## Fall 2010

1. Let $\mu > 0$ be an unknown parameter, and suppose that $X_1$ and $X_2$ are independent random variables, each having the exponential distribution with density function

$$p(x;\mu) = \frac{1}{\mu}\exp\left(-\frac{x}{\mu}\right), \qquad x > 0.$$

It is clear that since the distribution $p(x;\mu)$ has mean $\mu$ that an unbiased estimator of $\mu$ is given by $\overline{X} = (X_1 + X_2)/2$.

a) Calculate the variance of the estimator $\overline{X}$ of $\mu$.
*Solution.* Note that

$$\begin{aligned}\mathbb{E}X_1^2 &= \int_0^\infty x^2 \frac{1}{\mu} e^{-x/\mu}\,dx = \cancelto{0}{-x^2 e^{-x/\mu}\Big|_0^\infty} + \int_0^\infty 2x e^{-x/\mu}\,dx = \cancelto{0}{-2\mu x e^{-x/\mu}\Big|_0^\infty} + \int_0^\infty 2\mu e^{-x/\mu}\,dx \\ &= -2\mu^2 e^{-x/\mu}\Big|_0^\infty = 2\mu^2.\end{aligned}$$

Then,

$$\mathbb{E}\bar{X}^2 = \mathbb{E}\left(\frac{X_1+X_2}{2}\right)^2 = \frac{1}{4}\mathbb{E}X_1^2 + \frac{1}{2}\mathbb{E}X_1\mathbb{E}X_2 + \frac{1}{4}\mathbb{E}X_2^2 = \frac{\mu^2}{2} + \frac{\mu^2}{2} + \frac{\mu^2}{2} = \frac{3}{2}\mu^2,$$

and so,

$$\mathrm{Var}(\bar{X}) = \mathbb{E}\bar{X}^2 - \left(\mathbb{E}\bar{X}\right)^2 = \frac{3}{2}\mu^2 - \mu^2 = \boxed{\frac{\mu^2}{2}}.$$

b) Now consider the estimator of $\mu$ given by $T(X_1, X_2) = \sqrt{X_1 X_2}$. Calculate the bias of $T(X_1, X_2)$.
*Solution.* Note that

$$\mathbb{E}\sqrt{X_1} = \int_0^\infty \sqrt{x}\frac{1}{\mu}e^{-x/\mu}\,dx = \int_0^\infty \frac{2u^2}{\mu}e^{-u^2/\mu}\,du = \cancelto{0}{-ue^{-u^2/\mu}\Big|_0^\infty} + \int_0^\infty e^{-u^2/\mu}\,du = \frac{\sqrt{\pi}}{2}\sqrt{\mu}.$$

Then, $\mathbb{E}T = \mathbb{E}\sqrt{X_1X_2} = \mathbb{E}\sqrt{X_1}\mathbb{E}\sqrt{X_2} = \frac{\pi}{4}\mu$, and so the bias of $T$ is

$$\frac{\pi}{4}\mu - \mu = \boxed{\left(\frac{\pi}{4} - 1\right)\mu}.$$

c) Show that the mean square error of $T(X_1, X_2)$ as an estimator of $\mu$, that is

$$\mathrm{MSE}(T) = \mathbb{E}\left[T(X_1,X_2) - \mu\right]^2,$$

is strictly smaller than the mean squared error of the estimator $\overline{X}$ of $\mu$.
*Solution.* We have

$$\begin{aligned}\mathrm{MSE}(T) = \mathbb{E}[T-\mu]^2 &= \mathbb{E}T^2 - 2\mu\mathbb{E}T + \mu^2 \\ &= \mathbb{E}X_1X_2 - 2\left(\frac{\pi}{4}\right)\mu^2 + \mu^2 \\ &= \mu^2 - \frac{\pi}{2}\mu^2 + \mu^2 = \left(2 - \frac{\pi}{2}\right)\mu^2,\end{aligned}$$

and

$$\begin{aligned}\mathrm{MSE}(\bar{X}) = \mathbb{E}[\bar{X}-\mu]^2 &= \mathbb{E}\bar{X}^2 - 2\mu\mathbb{E}\bar{X} + \mu^2 \\ &= \frac{3}{2}\mu^2 - 2\mu^2 + \mu^2 = \frac{1}{2}\mu^2.\end{aligned}$$

Since $2 - \frac{\pi}{2} < \frac{1}{2}$, it follows that $\mathrm{MSE}(T) < \mathrm{MSE}(\bar{X})$. □

2. A random variable $X$ has the Weibull$(\alpha, \beta)$ distribution if

$$P_{\alpha,\beta}(X > x) = \exp\left[-(x/\alpha)^\beta\right] \qquad \text{for } x \geq 0.$$

Suppose that $X_1, \ldots, X_n$ are iid Weibull$(\alpha_0, \beta_0)$, where $(\alpha_0, \beta_0)$ is in the interior of a compact parameter space $\Theta \subseteq \mathbb{R}^+ \times \mathbb{R}^+$.

a) Show that the MLE of $\alpha$ is given by

$$\hat{\alpha} = \left(n^{-1} \sum_{i=1}^{n} X_i^{\hat{\beta}}\right)^{1/\hat{\beta}},$$

where $\hat{\beta}$ is the MLE of $\beta$.

*Solution.* Note that $P(X \leq x; \alpha, \beta) = 1 - \exp\left[-\left(\frac{x}{\alpha}\right)^\beta\right]$, and so, the probability density function of $X$ is

$$f(x; \alpha, \beta) = \beta \frac{x^{\beta-1}}{\alpha^\beta} \exp\left[-\left(\frac{x}{\alpha}\right)^\beta\right].$$

Then, the likelihood function is

$$L(\alpha, \beta; \boldsymbol{x}) = \prod_{i=1}^{n} \left(\beta \frac{x_i^{\beta-1}}{\alpha^\beta} \exp\left[-\left(\frac{x_i}{\alpha}\right)^\beta\right]\right) = \frac{\beta^n}{\alpha^{\beta n}} \left(\prod_{i=1}^{n} x_i\right)^{\beta-1} \exp\left[-\sum_{i=1}^{n} \left(\frac{x_i}{\alpha}\right)^\beta\right],$$

and so, the log-likelihood function is

$$\log L(\alpha, \beta; \boldsymbol{x}) = n \log \beta - \beta n \log \alpha + (\beta - 1) \log\left(\prod_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} \left(\frac{x_i}{\alpha}\right)^\beta.$$

The partial derivative of the log-likelihood function with respect to $\alpha$ is

$$\frac{\partial}{\partial \alpha} \log L(\alpha, \beta; \boldsymbol{x}) = -\frac{\beta n}{\alpha} + \beta \frac{\sum_{i=1}^{n} x_i^\beta}{\alpha^{\beta+1}},$$

and setting it equal to zero gives us

$$\frac{\beta n}{\alpha} = \beta \frac{\sum_{i=1}^{n} x_i^\beta}{\alpha^{\beta+1}} \implies \alpha^\beta = n^{-1} \sum_{i=1}^{n} x_i^\beta.$$

Thus, the MLE of $\alpha$ is

$$\hat{\alpha} = \left(n^{-1} \sum_{i=1}^{n} X_i^{\hat{\beta}}\right)^{1/\hat{\beta}}.$$

□

b) As an alternative to maximum likelihood, let $\tilde{\beta}$ be any estimator of $\beta$ and consider the estimator of $\alpha$ given by

$$\tilde{\alpha} = \left( n^{-1} \sum_{i=1}^{n} X_i^{\tilde{\beta}} \right)^{1/\tilde{\beta}}.$$

Show that $\tilde{\alpha}$ is a "pseudo-MLE" in the sense that it maximizes $l(\alpha, \tilde{\beta})$, where $l(\alpha, \beta)$ is the log-likelihood function.

*Solution.* From part (a), we have that

$$l(\alpha, \tilde{\beta}) = n \log \tilde{\beta} - \tilde{\beta} n \log \alpha + (\tilde{\beta} - 1) \log \left( \prod_{i=1}^{n} x_i \right) - \sum_{i=1}^{n} \left( \frac{x_i}{\alpha} \right)^{\tilde{\beta}}.$$

Taking the partial derivative of $l(\alpha, \tilde{\beta})$ with respect to $\alpha$ gives us

$$\frac{\partial}{\partial \alpha} l(\alpha, \tilde{\beta}) = -\frac{\tilde{\beta} n}{\alpha} + \tilde{\beta} \frac{\sum_{i=1}^{n} x_i^{\tilde{\beta}}}{\alpha^{\tilde{\beta}+1}},$$

and so, similarly as above, we see that

$$\tilde{\alpha} = \left( n^{-1} \sum_{i=1}^{n} X_i^{\tilde{\beta}} \right)^{1/\tilde{\beta}}$$

maximizes $l(\alpha, \tilde{\beta})$, and so, $\tilde{\alpha}$ is "pseudo-MLE." ☐

c) Prove that $\mathbb{E} X_1^{\beta_0} = \alpha_0^{\beta_0}$.

*Solution.* We have

$$\begin{aligned}
\mathbb{E} X_1^{\beta} &= \int_0^{\infty} x^{\beta} \beta \frac{x^{\beta-1}}{\alpha^{\beta}} \exp\left[ -\left( \frac{x}{\alpha} \right)^{\beta} \right] dx = \int_0^{\infty} \beta \frac{x^{2\beta-1}}{\alpha^{\beta}} \exp\left[ -\left( \frac{x}{\alpha} \right)^{\beta} \right] dx \\
&= \cancelto{0}{-x^{\beta} \exp\left[ -\left( \frac{x}{\alpha} \right)^{\beta} \right] \Bigg|_0^{\infty}} + \int_0^{\infty} \beta x^{\beta-1} \exp\left[ -\left( \frac{x}{\alpha} \right)^{\beta} \right] dx \\
&= -\alpha^{\beta} \exp\left[ -\left( \frac{x}{\alpha} \right)^{\beta} \right] \Bigg|_0^{\infty} = \alpha^{\beta},
\end{aligned}$$

and so, $\mathbb{E} X_1^{\beta_0} = \alpha_0^{\beta_0}$. ☐

d) Prove that if $\tilde{\beta}$ is any consistent estimator, then $\tilde{\alpha}$ given in (b) is consistent for $\alpha$.

[HINT: Letting $Y_n(\beta) = n^{-1} \sum_i X_i^{\beta}$, argue that it suffices to prove that $Y_n(\tilde{\beta}) \to \alpha^{\beta_0}$ in probability. Then, using the convexity of $Y_n(\beta)$, show that

$$|Y_n(\tilde{\beta}) - Y_n(\beta_0)| \xrightarrow{p} 0$$

and complete the argument by applying (c) and the law of large numbers.]