Spring 2009

1. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli distribution with parameter $p \in (0,1)$, that is,

$$P(X_i = 1) = p, \qquad \text{and} \qquad P(X_i = 0) = 1 - p.$$

a) Find a complete sufficient statistic $T_n(X_1, \ldots, X_n)$ for p.
Solution. The pdf is

$$f(x_1, \ldots, x_n; p) = p^{\sum x_i}(1-p)^{n-\sum x_i} = g_p\left(\frac{1}{n}\sum_{i=1}^n x_i\right) h(x_1, \ldots, x_n),$$

where $g_p(x) = p^{nx}(1-p)^{n-nx}$ and $h(x_1, \ldots, x_n) = 1$. Hence, by Fisher-Neyman Factorization Theorem, $T(\boldsymbol{X}) = \sum_{i=1}^n X_i$ is a sufficient statistic. Also, the pdf for X_1 is

$$f(x_1; p) = p^{x_1}(1-p)^{1-x_1} = (1-p)\left(\frac{p}{1-p}\right)^{x_1} = (1-p)\exp\left(x_1 \log \frac{p}{1-p}\right),$$

and since the set $\left\{\frac{p}{1-p} : p \in (0,1)\right\}$ contains an open set in $\mathbb{R}$, we see that $\boxed{T(\boldsymbol{X}) = \sum_{i=1}^n X_i}$ is also complete.

b) Justify $T_n(X_1, \ldots, X_n)$ is sufficient and complete using the definitions of sufficiency and completeness.
Solution. Note that $T_n(X_1, \ldots, X_n) \sim \text{Binomial}(n, p)$. Hence,

$$P(\boldsymbol{X} = \boldsymbol{x} \mid T(\boldsymbol{X}) = T(\boldsymbol{x})) = \frac{P(\boldsymbol{X} = \boldsymbol{x})}{P(T(\boldsymbol{X}) = T(\boldsymbol{x}))} = \frac{p^t(1-p)^{n-t}}{\binom{n}{t}p^t(1-p)^{n-t}} = \frac{1}{\binom{n}{t}},$$

which is independent of p, and so, $T(\boldsymbol{X})$ is sufficient for p. On the other hand, if g is a function such that $\mathbb{E}(g(T(\boldsymbol{X}))) = 0$, then

$$0 = \mathbb{E}g(T(\boldsymbol{X})) = \sum_{t=0}^n g(t)\binom{n}{t}p^t(1-p)^{n-t} = (1-p)^n \sum_{t=0}^n g(t)\binom{n}{t}\left(\frac{p}{1-p}\right)^t,$$

and since $(1-p)^n$ is not 0, it must be that

$$0 = \sum_{t=0}^n g(t)\binom{n}{t}\left(\frac{p}{1-p}\right)^t = \sum_{t=0}^n g(t)\binom{n}{t}r^t,$$

where $0 < r < \infty$. The expression is a polynomial of degree n is r, and for it to be identically 0, all the coefficients must be 0. Hence, $g(t) = 0$ for $t = 0, 1, \ldots, n$, and so, $P(g(T(\boldsymbol{X})) = 0) = 1$. Hence, $T(\boldsymbol{X})$ is complete. $\square$

c) Find the maximum likelihood estimator (MLE) of p, and determine its asymptotic distribution by a direct application of the Central Limit Theorem.

Solution. We see that the log-likelihood function is

$$\log \mathcal{L}(p; \boldsymbol{x}) = \sum x_i \log p + \left(n - \sum x_i\right) \log(1-p),$$

and so, taking the derivative with respect to p, we get

$$\frac{d}{dp} \log \mathcal{L}(p; \boldsymbol{x}) = \frac{\sum x_i}{p} - \frac{n - \sum x_i}{1-p}.$$

Setting the above expression, we get our MLE for p:

$$\hat{p} = \frac{\sum x_i}{n} = \bar{X},$$

since

$$\frac{d^2}{dp^2} \log \mathcal{L}(p; \boldsymbol{x}) = -\frac{\sum x_i}{p^2} - \frac{n - \sum x_i}{(1-p)^2} < 0.$$

Now, by the Central Limit Theorem,

$$\sqrt{n}\,(\hat{p} - p) = \frac{\sum_{i=1}^n X_i - \mathbb{E}X_i}{\sqrt{n}} \Rightarrow \mathcal{N}(0, \mathbb{E}X_1^2) = \mathcal{N}\left(0, p(1-p)\right).$$

□

d) Calculate the Fisher information for the sample, and verify that your result in part (c) agrees with the general theorem which provides the asymptotic distribution of MLE's.

Solution. From part (c), we have that

$$\frac{d^2}{dp^2} \log \mathcal{L}(p; \boldsymbol{x}) = -\frac{\sum x_i}{p^2} - \frac{n - \sum x_i}{(1-p)^2},$$

and so, the Fisher information is

$$\mathcal{I}(p) = -\mathbb{E}_p\left[-\frac{\sum x_i}{p^2} - \frac{n - \sum x_i}{(1-p)^2}\right] = \frac{np}{p^2} + \frac{n - np}{(1-p)^2} = \frac{n}{p} + \frac{n}{1-p} = \frac{n}{p(1-p)}.$$

Hence, the Cramer-Rao lower bound is $\mathcal{I}(p)^{-1} = p(1-p)/n$, and so,

$$\sqrt{n}\,(\hat{p} - p) \Rightarrow \mathcal{N}(0, p(1-p)).$$

□

e) Find a variance stabilizing transformation for p, that is, a function g such that

$$\sqrt{n}\,(g(\hat{p}) - g(p)) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$$

where $\sigma^2 > 0$ and does not depend on p; identify both g and σ^2.

Solution. By the Delta Method, we have that, for any function g such that $g'(p)$ exists and is not zero,

$$\sqrt{n}\,(g(\hat{p}) - g(p)) \Rightarrow \mathcal{N}\left(0, p(1-p)\left[g'(p)\right]^2\right).$$

So, if we take $\left[g'(p)\right]^2 = \frac{1}{p(1-p)}$, then we get what we desire. Hence,

$$\begin{aligned} g(x) &= \int \frac{1}{\sqrt{x(1-x)}}\,dx = \int \frac{1}{\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}}\,dx \qquad \left(x - \frac{1}{2} = \frac{1}{2}\sin\theta\right) \\ &= \int \frac{\frac{1}{2}\cos\theta\,d\theta}{\sqrt{\frac{1}{4} - \frac{1}{4}\sin^2\theta}} = \int d\theta = \theta = \sin^{-1}(2x-1). \end{aligned}$$

Thus, if we take $g(x) = \sin^{-1}(2x-1)$, then we have that

$$\sqrt{n}\,(g(\hat{p}) - g(p)) \Rightarrow \mathcal{N}\left(0, 1\right).$$

□

2. a) Let θ have a Gamma $\Gamma(\alpha, \beta)$ distribution with α, β positive,

$$p(\theta; \alpha, \beta) = \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\beta^\alpha \Gamma(\alpha)},$$

and suppose that the conditional distribution of X given θ is normal with mean zero and variance $1/\theta$.

Show that the conditional distribution of θ given X also has a Gamma distribution, and determine its parameters.

Solution. Note that the conditional pdf is

$$f(\theta \mid x) = \frac{f(\theta, x)}{f(x)} = \frac{f(x \mid \theta) f(\theta)}{f(x)}.$$

First, we compute $f(x)$:

$$\begin{aligned} f(x) = \int_0^\infty f(x \mid \theta) f(\theta)\, d\theta &= \int_0^\infty \frac{\sqrt{\theta}}{\sqrt{2\pi}} e^{-\frac{x^2}{2}\theta} \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\beta^\alpha \Gamma(\alpha)}\, d\theta \qquad \left(\text{set } u = \left(\frac{x^2}{2} + \frac{1}{\beta}\theta\right)\right) \\ &= \frac{1}{\sqrt{2\pi}\beta^\alpha \Gamma(\alpha)} \left(\frac{x^2}{2} + \frac{1}{\beta}\theta\right)^{-\left(\alpha+\frac{1}{2}\right)} \int_0^\infty u^{\alpha-\frac{1}{2}} e^{-u}\, du \\ &= \frac{1}{\sqrt{2\pi}\beta^\alpha} \left(\frac{x^2}{2} + \frac{1}{\beta}\theta\right)^{-\left(\alpha+\frac{1}{2}\right)} \frac{\Gamma\left(\alpha + \frac{1}{2}\right)}{\Gamma(\alpha)}. \end{aligned}$$

Then, we have that

$$f(\theta \mid x) = \frac{\frac{\sqrt{\theta}}{\sqrt{2\pi}} e^{-\frac{x^2}{2}\theta} \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\beta^\alpha \Gamma(\alpha)}}{\frac{1}{\sqrt{2\pi}\beta^\alpha} \left(\frac{x^2}{2} + \frac{1}{\beta}\theta\right)^{-\left(\alpha+\frac{1}{2}\right)} \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(\alpha)}} = \frac{\theta^{\alpha-\frac{1}{2}} e^{-\left(\frac{x^2}{2}+\frac{1}{\beta}\right)\theta}}{\left(\frac{x^2}{2} + \frac{1}{\beta}\right)^{-\left(\alpha+\frac{1}{2}\right)} \Gamma\left(\alpha + \frac{1}{2}\right)},$$

and so, the conditional distribution of θ given X has the distribution $\Gamma\left(\alpha + \frac{1}{2}, \left(\frac{x^2}{2} + \frac{1}{\beta}\right)^{-1}\right)$. $\square$

b) Conditional on θ as in part (a), suppose that a sample $X_1, \dots, X_n$ is composed of independent variables, normally distributed with mean zero and variance $1/\theta$. Find the conditional expectation $\mathbb{E}[\theta|X_1, \dots, X_n]$ of θ given the sample, and show that it is a consistent estimate of θ, that is, that

$$\mathbb{E}[\theta|X_1, \dots, X_n] \xrightarrow{p} \theta \qquad \text{as } n \to \infty.$$

Solution. Similarly as in part (a),

$$f(\theta \mid x_1, \dots, x_n) = \frac{f(\theta, x_1, \dots, x_n)}{f(x_1, \dots, x_n)}.$$

Now, we compute $f(x_1, \dots, x_n)$:

$$\begin{aligned}
f(x_1, \dots, x_n) &= \int_0^\infty f(x_1, \dots, x_n \mid \theta) f(\theta)\, d\theta = \int_0^\infty \left(\prod_{i=1}^n f(x_i \mid \theta) \right) f(\theta)\, d\theta \\
&= \int_0^\infty \frac{\theta^{n/2}}{(2\pi)^{n/2}} e^{-\frac{\sum x_i^2}{2}\theta} \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\beta^\alpha \Gamma(\alpha)}\, d\theta \\
&= \frac{1}{(2\pi)^{n/2}\beta^\alpha\Gamma(\alpha)} \int_0^\infty \theta^{\frac{n}{2}+\alpha-1} \exp\left(-\left(\frac{\sum x_i^2}{2} + \frac{1}{\beta} \right) \theta \right) d\theta \\
&= \frac{1}{(2\pi)^{n/2}\beta^\alpha\Gamma(\alpha)} \left(\frac{\sum x_i^2}{2} + \frac{1}{\beta} \right)^{-\left(\frac{n}{2}+\alpha\right)} \int_0^\infty u^{\frac{n}{2}+\alpha-1} e^{-u}\, du \\
&= \frac{\Gamma\left(\alpha + \frac{n}{2}\right)}{\Gamma(\alpha)} \frac{1}{(2\pi)^{n/2}\beta^\alpha} \left(\frac{\sum x_i^2}{2} + \frac{1}{\beta} \right)^{-\left(\frac{n}{2}+\alpha\right)},
\end{aligned}$$

and so,

$$f(\theta \mid x_1, \dots, x_n) = \frac{\theta^{\left(\alpha+\frac{n}{2}\right)-1} \exp\left(-\left(\frac{\sum x_i^2}{2} + \frac{1}{\beta} \right) \theta \right)}{\Gamma\left(\alpha+\frac{n}{2}\right) \left(\frac{\sum x_i^2}{2} + \frac{1}{\beta} \right)^{-\left(\alpha+\frac{n}{2}\right)}} \sim \Gamma\left(\alpha + \frac{n}{2}, \left(\frac{\sum x_i^2}{2} + \frac{1}{\beta} \right)^{-1} \right).$$

Hence,

$$\begin{aligned}
\mathbb{E}[\theta \mid X_1 = x_1, \dots, X_n = x_n] &= \int_0^\infty \theta f(\theta \mid x_1, \dots, x_n)\, d\theta = \int_0^\infty \theta \frac{\theta^{\left(\alpha+\frac{n}{2}\right)-1} \exp\left(-\left(\frac{\sum x_i^2}{2} + \frac{1}{\beta} \right) \theta \right)}{\Gamma\left(\alpha+\frac{n}{2}\right) \left(\frac{\sum x_i^2}{2} + \frac{1}{\beta} \right)^{-\left(\alpha+\frac{n}{2}\right)}}\, d\theta \\
&= \frac{1}{\Gamma\left(\alpha+\frac{n}{2}\right) \left(\frac{\sum x_i^2}{2} + \frac{1}{\beta} \right)^{-\left(\alpha+\frac{n}{2}\right)}} \int_0^\infty \theta^{\left(a+\frac{n}{2}\right)} \exp\left(-\left(\frac{\sum x_i^2}{2} + \frac{1}{\beta} \right) \theta \right) d\theta \\
&= \frac{\left(\frac{\sum x_i^2}{2} + \frac{1}{\beta} \right)^{-\left(\alpha+\frac{n}{2}\right)+1}}{\Gamma\left(\alpha+\frac{n}{2}\right) \left(\frac{\sum x_i^2}{2} + \frac{1}{\beta} \right)^{-\left(\alpha+\frac{n}{2}\right)}} \int_0^\infty u^{\alpha+\frac{n}{2}} e^{-u}\, du = \frac{\Gamma\left(\alpha+\frac{n}{2}+1\right)}{\Gamma\left(\alpha+\frac{n}{2}\right)} \left(\frac{\sum x_i^2}{2} + \frac{1}{\beta} \right)^{-1} \\
&= \frac{\alpha+\frac{n}{2}}{\sum \frac{x_i^2}{2} + \frac{1}{\beta}},
\end{aligned}$$

and so,

$$\mathbb{E}[\theta \mid X_1, \dots, X_n] = \frac{2\alpha+n}{\sum X_i^2 + 2\beta^{-1}} \to \frac{1}{\mathbb{E}X_1^2} = \theta \text{ a.s},$$

by the Law of Large Numbers. Since a.s convergence implies convergence in probability, the desired result follows. $\square$