Fall 2011

1. Let $Y_1, \ldots, Y_n$ be a random sample from the density $p(y|\theta) = 2y/\theta^2$, $0 < y < \theta$, where $\theta > 0$ is an unknown parameter.

   a) Let $\bar{Y} = (1/n)\sum_{i=1}^n Y_i$. Find the mean and variance of $\bar{Y}$.
   Solution. Note that
   $$\mathbb{E}\bar{Y} = \frac{1}{n}\sum_{i=1}^n \mathbb{E}Y_i = \int_0^\theta y\frac{2y}{\theta^2}\,dy = \frac{2}{3}\frac{y^3}{\theta^2}\bigg|_0^\theta = \boxed{\frac{2}{3}\theta},$$
   and
   $$\mathbb{E}Y_1^2 = \int_0^\theta y^2\frac{2y}{\theta^2}\,dy = \frac{1}{2}\frac{y^4}{\theta^2}\bigg|_0^\theta = \frac{1}{2}\theta^2.$$
   Hence,
   $$\operatorname{Var}(\bar{Y}) = \frac{1}{n^2}\sum_{i=1}^n \operatorname{Var}(Y_i) = \frac{1}{n}\left(\mathbb{E}Y_1^2 - (\mathbb{E}Y_1)^2\right) = \frac{1}{n}\left(\frac{1}{2}\theta^2 - \frac{4}{9}\theta^2\right) = \boxed{\frac{1}{18n}\theta^2}.$$

   b) Show that $\hat{\theta} = (3/2)\bar{Y}$ is an unbiased estimator of θ.
   Solution. As we saw in part (a), $\mathbb{E}\bar{Y} = \frac{2}{3}\theta$. Hence,
   $$\mathbb{E}\left(\frac{3}{2}\bar{Y}\right) = \frac{3}{2}\frac{2}{3}\theta = \theta,$$
   and so, $\frac{3}{2}\bar{Y}$ is an unbiased estimator of θ. □

   c) State the Cramer-Rao inequality for the above situation.
   Solution. The likelihood function is
   $$\mathcal{L}(\theta;\boldsymbol{y}) = \prod_{i=1}^n \frac{2y_i}{\theta^2} = \frac{2^n\prod y_i}{\theta^{2n}},$$
   and so, the log-likelihood function is
   $$\log\mathcal{L}(\theta;\boldsymbol{y}) = n\log 2 + \sum_{i=1}^n \log y_i - 2n\log\theta.$$
   Hence,
   $$\frac{d}{d\theta}\log\mathcal{L}(\theta;\boldsymbol{y}) = -\frac{2n}{\theta}, \text{ and } \frac{d^2}{d\theta^2}\log\mathcal{L}(\theta;\boldsymbol{y}) = \frac{2n}{\theta^2},$$
   and so, the Fisher information is
   $$\mathcal{I}(\theta) = -\frac{2n}{\theta^2}.$$
   Thus, if the regularity conditions hold, Cramer-Rao inequality states that for any unbiased estimator $W(\boldsymbol{Y})$,
   $$\operatorname{Var}(W(\boldsymbol{Y})) \geq -\frac{\theta^2}{2n}.$$

   d) Show that $\operatorname{Var}(\hat{\theta})$ violates the Cramer-Rao inequality. Explain why.

2. a) Let X have normal distribution with mean θ and variance σ^2, and let g be a differentiable function satisfying $\mathbb{E}|g'(X)| < \infty$. Show that

$$\mathbb{E}\left[g(X)(X-\theta)\right] = \sigma^2 \mathbb{E} g'(X).$$

[HINT: use integration by parts of Fubini's theorem.]

Solution. We have

$$\begin{aligned}
\mathbb{E}\left[g(X)(X-\theta)\right] &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} g(x)(x-\theta)\exp\left(-\frac{(x-\theta)^2}{2\sigma^2}\right)\,dx \\
&= -\frac{1}{\sqrt{2\pi\sigma^2}}\sigma^2 \left[\cancelto{0}{g(x)\exp\left(-\frac{(x-\theta)^2}{2\sigma^2}\right)\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} g'(x)\exp\left(-\frac{(x-\theta)^2}{2\sigma^2}\right)\,dx \right] \\
&= \sigma^2 \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} g'(x)\exp\left(-\frac{(x-\theta)^2}{2\sigma^2}\right)\,dx = \sigma^2 \mathbb{E} g'(X),
\end{aligned}$$

where the marked quantity vanishes to 0 since $\mathbb{E}|g'(X)| < \infty$. □

b) Let $g(x)$ be a function with $-\infty < \mathbb{E} g(X) < \infty$ and $g(-1)$ is finite. If X has a Poisson distribution with mean λ, show that

$$\mathbb{E}\left(\lambda g(X)\right) = \mathbb{E}\left(X(g(X-1))\right).$$

Solution. We have

$$\begin{aligned}
\mathbb{E}\left(\lambda g(X)\right) &= \sum_{x=0}^{\infty} \lambda g(x) \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty} g(x) \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}(x+1) \\
&\overset{(*)}{=} \sum_{x=0}^{\infty} x g(x-1) \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \mathbb{E}\left(X\left(g(X-1)\right)\right),
\end{aligned}$$

where the index shift in $(*)$ works because $g(-1)$ is finite. □