Fall 19 QE Math 525a

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1. Let f be a real valued function on a set X. Let $\mathcal{M}$ be the smallest σ-algebra on X for which f is measurable. Prove that $\{x\} \in \mathcal{M}$ for all $x \in X$ if and only if f is one-to-one.

   (Hint: " $\Leftarrow$ " is obvious. For " $\Rightarrow$ " prove that $\mathcal{M} = f^{-1}(\mathcal{N})$, where $\mathcal{N}$ is a collection of measurable sets of $\mathbf{R}$.)

Proof. For every $y \in f(X)$, $\{f^{-1}(y)\}$ is a single-point set. For every $x \in X$, there exists y such that $y = f(x)$. Thus, $\{x\} \in M$ for all $x \in X$. " $\Leftarrow$ " is proved.

Let $\mathcal{N}' = f^{-1}(\mathcal{N})$, where $\mathcal{N}$ is a collection of measurable sets of $\mathbf{R}$. $\mathcal{N}' \subset \mathcal{M}$ because f is measurable. Suffice to show that $\mathcal{N}'$ is σ-algebra. Obviously, $\mathcal{N}$ is σ-algebra. Let $\{N_i'\}_1^\infty \subset \mathcal{N}'$, then since $X = f^{-1}(\mathbf{R})$,

$$N_1'^c = f^{-1}(N_1)^c = f^{-1}(N_1^c) \in \mathcal{N}'.$$

$$\bigcup_{i=1}^{\infty} N_i' = \bigcup_{i=1}^{\infty} f^{-1}(N_i) = f^{-1}(\bigcup_{i=1}^{\infty} N_i) \in \mathcal{N}'.$$

$$N_1' \cap N_2' = f^{-1}(N_1) \cap f^{-1}(N_2) = f^{-1}(N_1 \cap N_2) \in \mathcal{N}'.$$

Therefore, $\mathcal{N}'$ is σ-algebra, which forces $\mathcal{N}' \supset \mathcal{M}$. Together we get $\mathcal{N}' = \mathcal{M}$. If $\{x\} \in M$, then $\{x\} = f^{-1}(f(x))$ for every $\{x\} \in M$, which means f is one-to-one. $\square$

2. Let m be the Lebesgue measure on $[0,1]$. Let $f_n : [0,1] \to \mathbf{R}$ be measurable functions. Assume that $\sum_{n=1}^{\infty} \int_0^1 |f_n(x)| dx \leq 1$. Prove that $f_n \to 0$ as $n \to \infty$ almost everywhere.

Proof.

$$A = \{x \in [0,1] | f_n \to 0\} = \bigcap_{r=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{x \in [0,1] || f_n| < \frac{1}{r}\}.$$

Since $\{x \in [0,1] || f_n| < \frac{1}{r}\}$ is measurable for every n, r, A is measurable, thus A^c is also measurable. If $m(A^c) > 0$, let $F_j(x) = \sum_{i=1}^{j} |f_n(x)|$. From Fatou Lemma,

$$\begin{aligned} \int_0^1 \varliminf_{n\to\infty} F_n(x) \leq \varliminf_{n\to\infty} \int_0^1 F_n(x) &= \varliminf_{n\to\infty} \sum_{i=1}^{n} \int_0^1 |f_i(x)| dx \\ &= \sum_{i=1}^{\infty} \int_0^1 |f_i(x)| dx \leq 1. \end{aligned}$$

For every $x \in A^c$, there exists ε_0 and $\{n_i\}_{i=1}^{\infty} \subset \mathbf{N}$ such that $f_{n_i}(x) > \varepsilon_0$. Since $F_n(x)$ increases as n increases,

$$\varliminf_{n\to\infty} F_n(x) \geq \sum_{i=1}^{[\frac{2}{\varepsilon_0 m(A^c)}]+1} |f_{n_i}(x)| > \varepsilon_0 \cdot \frac{2}{\varepsilon_0 m(A^c)} = \frac{2}{m(A^c)}.$$

Take integrals on this equality, and

$$\int_0^1 \varliminf_{n\to\infty} F_n(x) \geq \int_{A^c} \varliminf_{n\to\infty} F_n(x) \geq \int_{A^c} \frac{2}{m(A^c)} = 2.$$

This is a contradiction! Thus, $m(A^c) = 0$ and $m(A) = 1$, meaning $f_n \to 0$ a.e. . □

3. Let m be the Lebesgue measure on $[0,1]$. Prove that there does not exists a measurable set $A \subset [0,1]$ such that $m(A \cap [a,b]) = (b-a)/2$ for all $0 \leq a < b \leq 1$.

(Hint: generalization: for all measurable set $B \subset [0,1]$, $m(A \cap B) = m(B)/2$.)

Proof. First, for a=b, the conclusion is trivial. For any measurable $B \subset [0,1]$ and $m \in \mathbf{N}$, there exists finite disjoint union $\bigcup_{i=1}^{n}[a_i^{(m)}, b_i^{(m)}] \supset B$ such that $m(\bigcup_{i=1}^{n}[a_i^{(m)}, b_i^{(m)}]) < m(B) + \frac{1}{m}$. Thus, if A is measurable,

$$m(A \cap \bigcup_{i=1}^{n}[a_i^{(m)}, b_i^{(m)}]) \leq m(A \cap B) + \frac{1}{m}.$$

$m(A \cap \bigcup_{i=1}^{n}[a_i^{(m)}, b_i^{(m)}])$ goes to $m(A \cap B)$ when m goes to ∞. Besides,

$$\begin{aligned} m(A \cap \bigcup_{i=1}^{n}[a_i^{(m)}, b_i^{(m)}]) = m(\bigcup_{i=1}^{n}(A \cap [a_i^{(m)}, b_i^{(m)}])) = \sum_{i=1}^{n} m(A \cap [a_i^{(m)}, b_i^{(m)}]) \\ = \sum_{i=1}^{n} \frac{b_i^{(m)} - a_i^{(m)}}{2} = \frac{m(\bigcup_{i=1}^{n}[a_i^{(m)}, b_i^{(m)}])}{2}. \end{aligned}$$

This goes to $m(B)/2$ when m goes to ∞. Thus we prove that if A is measurable, for all measurable set $B \subset [0,1]$, $m(A \cap B) = m(B)/2$.

We have both $m(A) = m(A \cap [0,1]) = 1/2$ and $m(A) = m(A \cap A) = m(A)/2$. They cannot be true at the same time! □

4. Let $V_f(0,x)$ be the total variation of f on $[0,x]$. Prove that if $f(x)$ is absolutely continuous on $[0,1]$, then so is $V_f(0,x)$.

   (Hint: $V_f(0,x)$ is increasing function on $[0,1]$ and $V_f(0,1)$ is finite!)

   Proof. Same as the hint! $\square$