Spring 2010

1. Let $X_1, \dots, X_n$ be iid $\Gamma(p, 1/\lambda)$ with density $g_\theta(x) = \frac{1}{\Gamma(p)}\lambda^p x^{p-1}e^{-\lambda x}, x > 0, \theta = (p, \lambda), p > 0, \lambda > 0$.

   a) Find a moment estimate of the parameter.
   Solution. We have

$$\begin{aligned}\mathbb{E}X^1 &= \int_0^\infty x\frac{1}{\Gamma(p)}\lambda^p x^{p-1}e^{-\lambda x}\,dx = \frac{1}{\Gamma(p)}\int_0^\infty (\lambda x)^p e^{-\lambda x}\,dx = \frac{1}{\lambda\Gamma(p)}\int_0^\infty u^{(p+1)-1}e^{-u}\,dx \\ &= \frac{\Gamma(p+1)}{\Gamma(p)}\frac{1}{\lambda} = \frac{p}{\lambda},\end{aligned}$$

   and

$$\begin{aligned}\mathbb{E}X^2 &= \int_0^\infty x^2\frac{1}{\Gamma(p)}\lambda^p x^{p-1}e^{-\lambda x}\,dx = \frac{1}{\lambda\Gamma(p)}\int_0^\infty (\lambda x)^{p+1} e^{-\lambda x}\,dx = \frac{1}{\lambda^2\Gamma(p)}\int_0^\infty u^{(p+2)-1}e^{-u}\,dx \\ &= \frac{\Gamma(p+2)}{\Gamma(p)}\frac{1}{\lambda^2} = \frac{p(p+1)}{\lambda^2}.\end{aligned}$$

   Setting $\frac{p}{\lambda} = \bar{X}$ and $\frac{p(p+1)}{\lambda^2} = \frac{1}{n}\sum X_i^2$, and solving for λ and p gives us that the moment estimates are

$$\boxed{\tilde{\theta} = (\tilde{p}, \tilde{\lambda}) = \left(\frac{\bar{X}^2}{\frac{1}{n}\sum(X_i - \bar{X})^2}, \frac{\bar{X}}{\frac{1}{n}\sum(X_i - \bar{X})^2}\right)}.$$

   b) Show that the moment estimates, $\tilde{\theta}$, are asymptotically bi-variate normal and give their asymptotic mean and variance covariance matrix.
   Solution. Note that by the multivariate Central Limit Theorem, we have that

$$\sqrt{n}\left(\begin{bmatrix}\frac{1}{n}\sum X_i \\ \frac{1}{n}\sum X_i^2\end{bmatrix} - \begin{bmatrix}\frac{p}{\lambda} \\ \frac{p^2+p}{\lambda}\end{bmatrix}\right) \Rightarrow \mathcal{N}(0, \Sigma_\theta),$$

   where

$$\Sigma_\theta = \begin{bmatrix}\mathrm{Var}(X_i) & \mathrm{Cov}(X_i, X_i^2) \\ \mathrm{Cov}(X_i, X_i^2) & \mathrm{Var}(X_i^2)\end{bmatrix} = \begin{bmatrix}\frac{p}{\lambda^2} & \frac{2p(p+1)}{\lambda^3} \\ \frac{2p(p+1)}{\lambda^3} & \frac{2p(p+1)(2p+3)}{\lambda^4}\end{bmatrix}.$$

   Now, note that if $g(x, y) = \left(\frac{x^2}{y-x^2}, \frac{x}{y-x^2}\right)$, then $\tilde{\theta} = g\left(\frac{1}{n}\sum X_i, \frac{1}{n}\sum X_i^2\right)$. Also, we have that

$$\nabla g(x, y) = \frac{1}{(y-x^2)^2}\begin{bmatrix}2xy & -x^2 \\ y+x^2 & -x\end{bmatrix},$$

   and so,

$$\nabla g\left(\frac{p}{\lambda}, \frac{p(1+p)}{\lambda^2}\right) = \begin{bmatrix}2\lambda(1+p) & -\lambda^2 \\ \lambda^2\left(\frac{1}{p} + 2p\right) & -\frac{\lambda^3}{p}\end{bmatrix}.$$

   Hence, by the multivariate Delta Method, we have that

$$\sqrt{n}\left(\begin{bmatrix}\tilde{p} \\ \tilde{\lambda}\end{bmatrix} - \begin{bmatrix}p \\ \lambda\end{bmatrix}\right) \Rightarrow \mathcal{N}(0, \nabla g^T \Sigma_\theta \nabla g),$$

   where the asymptotic mean is $\boxed{(p, \lambda)}$, and the asymptotic variance-covariance matrix is

$$\nabla g^T \Sigma_\theta \nabla g = \boxed{\begin{bmatrix}2\lambda(1+p) & \lambda^2\left(\frac{1}{p} + 2p\right) \\ -\lambda^2 & -\frac{\lambda^3}{p}\end{bmatrix}\begin{bmatrix}\frac{p}{\lambda^2} & \frac{2p(p+1)}{\lambda^3} \\ \frac{2p(p+1)}{\lambda^3} & \frac{2p(p+1)(2p+3)}{\lambda^4}\end{bmatrix}\begin{bmatrix}2\lambda(1+p) & -\lambda^2 \\ \lambda^2\left(\frac{1}{p} + 2p\right) & -\frac{\lambda^3}{p}\end{bmatrix}}.$$

c) Compute the asymptotic variance covariance matrix of the maximum likelihood estimates. You may leave your answer in terms of Γ function derivatives.

Solution. Since MLEs are asymptotically efficient, we have that the Cramer-Rao lower bound matrix will be the asymptotic variance-covaraince matrix of the MLEs. The likelihood function is

$$\mathcal{L}(p, \lambda; \boldsymbol{x}) = \prod_{i=1}^{n} \frac{1}{\Gamma(p)} \lambda^p x_i^{p-1} e^{-\lambda x_i} = \Gamma(p)^{-n} \lambda^{np} \left(\prod_{i=1}^{n} x_i \right)^{p-1} e^{-\lambda \sum x_i},$$

and so, the log-likelihood function is

$$\log \mathcal{L}(p, \lambda; \boldsymbol{x}) = -n \log(\Gamma(p)) + np \log \lambda + (p-1) \sum \log x_i - \lambda \sum x_i.$$

Hence, the first partials are

$$\frac{\partial}{\partial p} \log \mathcal{L} = -n \frac{\Gamma'(p)}{\Gamma(p)} + n \log \lambda + \sum x_i, \text{ and } \frac{\partial}{\partial \lambda} \log \mathcal{L} = \frac{np}{\lambda} - \sum x_i,$$

and the second partials are

$$\frac{\partial^2}{\partial p^2} \log \mathcal{L} = -n \frac{\Gamma''\Gamma - \Gamma'^2}{\Gamma^2}, \frac{\partial^2}{\partial p \partial \lambda} \log \mathcal{L} = \frac{n}{\lambda}, \text{ and} \frac{\partial^2}{\partial \lambda^2} \log \mathcal{L} = -n \frac{p}{\lambda^2}.$$

Hence, the Fisher information matrix is

$$\mathcal{I}(p, \lambda) = -\mathbb{E} \begin{pmatrix} -n \frac{\Gamma''\Gamma - \Gamma'^2}{\Gamma^2} & \frac{n}{\lambda} \\ \frac{n}{\lambda} & -n \frac{p}{\lambda^2} \end{pmatrix} = n \begin{pmatrix} \frac{\Gamma''\Gamma - \Gamma'^2}{\Gamma^2} & -\frac{1}{\lambda} \\ -\frac{1}{\lambda} & \frac{p}{\lambda^2} \end{pmatrix}.$$

Thus, the Cramer-Rao lower bound matrix (and the asymptotic variance covariance matrix) is

$$\mathcal{I}(p, \lambda)^{-1} = \boxed{\frac{1}{n} \frac{\lambda^2 \Gamma^2}{p(\Gamma''\Gamma - \Gamma'^2) - \Gamma^2} \begin{pmatrix} \frac{p}{\lambda^2} & \frac{1}{\lambda} \\ \frac{1}{\lambda} & \frac{\Gamma''\Gamma - \Gamma'^2}{\Gamma^2} \end{pmatrix}}.$$

2. Recall that the t-distribution with $k > 0$ degrees of freedom, location parameter l, and scale parameter $s > 0$ has density

$$\frac{\Gamma((k+1)/2)}{\Gamma(k/2)\sqrt{k\pi s^2}}\left[1 + k^{-1}\left(\frac{x-l}{s}\right)^2\right]^{-(k+1)/2}.$$

Show that the t-distribution can be written as a mixture of Gaussian distributions by letting $X \sim N(\mu, \sigma^2), \tau = 1/\sigma^2 \sim \Gamma(\alpha, \beta)$ and integrating the joint density $f(x, \tau|\mu)$ to get the marginal desntiy $f(x|\mu)$. What are the parameters of the resulting t-distribution, as functions of μ, α, β?

Solution. Recall that $\Gamma(\alpha, \beta)$ has distribution

$$f(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)}\frac{1}{\beta^\alpha}x^{\alpha-1}e^{-x/\beta}.$$

Then, the joint density is

$$\begin{aligned} f(x, \tau \mid \mu) = f(x \mid \tau, \mu) \cdot f(\tau) &= \frac{\sqrt{\tau}}{\sqrt{2\pi}}\exp\left[-\frac{(x-\mu)^2}{2}\tau\right] \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha}\tau^{\alpha-1}\exp\left[-\frac{\tau}{\beta}\right] \\ &= \frac{1}{\sqrt{2\pi}\Gamma(\alpha)}\frac{\tau^{\alpha-1/2}}{\beta^\alpha}\exp\left[-\tau\left(\frac{(x-\mu)^2}{2} + \frac{1}{\beta}\right)\right]. \end{aligned}$$

Now, we integrate the joint density to get the marginal density:

$$\begin{aligned} f(x \mid \mu) = \int_0^\infty f(x, \tau \mid \mu)\, d\tau &= \frac{1}{\sqrt{2\pi}\Gamma(\alpha)\beta^\alpha}\int_0^\infty \tau^{\alpha-1/2}\exp\left[-\tau\left(\frac{(x-\mu)^2}{2} + \frac{1}{\beta}\right)\right] d\tau \\ &= \left(\sqrt{2\pi}\Gamma(\alpha)\beta^\alpha\right)^{-1}\left(\frac{(x-\mu)^2}{2} + \frac{1}{\beta}\right)^{-(\alpha+1/2)}\int_0^\infty t^{\alpha-1/2}e^{-t}\, dt \\ &= \left(\sqrt{2\pi}\Gamma(\alpha)\beta^{-1/2}\right)^{-1}\left(\beta\frac{(x-\mu)^2}{2} + 1\right)^{-(\alpha+1/2)}\Gamma\left(\alpha + \frac{1}{2}\right) \\ &= \frac{\Gamma((2\alpha+1)/2)}{\Gamma((2\alpha)/2)\sqrt{(2\alpha)\pi\left(\sqrt{\frac{1}{\alpha\beta}}\right)^2}}\left[1 + (2\alpha)^{-1}\left(\frac{x-\mu}{\sqrt{\frac{1}{\alpha\beta}}}\right)^2\right]^{-(2\alpha+1)/2}, \end{aligned}$$

which is the density of a t-distribution with $k = 2\alpha$ degrees of freedom, locationa parameter $l = \mu$, and scale parameter $s = (\alpha\beta)^{-1/2}$. □