USC Graduate Exams Real Analysis

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1 Spring 1992

1. Let (X, Σ, μ) be a measure space and $\{f_n\}$ a sequence in $L^1(d\mu)$ which converges a.e. to $f \in L^1(X, \mu)$ Prove: $f_n \to f$ in $L^1(X, \mu)$ iff $\int |f_n| d\mu \to \int |f| d\mu$.

   Proof. Let us first suppose that $\int |f_n| d\mu \to \int |f| d\mu$ in $L^1(X, \mu)$. We consider the function $|f| + |f_n| - |f - f_n|$ is non-negative due to the triangle inequality and measurable because f_n, f are both measurable. Thus, we can apply Fatou's lemma to obtain:

$$\int \liminf_{n \to \infty} |f| + |f_n| - |f - f_n| d\mu \le \liminf_{n \to \infty} \int |f| + |f_n| - |f - f_n| d\mu \tag{1.1}$$

   The RHS of the above equation is $2 \int |f| d\mu + \liminf_{n \to \infty} - \int |f - f_n| d\mu$ while the LHS of the above is $2 \int |f| d\mu$. Thus, we have

$$\liminf_{n \to \infty} - \int |f - f_n| d\mu = - \limsup_{n \to \infty} \int |f - f_n| d\mu \ge 0 \tag{1.2}$$

   proving that $\lim_{n \to \infty} \int |f - f_n| d\mu = 0$ as desired.

   Conversely, suppose that $f_n \to f$ in $L^1(X, \mu)$. It's easy to prove that ove the reals, $||f_n| - |f|| \le |f - f_n|$. Thus if $\int_X |f_n - f|$

$$\left| \int_X |f_n| d\mu - \int_X |f| d\mu \right| \le \int_X |f_n - f| d\mu \tag{1.3}$$

   proving that $\int |f_n| d\mu \to \int |f| d\mu$ as desired. □

2. Let $\{f_n\}$ be a sequence of Lebesgue-measurable real-valued functions on $[0, 1]$ such that

$$\lim_{n \to \infty} \int_0^1 |f_n(x)| dx = 0 \tag{1.4}$$

   Prove there exists a subsequence of $\{f_n\}$ such that $\{f_{n_i}(x)\}$ converges to 0 for a.e. x.

   Proof. Suppose the opposite was true and there was a set $S \subset [0, 1]$ with nonzero measure such that there is no subsequence $f_{n_i}(x)$ that converges to 0. Then there is an $\epsilon > 0$ such that $\forall x \in S$ there is

   Letting μ be the standard euclidean measure on $[0, 1]$, we know that $\forall \epsilon > 0$, it is true that $\int_0^1 |f_n(x)| dx > \mu(\{x : |f_n(x)| > \epsilon\})\epsilon > \mu(S)\epsilon$. □

3. Prove that Lebesgue measure λ on $\mathbb{R}$ is translation-invariant: if A is Lebesgue-measurable subset of $\mathbb{R}$ then for each $u \in \mathbb{R}$, $u + A$ is also Lebesgue-measurable and $\lambda(u + A) = \lambda(A)$.

   Proof. Suppose we had an open cover of intervals U_i which cover A, then $U_i + u$ is an open cover of intervals which cover $u + A$. Conversely, any cover V_i of $u + A$ gives rise to a cover $V_i - u$ of A. This bijection of covers proves that A being measurable implies $u + A$ is measurable and that they have the same measure. □

4. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be lower semi-continuous provided

$$f(x) \leq \liminf_{n\to\infty} f(x_n) \tag{1.5}$$

whenever $\lim_n x_n = x$. Show that every lower semi-continuous function is Borel measurable.

Proof. □

5. Show that the function φ defined by

$$\varphi(p) = \int_0^\infty x^p e^{-x} dx \quad (p \geq 0) \tag{1.6}$$

is well-defined and differentiable on $(0, \infty)$.

Proof. Proving φ is well defined is same as proving the integral is bounded. We can evaluate this integral by integrating by parts $\lfloor p \rfloor$ times to obtain:

$$\int_1^\infty x^p e^{-x} dx = p(p-1)\cdots(p - \lfloor p \rfloor + 1) \int_0^\infty \frac{x^{p-\lfloor p \rfloor}}{e^x} dx < \int_0^\infty \frac{x}{e^x} dx = 1 \tag{1.7}$$

We can now prove differentiability by differentiating under the integral sign and proving the result is well defined. Doing so yields:

$$\varphi'(p) = \int_0^\infty p x^{p-1} e^{-x} dx \tag{1.8}$$

which we can apply the reasoning from before to prove that this is bounded and thus well defined. □

2 Fall 1993

1. Define $D_r = \{z \in \mathbb{C} : |z| < r\}$, the open r-disk. Let $M > 0$ and $f_n : D_1 \to D_M$ for $n = 1, 2, ...$ be a sequence of analytic functions. Prove there is a subsequence which converges uniformly on $D_{1/2}$.

Proof. Since f_n analytic on the complex space, it is also holomorphic. It's also clear that D_r is uniformly bounded Since $D_{1/2}$ is compact, can apply Montel's therem if we can prove that f_n is □

2. Prove or find a counterexample: Let D be a coutable dense subset of $(0, 1)$ and let G be an open subset of $\mathbb{R}$ such that $G \supset D$, then $G \supset (0, 1)$.

Proof. We shall find a counterexample. Let D be the rational numbers in $(0, 1)$, which is dense and countable in $(0, 1)$. Now let $G = (0, \frac{\sqrt{2}}{2}) \cup (\frac{\sqrt{2}}{2}, 1)$. It's clear that $G \supset D$, and that G is open. However, G doesn't contain $(0, 1)$, thus giving us our counterexample. □

3. Let f be a non-constant meromorphic function with is doubly periodic (i.e. has two periods linearly independent over the reals). Prove that f has at least one singularity.

Proof. Since the reals is one dimensional vector space, I assume they mean "has two periods linearly independent over the complex numbers". Let the periods be $a, b \in \mathbb{C}$. Then every point in the complex plane can be expressed as $z = xa + yb$ for some $x, y \in \mathbb{R}$. Since f is periodic, we have $f(xa + yb) = f((x \pm 1)a, y(\pm 1)b)$. Thus, we have $f(z) = f(z')$ for some z' within the parallelogram with vertices $0, a, a+b, b$. If there is no singularity, since this is a compact set f will reach its maximum on this parallelogram, and thus f will have a global maximum. However, by the maximum principles, f, being a complex valued function, cannot attain a maximum on an open set. Thus f has at least one singularity. □

4. How many roots of the equation $f(z) = 0$ lie in the right half-plane, where

$$f(z) = z^4 + \sqrt{2}z^3 + 2z^2 - 5z + 2 \tag{2.1}$$

Proof. We let our contour be the curve from iR to $-iR$, and around the semicircle of radius R in the right half plane. If we set $g(z) = z^4 + 2$ then $|g(z)| \geq R^4 - 2$ when $|z| = R$ On the other hand, when z is on the imaginary axis, we have:

$$f(it) = t^4 - i\sqrt{2}t^3 - 2t^2 - 5it + 2 \tag{2.2}$$

for $t \in \mathbb{R}$. We let our o$|f(it)| = \sqrt{(t^4 - 2t^2 + 2)^2 + (\sqrt{2}t^3 + 5t)^2}$. Let $g(z) = z^4 - 5z$ Let $g(z) = z^4 + 2$ □

5. Show that a function $f : (a, b) \to \mathbb{R}$ which is absolutely continuous is both uniformly continuous and of bounded variation.

Proof. Since f is absolutely continuous, for all $\epsilon > 0$, there exists $\delta > 0$ such that if a sequence of pairwise disjoint subintervals (x_k, y_k) of (a, b) satisfy $\sum_k (y_k - x_k) < \delta$, then $\sum_k |f(y_k) - f(x_k)| < \epsilon$. Letting $k = 1$ proves that f is uniformly continuous.

Now we prove that f has bounded variation. Let define $var_{(a,b)}f$ to be the variation of f on the interval (a, b). By hypothesis, there is a $\delta > 0$ such that $\sum_k |f(y_k) - f(x_k)| < 1$ for all disjoint subintervals (x_i, y_i) and $\sum_k |y_k - x_k| < \delta$. Let N be an integer greater than $\frac{b-a}{\delta}$, and partition (a, b) into N evenly space intervals $\left(a + \frac{j(b-a)}{N}, a + \frac{(j+1)(b-a)}{N}\right)$. Thus, $var_{(a,b)}f = \sum_{j=1}^{N} var_{\left(a+\frac{j(b-a)}{N}, a+\frac{(j+1)(b-a)}{N}\right)} < \sum_{j=1}^{N} 1 = N$. □

6. Show that $\frac{\sin x}{x} \in L^2(\mathbb{R}^+)$ and evaluate its L^2 norm.

Proof. We first show $\left(\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx\right)^{\frac{1}{2}} < \infty$. By Lhopital's rule, $\lim_{x\to 0} \frac{\sin x}{x} = 1$ so by continuity of $\frac{\sin x}{x}$, it follows that $\left(\int_0^1 \left(\frac{\sin x}{x}\right)^2 dx\right)^{\frac{1}{2}} < \infty$. Now we just have to show that $\left(\int_1^\infty \left(\frac{\sin x}{x}\right)^2 dx\right)^2 < \infty$.

This follows from

$$\left(\int_1^\infty \left(\frac{\sin x}{x}\right)^2 dx\right)^{\frac{1}{2}} < \left(\int_1^\infty \frac{1}{x^2} dx\right)^{\frac{1}{2}} < \infty = 1 < \infty \tag{2.3}$$

Now to evaluate this integral, we can use calculus of residues. Since this function is even we can Evaluated it from $-\infty$ to ∞ and divide the result by 2. □

7. Suppose f is a non-negative function which is Lebesgue integrable on $[0,1]$, and $\{r_n : n = 1, 2, ...\}$ is an enumeration of the rational numbers in $[0,1]$. Show that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} f(|x - r_n|) \tag{2.4}$$

converges for a.e. $x \in [0,1]$.

Proof. We will set $g(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f(|x - r_n|)$, and prove that $\int_0^1 g(x)dx < \infty$. From this it will immediately follow that $g(x) < \infty$ a.e. First, we note that $\int_0^1 f(|x - r_n|) \leq \int_{-1}^1 f(|x - r_n|) = 2\int_0^1 f(x)$. Thus:

$$\int_0^1 g(x)dx = \sum_{n=1}^{\infty} \int_0^1 \frac{1}{2^n} f(|x - r_n|) \tag{2.5}$$

$$\leq 2\int_0^1 f(x)dx \sum_{n=1}^{\infty} \frac{1}{2^n} \tag{2.6}$$

$$= 2\int_0^1 f(x) < \infty \tag{2.7}$$

as desired. □

3 Spring 1994

1. Evaluate $\int_0^\infty \frac{\log x}{1+x^2} dx$.

Proof. Let us integrate this in the complex plane with respect to complex variable z, consider the path γ from $-R$ to R, then moving along Re^{it} for $t \in [0, \pi]$ back to $-R$. Let us consider the contribution of the integral around the path $z = Re^{it}$ for $t \in [0, \pi]$. Changing the variable of integration from z to t yields:

$$\left|\int_0^\infty \frac{\log x}{1 + x^2} dx\right| < M \int_0^{2\pi} \frac{\log R}{|1 + R^2|} dt \tag{3.1}$$

$$\leq M \frac{\log R}{R} \tag{3.2}$$

for some constant M and sufficiently large R. Thus, the integral around γ reduces to the portion of the integral on the real number line as $R \to \infty$. We can use the calculus of residues to evaluate this integral. There is only 1 reside at $z = i$ evaluated as follows:

$$Res(f(z); z = i) = \lim_{z \to i}(z - i)\frac{\log z}{(z - i)(z + i)} \tag{3.3}$$

$$= \tag{3.4}$$

□

2. Show that $[0, 1]$ cannot be written as the countably infinite union of disjoint nonempty closed intervals.

   Proof. We prove this statement by contradiction. Suppose $[0, 1]$ was the union of countably many closed intervals. Then removing the endpoints of each interval we get that there is a sequence of disjoint open intervals I_n such that

   $$[0, 1] = \cup_{n=1}^{\infty} I_n \tag{3.5}$$

   Letting $I_n = [x_n, y_n]$ we consider the union of the endpoints;

   $$U = \cup_{n=1}^{\infty}\{x_n, b_n\} \tag{3.6}$$

   U is clearly closed, and we can see that U is also perfect since every point in U is a limit point. We can now apply the Baire category theorem which shows that a perfect subset of a complete metric space can't be countable infinite. The result follows. □

3. Let $f : D \to \mathbb{C}$ be analytic such that $\Re f(z) > 0$ for all z. Prove

   $$|f(z)| \leq |f(0)|\frac{1 + |z|}{1 - |z|} \tag{3.7}$$

   Proof. We note the map $g(z) = \frac{f(0)-z}{f(0)+z}$ maps the right half complex plane conformally onto the unit disk such that $g(f(0)) = 0$. Thus, we can apply Schwarz's lemma to the function $g(f(z))$ to obtain $|g(f(z))| \leq |z|$. We also have

   $$g^{-1}(z) = f(0)\frac{1 - z}{1 + z} \tag{3.8}$$

   Thus, $|f(z)| \leq g^{-1}(|z|) = f(0)\frac{1-|z|}{1+|z|}$ as desired. □

4. Let $f : [1, +\infty) \to [0, +\infty)$ be Lebesgue measurable. Prove:

   $$\int_1^{\infty} \frac{f(x)^2}{x^2} < +\infty \Rightarrow \int_1^{\infty} \frac{f(x)}{x^2}dx < +\infty \tag{3.9}$$

Proof. Define $S_0 = \{x : f(x) < 1\}$, and $S_1 = \{x : f(x) \geq 1\}$. It's clear that $\int_1^\infty \frac{f(x)}{x^2}dx = \int_{S_0} \frac{f(x)}{x^2}dx + \int_{S_1} \frac{f(x)}{x^2}$, so if we can bound each of the integrals then we are done.

First, we have $\int_{S_0} \frac{f(x)}{x^2}dx \leq \int_{S_0} \frac{1}{x^2}dx < \int_1^\infty \frac{1}{x^2}dx < \infty$.

On the other hand, we have $\int_{S_1} \frac{f(x)}{x^2} < \int_{S_1} \frac{f(x)^2}{x^2} < \int_1^\infty \frac{f(x)^2}{x^2} < \infty$ by hypothesis. Putting everything together yields our desired result. □

5.

6. Let $([0,1], \mathcal{A}, \mu)$ denote the Lebesgue space on $f : [0,1] \to \mathbb{R}$ the condition "f is continuous a.e." neither implies, nor is implied by, the condition "there exists a continuous function $g : [0,1] \to \mathbb{R}$ such that $f = g$ a.e."

   Proof. Let $g(x) = 0$ and define $f(x)$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \tag{3.10}$$

   Then because $\mathbb{Q}$ has Lebesgue measure 0, it follows that $g(x) = f(x)$ a.e. However, $f(x)$ is nowhere continuous.

   Conversely, let

$$f(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{2} \\ 1 & \text{if } x > \frac{1}{2} \end{cases} \tag{3.11}$$

   which will always differ from a continuous function g around an interval centered at $x = \frac{1}{2}$, and thus not equal to g a.e. □

7. An entire function is said to have finite order if there exists $c > 0$ such that $|f(z)| \leq exp(|z|^c)$ for all $|z|$ sufficiently large; the order of f is the infimum of all such $c > 0$. Prove that the following function is entire and has order $\frac{1}{2}$.

$$f(z) = \prod_{k=1}^{\infty} \left(1 + \frac{z}{k^2}\right) \tag{3.12}$$

   Proof. □

8. Let $\{f_n\}$ be a sequence of measurable functions on some measure space $(X, \mathcal{A}, \mu)$ with $\mu(X) < \infty$. We say the sequence is uniformly integrable if

$$\lim_{n \to \infty} \sum_n \int_{|f_n| > R} |f_n| d\mu = 0 \tag{3.13}$$

   (a) Show that if there exists $g \in L^1(X)$ such that $|f_n(x)| \leq |g(x)|$ for all x, n then the $\{f_n\}$ are uniformly integrable.

Proof. Since $\mu(X) < \infty$, and $g \in L^1(X)$, it follows that $ess\sup_{x\in X} g(x) < \infty$. Thus, whenever $R > ess\sup_{x\in X} g(x)$, $\int_{|f_n|>R} |f_n| d\mu = 0$ for all n, and thus $\lim_{n\to\infty} \sum_n \int_{|f_n|>R} |f_n| d\mu = 0$ as desired. □

(b) Prove that if $f_n \to f$ pointwise and the $\{f_n\}$ are uniformly integrable then $f \in L^1(\mathbb{R})$ and

$$\lim_n \int f_n d\mu = \int f d\mu \tag{3.14}$$

Proof. We have the following inequality:

$$\int_X |f| d\mu = \int_{|f|\le R} |f| + \int_{|f|>R} |f| d\mu < \int_{|f|\le R} |f| + \sum_n \int_{|f_n|>R} |f_n| d\mu < \infty \tag{3.15}$$

where in the last inequality follows we assume R is sufficiently large such that $\sum_n \int_{|f_n|>R} |f_n| < \infty$. Thus $f \in L^1(X)$ and by the Lebesgue dominated convergence theorem, it follows that $\lim_n \int f_n d\mu = \int f d\mu$ as desired. □