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Real Analysis Exam Cheat Sheet

Basic Review

Definition 1. *Compact Support*

A function has compact support if it vanishes outside of some compact set.

Definition 2. *Semi-Continous*

- f is upper semi-continuous if for all x and all $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(y) < f(x) + \varepsilon$ for all $|y - x| < \delta$
- f is lower semi-continuous if for all x and all $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x) < f(y) + \varepsilon$ for all $|y - x| < \delta$

Lemma 1. *Facts about USC and LSC*

Immediately from the definitions:

- f is upper semi-continuous $\iff \limsup_{y\to x} f(y) \le f(x)$ for all x.
- f is lower semi-continuous $\iff f(x) \le \liminf_{y\to x} f(y)$ for all x.

Theorem 1. *Weierstrass Approximation Theorem*

If: f is continuous and real valued on $[a, b]$ a closed interval

Then: f can be uniformly approximated by polynomials. (For all $\varepsilon > 0$ There exists $p(x)$ so $|f(x) - p(x)| < \varepsilon$ for all x.)

Lemma 2. *Monotone Convergence of a Sequence*

If: $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence with $a_n \leq a_{n+1}$ for all n

Then: $\lim_{n\to\infty} a_n = \sup_n a_n$ and so namely, the limit exists.

Example 1. *Understanding Limsups and Liminfs*

- Let $\{A_n\}_{n=1}^{\infty}$ be sets. Then $\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k$ is the set where each element belongs to all but finitely many of the A_n.
- Let $\{A_n\}_{n=1}^{\infty}$ be sets. Then $\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k$ is the set where each element belongs to infinitely many of the A_k (but could also not belong in infinitely many).
- Let $\{f_n\}_{n=1}^{\infty}$ be functions. Then $\liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty} \inf_{k\geq n} f_k(x)$.
- Let $\{f_n\}_{n=1}^{\infty}$ be functions. Then $\limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sup_{k\geq n} f_k(x)$.
- Let f be a function. Then $\liminf_{y\to x} f(y) = \sup_{\varepsilon>0} \inf_{|y-x|<\varepsilon} f(y)$.
- Let f be a function. Then $\limsup_{y\to x} f(y) = \inf_{\varepsilon>0} \sup_{|y-x|<\varepsilon} f(y)$.

Lemma 3. *Facts from Topology*

- A union of open sets (countable or uncountable) is open
- An intersection of closed sets (countable or uncountable) is closed
- In $\mathbb{C}$ a set is compact $\iff$ it is closed and bounded
- The Cantor set C is compact and has the cardinality of $\mathbb{R}$.

Algebras and σ-Algebras

Definition 3. *Algebras and σ-Algebras*

- An *algebra* $\mathcal{A} \subset \mathcal{P}(X)$ on a set X is a subset of the powerset of X which contains X and is closed under compliments and *finite* unions and *finite* intersections.
- A σ-algebra $\mathcal{A} \subset \mathcal{P}(X)$ on a set X is a subset of the powerset of X which contains X and is closed under compliments and *countable* unions and *countable* intersections.

Example 2.

- $\mathcal{P}(X)$ and $\{\varnothing, X\}$ are always σ-algebras (and algebras)
- Borel σ-algebra $\mathcal{B}_X$ is the σ-algebra generated by all open subsets of X.
- $\mathcal{B}_\mathbb{R}$ is generated by sets of any of the following forms:

$$\begin{array}{cc} (a,b) & (a,\infty) \\ [a,b) & [a,\infty) \\ (a,b] & (-\infty,b) \\ [a,b] & (-\infty,b] \end{array}$$

- If X is infinite, $\mathcal{A} = \{E \subset X \,|\, E \text{ is finite or } E^c \text{ is finite}\}$ is an algebra but *not* a σ-aglebra.
- If X is infinite, $\mathcal{A} = \{E \subset X \,|\, E \text{ is countable or } E^c \text{ is countable}\}$ *is* a σ-algebra.

Definition 4. *Types of Sets in a σ-Algebra*

- G_δ-sets are intersections of open sets ($\bigcap\{\text{open}\}$)
- F_σ-sets are unions of closed sets ($\bigcup\{\text{closed}\}$)
- $G_{\delta\sigma}$-sets are unions of G_δ-sets, ($\bigcup\bigcap\{\text{open}\}$)
- $F_{\sigma\delta}$-sets are intersections of F_σ sets ($\bigcap\bigcup\{\text{closed}\}$)

Mnumonic: σ is sum, and F is closed.

Measures

Definition 5. *Measure*

$\mu : \mathcal{A} \to [0, \infty]$ from a σ-algebra is a measure if

- $\mu(\varnothing) = 0$
- μ is countably additive: for all disjoint collections $\{E_i\}_{i=1}^{\infty} \subset \mathcal{A}$

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

Lemma 4. *Facts about Measures*

Immediately from the definitions:

- if $E \subset F$ then $\mu(E) \leq \mu(F)$
- $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$ for any collection $\{E_i\}_{i=1}^{\infty} \subset \mathcal{A}$
- continuity from below: if $E_1 \subset E_2 \subset \cdots$ then $\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$
- continuity from above: if $E_1 \supset E_2 \supset \cdots$ *and* $\mu(E_1) < \infty$, then $\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right)$

Example 3. *Disjointification*

Let $\{E_i\}_{i=1}^{\infty} \subset \mathcal{A}$. Then let

$$\begin{aligned}
F_1 &= E_1 \\
F_2 &= E_2 \backslash E_1 \\
F_2 &= E_3 \backslash (E_2 \cup E_1) \\
&\vdots \\
F_n &= E_n \backslash \left(\bigcup_{i=1}^{n-1} E_i\right)
\end{aligned}$$

Then $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$ but the F_i are disjoint.

Example 4. *Examples of Measures*

- The counting measure on a set $\mu(E) = |E|$, often defined on the σ-algebra $\mathbb{N}$
- The durac or pointmass measure at some point x_0,

$$\mu_{x_0}(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E \end{cases}$$

- The Lebesgue measure

Lemma 5. *Facts about the Lebesgue Measure*

- m is outer regular: $m(E) = \inf\{m(U) \,|\, E \subset U \text{ open}\}$.
- m is inner regular: $m(E) = \sup\{m(K) \,|\, K \text{ compact } \subset E\}$.
- $m(Q) = 0$ for any countable set Q, namely, $m(\mathbb{Q}) = 0$
- $m(C) = 0$ where C is the Cantor-set.
- $m(E + s) = m(E)$ where $E + s = \{x + s \,|\, x \in E\}$
- $m(rE) = |r|m(E)$ where $rE = \{rx \,|\, x \in E\}$
- $\mathcal{L}$ is the completion of $\mathcal{B}_{\mathbb{R}}$ (the Borel σ-algebra for $\mathbb{R}$) and it is the domain of m. Namely, m is complete measure.

Definition 6. *Premeasure*

$\mu_0 : \mathcal{A} \to [0, \infty]$ from an *algebra* (not σ-algebra) satisfies

- $\mu_0(\varnothing) = 0$
- $\mu_0\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu^*(E_i)$ for all disjoint collections $\{E_i\}_{i=1}^{\infty}$ where $\bigcup E_i \subset \mathcal{A}$ (which does not always happen in algebras).

Definition 7. *Outer Measure*

$\mu^* : \mathcal{P}(X) \to [0, \infty]$ from the power set satisfies

- $\mu^*(\varnothing) = 0$
- if $A \subset B$ $\mu^*(A) \leq \mu^*(B)$

- $\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$ for all collections $\{E_i\}_{i=1}^{\infty}$

Sets A satisfying

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \qquad \forall E \subset X$$

are called μ^*-measurable.

Theorem 2. *Caratheodory's*

If: μ^* is an outer measure

Then: $\mathcal{M}$, the set of all μ^*-measurable sets, is a σ-algebra and $\mu^*|_{\mathcal{M}}$ is a complete measure.

Definition 8. *Other Types of Measures*

$\mu : \mathcal{A} \to [0, \infty]$

- Finite measure: $\mu(X) < \infty$
- σ-Finite measure: There exists a disjoint collection $\{E_i\}_{i=1}^{\infty} \subset \mathcal{A}$ such that $X = \bigcup_{i=1}^{\infty} E_i$ and $\mu(E_i) < \infty$ for all i.
- Semi-finite: for all E where $\mu(E) = \infty$, there exists $F \subset E$ so $0 < \mu(F) < \mu(E) = \infty$.

Lemma 6. *Trick for Borel Measures*

If: E is measurable and $\mu(E) < \infty$,

Then: for every $\varepsilon > 0$, there exists A which is a finite union of disjoint open intervals such that $\mu(E \Delta A) < \varepsilon$.

Example 5. *The Construction of a Measure*

(1) Start with an algebra $\mathcal{A}$ and a premeasure μ_0 on that algebra.
(2) Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be defined by

$$\mu^*(E) = \inf\left\{\sum_{i=1}^{\infty} \mu_0(A_i) \mid A_i \in \mathcal{A}, E \subset \bigcup_{i=1}^{\infty} A_i\right\}.$$

Then μ^* defines an outer measure.

(3) Apply Caratheodory to obtain $\mu = \mu^*|_{\mathcal{M}}$ the outer measure restricted to the σ-algebra of μ^*-measurable sets.

(4) μ is now a complete measure.

A shortcut to this process: if $f : X \to [0\infty]$, then defining

$$\mu(E) = \sum_{x\in E} f(x) = \sup\left\{\sum_{x\in F} f(x) \,|\, F \text{ finite} \subset E\right\}$$

defines a measure.

Example 6. *Construction of Measures from Functions*

We utilize the outline the previous example to adapt functions into measures.

Let $F : \mathbb{R} \to \mathbb{R}$ be any increasing right continuous function. Let $\mathcal{A}$ be the algebra generated by half-open invervals of the real line $\{(a, b]\}$. Then

$$\mu_0\left(\bigcup_{i=1}^{n}(a_i, b_i]\right) = \sum_{i=1}^{n}[F(b_i) - F(a_i)]$$

defines a premeasure on $\mathcal{A}$.

Going through the process described in the previous example, we finally obtain a *unique* regular Borel measure μ_F which is defined on $\mathcal{B}_{\mathbb{R}}$ by

$$\mu_F((a, b]) = F(b) - F(a)$$

***The function $F(x) = x$ defines the Lebesgue Measure.

Conversely, any finite Borel measure μ can be used to define an increasing and right continuous function by the formula

$$F(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\mu((x, 0]) & \text{if } x < 0 \end{cases}$$

Measurable Functions

Definition 9. *Measurable Functions*

A function $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is called $(\mathcal{M}, \mathcal{N})$-measurable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

***Continuous functions are Borel measurable by definition.

***To check if $f : (X, \mathcal{M}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is measurable, it suffices to check that $f^{-1}(E) \in \mathcal{M}$ for $E = (a, \infty), [a, \infty), (-\infty, b), (-\infty, b]$.

Example 7.

Borel measurable implies Lebesgue measurable, since if $f : \mathbb{R} \to \mathbb{R}$, then $f^{-1}(E) \in \mathcal{B}_{\mathbb{R}} \subset \mathcal{L}$ for all $E \in \mathcal{B}_{\mathbb{R}}$.

However, the converse is not true.

Take a null set $N \in \mathcal{L}$ such that $N \notin \mathcal{B}_{\mathbb{R}}$. Then χ_N be characteristic function of N. Then $\{-1\} \in \mathcal{B}_{\mathbb{R}} \subset \mathcal{L}$ but $\chi_N^{-1}(\{1\}) = N \notin \mathcal{B}_{\mathbb{R}}$. So χ_N is not Borel measurable.

However, $N \in \mathcal{L}$ so χ_N *is* Lebesgue measurable.

Lemma 7. *Combining Measurable Functions*

- If f, g are measurable, then $f + g$, $f - g$, fg, $\max\{f, g\}$, $\min\{f, g\}$ are measurable.
- If $\{f_i\}_{i=1}^{\infty}$ is a sequence of measurable functions, $\sup f_i$, $\inf f_i$, $\limsup f_i$, $\liminf f_i$ are all measurable.
- If $\lim\limits_{i\to\infty} f_i(x)$ exists for every $x \in X$, then the limit is measurable.

Definition 10. *Simple Functions*

A simple function is a finite sum of characteristic functions

$$\varphi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x).$$

Theorem 3. *Approximating Measurable Functions*

If: $f \in L^+$

Then: there exists a sequence of $\{\varphi_n\}_{n=1}^\infty$ approximating f pointwise from below, namely,

$$0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq f$$

$\varphi_n \to f$ pointwise and $\varphi_n \to f$ uniformly from below on any set which f is bounded.

***If $f : X \to \mathbb{C}$ is measurable, then there exists $\{\varphi_n\}_{n=1}^\infty$ so

$$0 \leq |\varphi_1| \leq |\varphi_2| \leq \cdots \leq |f|$$

with $\varphi_n \to f$ pointwise and $\varphi_n \to f$ uniformly on any set where f is bounded.

Definition 11. *Absolute Continuity*

A function f is absolutely continuous on $[a, b]$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any finite collection $\{[a_i, b_i]\}_{i=1}^n$ of subintervals of $[a, b]$,

$$\sum_{i=1}^n |b_i - a_i| < \delta \implies \sum_{i=1}^n |f(b_i) - f(a_i)| < \varepsilon.$$

Example 8.

- Uniform Continuity $\not\Longrightarrow$ Absolute Continuity: Let $f(x)$ be the Cantor-Lebesgue function on $[0, 1]$. Then f is continuous on a compact set so it is uniformly continuous. Assume f is absolutely continuous, then by **FTOLI**,

$$1 = f(1) = f(1) - f(0) = \int_0^1 f'(x)dm = 0$$

- Absolute Continuity $\not\Longrightarrow$ Lipschitz Continuity: Let $f(x) = \sqrt{x}$ on $[0, 1]$. Then f is discontinuous only at 0, so by the comparison theorem, f is Riemmann integrable, and its Riemann and Lebesgue integerals coincide. Namely,

$$f(x) = \int_0^x f'(x)dx \qquad \int_0^1 |f'(x)|dx = 1 < \infty$$

by techniques of Riemann integration, so f is absolutely continuous by FTOLI.

However, $\dfrac{|\sqrt{x} - \sqrt{y}|}{|x - y|} = \dfrac{1}{\sqrt{x} + \sqrt{y}}$ which can grow arbitrarily large for x, y near 0. Namely, there is no M so $|f(x) - f(y)| \leq M|x - y|$ so f is not Lipschitz continuous.

Integration

Definition 12. *Integral*

For measurable, non-negative functions ($f \in L^+$),

$$\int_X f d\mu = \sup\left\{\int_X \varphi d\mu \,|\, 0 \le \varphi \le f, \varphi \text{ simple}\right\}$$

where

$$\int_X \varphi d\mu = \sum_{i=1}^n a_i \mu(E_i).$$

For measurable, functions,

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$$

where $f = f^+ - f^-$ its positive and negative parts (which are both non-negative measurable functions).

A function $f \in L^1$ if its measurable and

$$\int |f| d\mu < \infty.$$

Lemma 8. *Facts about Integration*

- If $a \in \mathbb{R}$, $\displaystyle\int a f d\mu = a \int f d\mu$
- $f, g \in L^1$, $\displaystyle\int f \pm g d\mu = \int f d\mu \pm \int g d\mu$
- $f \le g$, then $\displaystyle\int f d\mu \le \int g d\mu$
- $f \in L^+$, then $\displaystyle\int f d\mu = 0$ if and only if $f = 0$ a.e.
- $f \in L^1$, then $|f(x)| < \infty$ a.e. and $\{x : f(x) \neq 0\}$ is σ-finite.
- $|\int f d\mu| \le \int |f| d\mu$
- $f_n \in L^+$ for all n then $\displaystyle\sum_{n=0}^\infty \int f_n d\mu = \int \sum_{n=0}^\infty f_n d\mu$

Theorem 4. *Monotone Convergence Theorem*

If:

- $f_n \in L^+$ for all n
- $f_n \leq f_{n+1}$ for all n (and all x)

Then:

$$\int \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

Theorem 5. *Fatou's Lemma*

If: $f_n \in L^+$ for all n,

Then:

$$\int \liminf f_n d\mu \leq \liminf \int f_n d\mu.$$

Theorem 6. *Dominated Convergence Theorem*

If:

- f_n measurable for all n
- $\lim_{n\to\infty} f_n(x)$ exists for a.e. x (pointwise convergence)
- there exists $g \in L^1$ such that $|f_n(x)| \leq g(x)$ for all n and a.e. x.

Then: the limit in in L^1 and

$$\int \lim_{n\to\infty} f_n d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

Theorem 7. *Integral Approximation Theorem*

If: $f \in L^1$

Then: for all $\varepsilon > 0$, there exists a simple function such that $\int |f - \varphi| d\mu < \varepsilon$.

***If $f \in L^1(m)$ (where m is the Lebesgue measure) then there exists a continuous function with compact support such that $\int |f - g| dm < \varepsilon$.

Lemma 9. *Tricks for Integrating Functions*

- Show the result holds for a characteristic function. By linearity, it holds for all simple functions. If f is measurable, it can be uniformly approximated by simple functions (Theorem 3).
- If f is L^1, the integral of f can be approximated by the integral of some simple function (Theorem 7).
- If f is $L^1(m)$, the integral of f can be approximated by the integral of some continuous function with compact support (Theorem 7).

Theorem 8. *Comparing Riemann and Lebesgue Integerals*

-
- **If:** f is bounded and real valued on a bounded interval $[a, b]$

  Then: if f is Riemann integrable, f is Lebesgue measurable (and hence integrable) and the two integrals agree $\int_a^b f(x)dx = \int_{[a,b]} f dm$

- **If:** f is bounded on $[a, b]$

  Then: f is Riemann integrable $\iff$ $\{x \in [a, b] : f \text{ is discontinuous at } x\}$ is Lebesgue null.

- **If:** $f : (a, b] \to [0, \infty)$ is a nonnegative continuous function (where $\lim_{\alpha\to a} f(\alpha) = \infty$ which has a finite (although perhaps improper) Riemann integral

  Then: $f \in L^1(a, b]$ and the Riemann and Lebesgue integrals agree.

***The last bullet is because on $[\alpha, b]$ for $\alpha > a$, f is bounded and so by the first part of this theorem,

$$\int_{(a,b]} f(x)dm(x) = \lim_{\alpha\to a} \int_{[\alpha,b]} f(x)dm(x) = \lim_{\alpha\to a} \int_\alpha^b f(x)dx < \infty.$$

Theorem 9. *Fundamental Theorem of (Riemann) Integerals*

If: f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)dt$

Then: F is well defined and continuous for all $x \in [a, b]$, (so F is uniformly continuous) and $F'(x) = f(x)$ for all $x \in (a, b)$.

***Conversely, if f is Riemann integrable on $[a, b]$ and has an antiderivative F on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Theorem 10. *Fundamental Theorem of (Lebesgue) Integerals*

$F : [a, b] \to \mathbb{C}$, TFAE:

- F is absolutely continuous on $[a, b]$
- $F(x) - F(a) = \int_a^x f(t)dt$ for some $f \in L^1([a, b], m)$.
- F is differentiable a.e. on $[a, b]$, $F' \in L^1([a, b], m)$, and $F(x) - F(a) = \int_a^x F'(t)dt$.

Theorem 11. *Tonelli*

If:

- If $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite
- $f \in L^+(X \times Y)$ (positive and measurable)

Then:

$$\int f d(\mu \times \nu) = \int \int f d\mu d\nu = \int \int f d\nu d\mu.$$

Theorem 12. *Fubini*

If:

- If $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite
- $f \in L^1(\mu \times \nu)$ (which can be checked by looking at $|f|$ and using Tonelli)

Then:

$$\int f d(\mu \times \nu) = \int \int f d\mu d\nu = \int \int f d\nu d\mu.$$

Modes of Convergence

Definition 13. *Modes of Convergence*

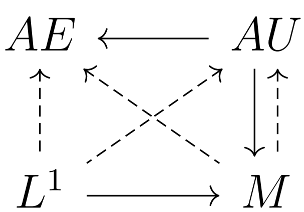
- Convergence in Measure: $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \to 0$ as $n \to \infty$
- Convergence in L^1: $\int |f_n(x) - f(x)| d\mu \to 0$ as $n \to \infty$.
- Almost Uniform (AU): If $\mu(X) < \infty$, for every $\varepsilon > 0$, there exists a set E_ε such that $\mu(E_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on E_ε^c.

Theorem 13. *Implication Diagrams for Convergence*

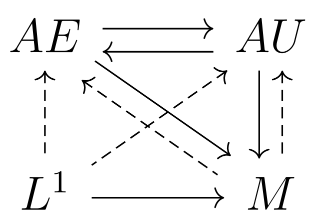
$\longrightarrow$ Represents implication (e.g. almost uniform convergence implies a.e. convergence).

$\dashrightarrow$ Represents existence of subsequence which converges (e.g. convergence in L^1 implies existence of a subsequence which converges a.e.).

General Case:



Finite measure space:



> ***Almost uniform convergence being equivalent to a.e. convergence in a finite measure space is a result of *Egoroff's Theorem*.

Example 9. *Classic Counter Examples*

(1) Uniform Convergence $\not\Longrightarrow$ Convergence in L^1: Let $f_n(x) = \frac{1}{n}\chi_{(0,n)}(x)$. Then $f_n \to 0$ uniformly since f_n is clearly bounded everywhere. However,

$$\int |f_n - 0| = \frac{1}{n} m((0,n)) = 1 \qquad \forall n$$

so this clealry does not converge to 0 in L^1.

(2) Convergence in $L^1 \not\Longrightarrow$ Convergence a.e.: Let f_n be the moving box example. Namely, for each n, there exists k so $2^k \leq n < 2^{k+1}$, let

$$f_n(x) = \chi_{[\frac{n}{2^k}-1, \frac{n+1}{2^k}-1]}(x).$$

Namely,

$$\begin{aligned} f_1(x) &= \chi_{[0,1]} & 2^0 = 1 \leq 1 \\ f_2(x) &= \chi_{[0,\frac{1}{2}]} & 2^1 = 2 \leq 2 \\ f_3(x) &= \chi_{[\frac{1}{2},1]} & 2^1 = 2 \leq 3 \\ &\vdots \end{aligned}$$

then

$$\int |f_n| = \frac{1}{2^k} \to 0$$

but $f_n(x)$ doesn't converge for any x since there are an infinite number of n where $f_n(x) = 1$ and an infinite number of n where $f_n(x) = 0$.

(3) Convergence in Measure $\not\Longrightarrow$ Convergence in L^1: Let $f_n(x) = n\chi_{[0,\frac{1}{n}]}(x)$. Then $f_n \to 0$ in measure, since the measure of the set where f_n is large shrinks to nothing as $n \to \infty$.

However,

$$\int |f_n| = 1 \qquad \forall n$$

so $f_n \not\to 0$ in L^1.

(4) Convergence a.e. $\not\Longrightarrow$ Convergence in measure: Let $f_n(x) = \frac{x}{n}$. Then $f_n(x) \to 0$ for all $x \in \mathbb{R}$, however,

$$m(\{x : |f_n(x)| \geq \varepsilon\}) = m(\{x : x \geq n\varepsilon\}) = m([n\varepsilon, \infty)) = \infty$$

for all n.

Signed Measures

Definition 14. *Signed Measure*

$\nu : \mathcal{M} \to [-\infty, \infty]$

- $\nu(\varnothing) = 0$
- ν assumes at most one of the $\pm\infty$
- $\nu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \nu(E_i)$ for all disjoint collections $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$ (where the sum converges absolutely if $\nu\left(\bigcup_{i=1}^{\infty} E_i\right) < \infty$)

***Signed measures are also continuous from above and below, just like positive measures.

Definition 15. *Singular and Absolutely Continuous*

- If μ and ν are measures (signed or otherwise) on $(X, \mathcal{M})$, then μ and ν are mutually singular (write $\mu \perp \nu$) if there exists $E, F \in \mathcal{M}$ such that $E \cap F = \varnothing$, $E \cup F = X$, $\mu(E) = 0$ and $\nu(F) = 0$.
- If μ and ν are measures (where at most ν is singed) on $(X, \mathcal{M})$, then ν is absolutely continuous with respect to μ (write $\nu << \mu$) if $\mu(E) = 0$ implies $\nu(E) = 0$ for all $E \in \mathcal{M}$.
- $\nu << \mu$ if and only if $\nu(E) = \int_E f d\mu$ for some $f \in L^1(\mu)$ (write $d\nu = f d\mu$).

Theorem 14. *Hahn Decomposition*

If: ν is a signed measure on $(X, \mathcal{M})$,

Then: there exists a positive set P and neative set N for ν such that $P \cup U = X$, $P \cap U = \varnothing$, and these choices are unique up to null set.

Definition 16. *Locally Integrable*

$f \in L^1_{loc}$ if $\int_K |f(x)| d\mu < \infty$ for all bounded measurable sets K.

Definition 17. *Shrinks Nicely*

$\{E_r\}_{r\geq 0} \subset \mathcal{B}_{\mathbb{R}^n}$ is said to shrink nicely to a point x if

- $E_r \subset B_r(x)$ for all r
- there exists a constant $\alpha > 0$ (independent of r) so $m(E_r) > \alpha m(B(r,x))$ for all r.

Theorem 15. *Lebesgue-Radon-Nikodym*

If:

- ν is a signed and σ-finite measure
- m is a σ-finite measure (usually taken to be the Lebesgue measure)

Then: there exists a measure λ and function $f \in L^1(m)$ such that $\lambda \perp m$ and $d\nu = d\lambda + f dm$.

***Furthermore, when m is the Lebesgue measure, for m-a.e. x, and for every family $\{E_r\}_{r\geq 0}$ that shrinks nicely to x,

$$\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = f(x).$$

Theorem 16. *Generalized Lebesgue-Radon-Nikodym*

If:

- ν is a complex measure
- μ is a σ-finite measure

Then: there exists a measure λ and function $f \in L^1(m)$ such that $\lambda \perp \mu$ and $d\nu = d\lambda + f d\mu$.

Theorem 17. *Lebesgue Differentiation Theorem*

If: $f \in L^1_{loc}$,

Then: for a.e. x, and for every family $\{E_r\}_{r\geq 0}$ that shrinks nicely to x.

$$\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} f(y)dy = f(x).$$