Spring 2011

1. Let $X_1, \dots, X_n$ be iid with distribution $\mathcal{N}(\mu, \sigma^2)$ and $n \geq 2$.

   a) Find UMVU estimates $\hat{\mu}$ and $\hat{\sigma}^2$ of μ and σ^2, respectively, and prove that they are such.
   Solution. The joint density is

$$f(\boldsymbol{x}; \theta) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2\right)$$
$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{n\mu^2}{2\sigma^2}\right) \exp\left(\frac{\mu}{\sigma^2}\sum_{i=1}^n x_i\right) \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2\right).$$

Hence, by the Factorization theorem, $\left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2\right)$ are sufficient for μ and σ^2. Also, since $\left\{\left(\frac{\mu}{\sigma^2}, -\frac{1}{2\sigma^2}\right) : \mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}^+\right\}$ contains an open set in $\mathbb{R}^2$, it follows that $\left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2\right)$ are complete as well. Now, note that

$$\mathbb{E}\sum_{i=1}^n X_i = n\mathbb{E}X_1 = n\mu.$$

Hence,

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n X_i\right] = \mu,$$

and so, $\hat{\mu} := \boxed{\frac{1}{n}\sum_{i=1}^n X_i}$ is an UMVUE for μ. Now, define

$$\hat{\sigma}^2 := \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2 = \frac{1}{n-1}\sum_{i=1}^n \left(X_i^2 - 2X_i\bar{X} + \bar{X}^2\right)$$
$$= \frac{1}{n-1}\left[\left(\sum_{i=1}^n X_i^2\right) - \frac{2}{n}\left(\sum_{i=1}^n X_i\right)^2 + \frac{1}{n}\left(\sum_{i=1}^n X_i\right)^2\right]$$
$$= \boxed{\frac{1}{n-1}\left(\sum_{i=1}^n X_i^2 - \frac{1}{n}\left(\sum_{i=1}^n X_i\right)^2\right)},$$

which is a function of our complete and sufficient statistics. Since

$$\mathbb{E}\hat{\sigma}^2 = \frac{1}{n-1}\mathbb{E}\left[\sum_{i=1}^n (X_i-\bar{X})^2\right] = \frac{1}{n-1}\mathbb{E}\left[\sum_{i=1}^n X_i^2 - n\bar{X}^2\right]$$
$$= \frac{1}{n-1}\left[n\left(\mu^2+\sigma^2\right) - n\left(\mu^2 + \frac{\sigma^2}{n}\right)\right] = \frac{1}{n-1}\frac{n-1}{n}\sigma^2 = \sigma^2,$$

by Lehmann-Scheffe theorem, we have that $\hat{\sigma}^2$ is an UMVUE for σ^2.

b) Derive the marginal distributions of $\hat{\mu}$ and $\hat{\sigma}^2$, and prove that these estimators are independent.

Solution. Since $\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2)$, we have that

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right).$$

Now, define $P = I - \frac{1}{n}\mathbf{1}\mathbf{1}^T$. Note that P is an orthogonal projection, i.e. $P^2 = P$, and $P^T = P$. Since P is symmetric, there exists T and Λ such that $P = T^T \Lambda T$, where T is orthogonal and Λ is diagonal. Also, since P is an orthogonal projection, the entries of Λ are either 0 or 1. Then,

$$\begin{aligned}\sum_{i=1}^{n}(X_i - \bar{X})^2 &= \left\| \begin{bmatrix} X_1 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{bmatrix} \right\|^2 = \|P\boldsymbol{X}\|^2 = (P\boldsymbol{X})^T P\boldsymbol{X} = \boldsymbol{X}^T P^T P \boldsymbol{X} = \boldsymbol{X}^T P \boldsymbol{X} \\ &= \boldsymbol{X}^T T^T \Lambda T \boldsymbol{X} = (T\boldsymbol{X})^T \Lambda (T\boldsymbol{X}).\end{aligned}$$

Now,

$$\begin{aligned}\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \bar{X})^2 &= \sum_{i=1}^{n}\left[\frac{X_i - \mu - (\bar{X} - \mu)}{\sigma}\right]^2 = \sum_{i=1}^{n}(Z_i - \bar{Z})^2, \qquad Z_i \sim \mathcal{N}(0,1) \\ &= \boldsymbol{Z}^T P \boldsymbol{Z} = (T\boldsymbol{Z})^T \Lambda (T\boldsymbol{Z}) =_d \boldsymbol{Z}^T \Lambda \boldsymbol{Z} \\ &\qquad \text{(since normal distributions are invariant under reflection and rotation)} \\ &= \sum_{i=1}^{n} \lambda_i Z_i^2 = (n-1) Z_1^2 \sim \chi^2_{n-1},\end{aligned}$$

since

$$\sum_{i=1}^{n} \lambda_i = \operatorname{tr}(P) = tr\left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right) = n - \frac{1}{n}\operatorname{tr}\left(\mathbf{1}\mathbf{1}^T\right) = n - 1.$$

Now, since

$$\begin{aligned}\operatorname{Cov}\left(\frac{1}{n}\mathbf{1}^T\boldsymbol{X}, P\boldsymbol{X}\right) &= \frac{1}{n}\mathbf{1}^T \operatorname{Var}(\boldsymbol{X}) P^T = \frac{1}{n}\mathbf{1}^T(\sigma^2 I)P = \frac{\sigma^2}{n}\mathbf{1}^T\left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right) \\ &= \frac{\sigma^2}{n}\left(\mathbf{1}^T - \frac{1}{n}\mathbf{1}^T\mathbf{1}\mathbf{1}^T\right) = \frac{\sigma^2}{n}\left(\mathbf{1}^T - \frac{1}{n}n\mathbf{1}^T\right) = 0,\end{aligned}$$

we have that $\bar{X} = \frac{1}{n}\mathbf{1}^T\boldsymbol{X}$ and $P\boldsymbol{X} = \boldsymbol{X} - \mathbf{1}\bar{X}$ are independent. Now, for any measurable f, we know that $\bar{X}$ and $f(\boldsymbol{X} - \mathbf{1}\bar{X})$ will be independent. In particular, take $f(\boldsymbol{x}) = \frac{1}{n-1}\boldsymbol{x}^T\boldsymbol{x}$. Then,

$$f\left(\boldsymbol{X} - \mathbf{1}\bar{X}\right) = \frac{1}{n-1}\begin{bmatrix} X_1 - \bar{X} & \cdots & X_n - \bar{X} \end{bmatrix}\begin{bmatrix} X_1 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{bmatrix} = \hat{\sigma}^2,$$

and so, $\hat{\mu}$ and $\hat{\sigma}^2$ are independent. □

2. For $\theta \in \mathbb{R}$, let $X_1, X_2, \ldots, X_n$ be independent continuous random variables, each having density function

$$p(x;\theta) = \exp\left(-(x-\theta)\right) I\{x > \theta\},$$

where $I(x) = 1$ if $x > 0$ and $I(x) = 0$ otherwise. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the corresponding order statistics.

a) Find the joint density function of $\left(X_{(1)}, X_{(2)}\right)$, and the marginal densities of $X_{(1)}$ and $X_{(2)}$.

Solution. First, note that the cumulative distribution function is

$$F(x) = P(X \leq x) = \int_\theta^x e^{-(x-\theta)}\, dx = 1 - e^{-(x-\theta)},$$

for $x > \theta$. Then, the joint density function is

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= \binom{n}{1,1,n-2} f_{X_1}(x_1) f_{X_2}(x_2) \left(1 - F(x_2)\right)^{n-2} \\ &= \frac{n!}{(n-2)!} e^{-(x_1-\theta)} e^{-(x_2-\theta)} \left(e^{-(x_2-\theta)}\right)^{n-2} \\ &= n(n-1) e^{-(x_1-\theta)} e^{-(n-1)(x_2-\theta)}. \end{aligned}$$

As for the marginal densities, we have

$$f_{X_{(1)}}(x_1) = \binom{n}{n-1} f_{X_1}(x_1) \left(1 - F(x_1)\right)^{n-1} = n e^{-n(x_1-\theta)},$$

and

$$f_{X_{(2)}}(x_2) = \binom{n}{1,1,n-2} F(x_2) f_{X_2}(x_2) \left(1 - F(x_2)\right)^{n-2} = n(n-1)\left(1 - e^{-(x_2-\theta)}\right) e^{-(n-1)(x_2-\theta)}.$$

b) Show that

$$T = X_{(1)} - (n-1)\left(X_{(2)} - X_{(1)}\right)/n$$

is an unbiased estimator of θ.

Solution. Note that

$$\mathbb{E}X_{(1)} = \int_\theta^\infty x n e^{-n(x-\theta)}\, dx = -\, x e^{-n(x-\theta)}\Big|_\theta^\infty + \int_\theta^\infty e^{-n(x-\theta)}\, dx = \theta - \frac{e^{-n(x-\theta)}}{n}\Big|_\theta^\infty = \theta + \frac{1}{n},$$

and

$$\begin{aligned} \mathbb{E}X_{(2)} &= \int_\theta^\infty x n(n-1) e^{-(n-1)(x-\theta)}\, dx - \int_\theta^\infty x n(n-1) e^{-n(x-\theta)}\, dx \\ &= \left(-\, n x e^{-(n-1)(x-\theta)}\Big|_\theta^\infty + \int_\theta^\infty n e^{-(n-1)(x-\theta)}\, dx\right) \\ &\quad - \left(-\,(n-1) x e^{-n(x-\theta)}\Big|_\theta^\infty + \int_\theta^\infty (n-1) e^{-n(x-\theta)}\, dx\right) \\ &= \left(n\theta + \frac{n}{n-1}\right) - \left((n-1)\theta + \frac{n-1}{n}\right) = \theta + \frac{2n-1}{n(n-1)}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}T &= \mathbb{E}X_{(1)} - \frac{n-1}{n}\left(\mathbb{E}X_{(2)} - \mathbb{E}X_{(1)}\right) = \left(\theta + \frac{1}{n}\right) - \frac{n-1}{n}\left(\theta - \frac{2n-1}{n(n-1)} - \theta - \frac{1}{n}\right) \\ &= \left(\theta + \frac{1}{n}\right) - \frac{n-1}{n}\frac{n}{n(n-1)} = \theta, \end{aligned}$$

and so, T is an unbiased estimator of θ. □

c) Find the maximum likelihood estimate of θ.

Solution. The likelihood function is

$$\mathcal{L}(\theta; \boldsymbol{x}) = \prod_{i=1}^{n} \exp\left(-(x_i - \theta)\right) = \exp\left(-\sum_{i=1}^{n} x_i\right) \exp\left(n\theta\right),$$

and so, the log-likelihood function is

$$\log \mathcal{L}(\theta; \boldsymbol{x}) = -\sum_{i=1}^{n} x_i + n\theta.$$

Hence, the derivative of the log-likelihood function with respect to θ is

$$\frac{d}{d\theta} \log \mathcal{L}(\theta; \boldsymbol{x}) = n > 0,$$

which means that the log-likelihood function is always increasing. Hence, the MLE is the maximum possible value of θ, which is

$$\hat{\theta} = \boxed{X_{(1)}}.$$