 0$ $\forall i$):

$\max \{ P(Y_1, \ldots, Y_n | \theta) \mid \theta \in \Omega \}$

$$= \max_{0 \leq \theta \leq \theta_0} \frac{1}{\theta^n} \prod_{i=1}^{n} \mathbb{1}\{Y_i \in [0, \theta]\}$$

$$= \max_{0 \leq \theta \leq \theta_0} \frac{1}{\theta^n} \mathbb{1}\{\theta \geq Y_1, \ldots, Y_n\}$$

$$= \max_{0 \leq \theta \leq \theta_0} \frac{1}{\theta^n} \mathbb{1}\{\theta \geq Y_{max}\}$$

$$= \begin{cases} 0, & \theta_0 < Y_{max} \\ \frac{1}{Y_{max}^n}, & \theta_0 \geq Y_{max} \end{cases}$$

since $\frac{1}{\theta^n}$ decreasing on $[0, Y_{max}]$.

(1)

i) $$\lambda(Y) = \frac{\sup_{\theta = \theta_0} L(\theta \mid Y_1, \dots, Y_n)}{\sup_{\theta < \theta_0} L(\theta \mid Y_1, \dots, Y_n)}$$

$$= \frac{L(\theta_0 \mid Y_1, \dots, Y_n)}{\quad}$$

$$= \frac{\frac{1}{\theta_0^n} \mathbb{1}\{\theta_0 \geq Y_{max}\}}{\frac{1}{Y_{max}^n} \mathbb{1}\{\theta_0 \geq Y_{max}\}}$$

$$= \left(\frac{Y_{max}}{\theta_0}\right)^n$$

1

d) $B(\theta) = \mathbb{P}_\theta(X \in R)$ where R is rejection region

$$\alpha = \mathbb{P}_{\theta_0}(\lambda(X) \le c_\alpha) = \alpha$$

$$= \mathbb{P}_{\theta_0}\left(\left(\frac{Y_{max}}{\theta_0}\right)^n \le c_\alpha\right) = \alpha$$

$$= \mathbb{P}_{\theta_0}\left(Y_{max} \le \theta_0 \sqrt[n]{c_\alpha}\right)$$

$$= \mathbb{P}_{\theta_0}\left(Y_1, \dots, Y_n \le \theta_0 \sqrt[n]{c_\alpha}\right)$$

$$= \prod_{i=1}^{n} \mathbb{P}_{\theta_0}\left(Y_i \le \theta_0 \sqrt[n]{c_\alpha}\right)$$

$$= \prod_{i=1}^{n} \sqrt[n]{c_\alpha}$$

$$= c_\alpha$$

LRT Rejects if $\left(\frac{Y_{max}}{\theta_0}\right)^n \le c_\alpha = \alpha$

$\Rightarrow Y_{max} \le \theta_0 \sqrt[n]{\alpha}$

(2) a) $\mathbb{P}((I_1, Y_1) \cdots (I_n, Y_n) \mid \lambda, \mu) = L_c(\lambda, \mu)$

$$= \prod_{i=1}^{n} \mathbb{P}(I_i, Y_i \mid \theta)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \left(\frac{\lambda}{\sigma_1} I_i + \frac{1-\lambda}{\sigma_0} (1 - I_i) \right) e^{-\frac{(Y_i - \mu)^2}{2[\sigma_1^2 I_i + (1-I_i)\sigma_0^2]}}$$

b) $\max L_c(\lambda, \mu)$

$$= \max \left(\log L_c(\lambda, \mu) \right)$$

$$= \max \left[\sum_{i=1}^{n} \left[\frac{-(Y_i - \mu)^2}{2[\sigma_1^2 I_i + (1-I_i)^2 \sigma_0^2]} + \log \left[\frac{\lambda}{\sigma_1} I_i + \frac{1-\lambda}{\sigma_0} (1 - I_i) \right] \right] \right]$$

Continuous in μ, λ, bounded above, $\lim_{\mu \to \pm\infty} = -\infty$

$\lambda \in [0, 1]$

So take derivative

$$\frac{d}{d\mu} \log L_c(\lambda, \mu) = \sum_{i=1}^{n} \frac{-2(Y_i - \mu)}{2\underbrace{(\sigma_1^2 I_i + (1-I_i)^2 \sigma_0^2)}_{A_i}}$$

$$= \mu \sum_{i=1}^{n} \frac{1}{A_i} - \sum_{i=1}^{n} \frac{Y_i}{A_i} = 0$$

$$\Rightarrow \hat{\mu} = \frac{\sum_{i=1}^{n} \frac{Y_i}{A_i}}{\sum_{i=1}^{n} \frac{1}{A_i}}$$

(2) b) $\frac{d}{d\lambda} \log L_c(\lambda, n)$

cont.

$$= \sum_{i=1}^{n} \left[\frac{1}{\frac{\lambda}{\sigma_1} I_i + \frac{(1-\lambda)}{\sigma_2}(1-I_i)} \cdot \left(\frac{I_i}{\sigma_1} - \frac{(1-I_i)}{\sigma_2} \right) \right]$$

$$= \sum_{i=1}^{n} \frac{\left(\frac{I_i}{\sigma_1} - \frac{(1-I_i)}{\sigma_2} \right)}{\lambda\left(\frac{I_i}{\sigma_1} - \frac{1-I_i}{\sigma_2} \right) + \frac{(1-I_i)}{\sigma_2}}$$

$$= \frac{\frac{1}{\sigma_1}}{\lambda/\sigma_1} \sum_{i=1}^{n} I_i + \frac{-\frac{1}{\sigma_2}}{(1-\lambda)\frac{1}{\sigma_2}} \sum_{i=1}^{n} (1-I_i)$$

$$= \frac{1}{\lambda} \sum I_i - \frac{1}{1-\lambda} \sum (1-I_i) = 0$$

$$\Rightarrow (1-\lambda)\sum I_i - \lambda \sum (1-I_i) = 0$$

$$\sum I_i - \lambda \sum I_i - \lambda \sum 1 + \lambda \sum I_i = 0$$

$$n\lambda = \sum I_i$$

$$\lambda = \frac{1}{n} \sum I_i$$

a) c) E-step: Find $\mathbb{E}\left[\log L(\lambda, \mu \mid I, Y) \mid \lambda^r, \mu^r, Y\right]$

$$= \mathbb{E}\left[\sum_{i=1}^{n} \frac{-(Y_i-\mu)^2}{2[\sigma_1^2 I_i + (1-I_i)\sigma_0^2]} + \log\left[\frac{\lambda}{\sigma_1} I_i + \frac{1-\lambda}{\sigma_0}(1-I_i)\right] \,\middle|\, \lambda^r, \mu^r, Y\right]$$

$$= \sum_{i=1}^{n} \mathbb{E} \text{ "}$$

$$= \sum_{i=1}^{n} \mathbb{P}(I_i=1 \mid \lambda^r, \mu^r, Y_i)\left[\frac{-(Y_i-\mu^r)^2}{2\sigma_1^2} + \log\frac{\lambda^r}{\sigma_1}\right] + \mathbb{P}(I_i=0 \mid \lambda^r, \mu^r, Y_i) \left[\frac{-(Y_i-\mu^r)^2}{2\sigma_0^2} + \log\frac{1-\lambda^r}{\sigma_1}\right]$$

$$P_{1i} := \mathbb{P}(I=1 \mid \lambda^r, \mu^r, Y_i) = \frac{\mathbb{P}(I_i=1, Y_i \mid \mu^r, \lambda^r)}{\mathbb{P}(Y \mid \mu^r, \lambda^r)}$$

$$= \frac{\lambda^r \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(\frac{-(Y_i-\mu^r)^2}{2\sigma_1^2}\right)}{\lambda^r \cdot \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(\frac{-(Y_i-\mu^r)^2}{2\sigma_1^2}\right) + (1-\lambda)\frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(\frac{-(Y_i-\mu^r)^2}{2\sigma_0^2}\right)}$$

$$P_{0i} := \mathbb{P}(I=0 \mid \lambda^r, \mu^r, Y_i) = \frac{(1-\lambda^r)\frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(\frac{-(Y_i-\mu^r)^2}{2\sigma_0^2}\right)}{\lambda^r \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(\frac{-(Y_i-\mu^r)^2}{2\sigma_1^2}\right) + (1-\lambda)\frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(\frac{-(Y_i-\mu^r)^2}{2\sigma_0^2}\right)}$$

(2) c) M- Step $\lambda^{r+1} = \max_{\lambda} \mathbb{E}\left[\log L(\lambda, \mu \mid I, Y) \mid \lambda^r, \mu^r, Y\right]$

$$\mu^{r+1} = \max_{\mu} \mathbb{E}\left[\log L(\lambda, \mu \mid I, Y) \mid \lambda^r, \mu^r, Y\right]$$

assuming P_1, P_0 constant

$$\frac{d}{d\mu} \text{ `` } = \sum_{i=1}^{n} \frac{P_{1i}(Y_i - \mu)}{\sigma_1^2} + P_0\left(\frac{(Y_i - \mu)}{\sigma_0^2}\right) = 0$$

$$\mu \sum\left(\frac{P_{1i}}{\sigma_1^2} + \frac{P_{0i}}{\sigma_0^2}\right) = \sum\left(\frac{P_{1i}}{\sigma_1^2} + \frac{P_{0i}}{\sigma_0^2}\right) Y_i$$

$$\boxed{\mu^{r+1} = \frac{\sum\left(\frac{P_{1i}}{\sigma_1^2} + \frac{P_{0i}}{\sigma_0^2}\right) Y_i}{\sum\left(\frac{P_{1i}}{\sigma_1^2} + \frac{P_{0i}}{\sigma_0^2}\right)}}$$

$$\frac{d}{d\lambda} = \sum_{i=1}^{n} \frac{P_{1i}}{\lambda} - \frac{P_{0i}}{1-\lambda} = 0$$

$$(1-\lambda)\sum P_{1i} - \lambda \sum P_{0i} = 0$$

$$\lambda = \frac{\sum P_{1i}}{\sum P_{1i} + \sum P_{0i}}$$