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Real Analysis Exam Spring 2018

Problem 1. Let $-\infty < a < b < \infty$ and suppose $\mathcal{B}$ is a countable collection of closed subintervals of (a, b). Give the proof that there is a countable pairwise-disjoint subcollection $\mathcal{B}' \subset \mathcal{B}$ such that

$$\bigcup_{I \in \mathcal{B}} I \subset \bigcup_{I \in \mathcal{B}'} \tilde{I},$$

where $\tilde{I}$ denotes the 5-times enlargement of I; thus if $I = [x - \rho, x + \rho]$ then $\tilde{I} = [x - 5\rho, x + 5\rho]$.

Solution. Let each $I \in \mathcal{B}$ be written as $I = [x - r, x + r]$ some $x \in (a, b)$ some $r < b - a$. Let $R = \sup_r [x - r, x + r]$. Then $m(I) < 2R < \infty$ for all $I \in \mathcal{B}$, where m is the Lebesgue measure.

Now, let $F_n \subset \mathcal{B}$ be the collection of I such that

$$\text{Radius of } I = \frac{m(I)}{2} \in \left(\frac{R}{2^{n+1}}, \frac{R}{2^n} \right].$$

Now, we define G_n as follows, let $H_0 = F_0$ and G_0 be a maximal disjoint subcollection of H_0.

Now, for each n define

$$H_{n+1} = \{ I \in F_{n+1} \,|\, I \cap J = \varnothing, \forall J \in G_0 \cup G_1 \cup \cdots \cup G_n \}.$$

Let G_{n+1} be a maximal disjoint subcollection of H_{n+1}.

Let

$$\mathcal{B}' = \bigcup_{n=1}^{\infty} G_n.$$

First, $\mathcal{B}'$ is pairwise disjoint by construction. This is because if $I, J \in \mathcal{B}'$, then there exists n, m so $I \in G_n$ and $J \in G_m$, namely, $I \in H_n$ and $J \in H_m$. If $n = m$ then $I, J \in G_n$ which is a disjoint collection, so $I \cap J = \varnothing$. If, WLOG, $n > m$, I cannot intersect J since $J \in G_0 \cup G_1 \cup \cdots \cup G_m \cup \cdots \cup G_{n-1}$ and $I \in H_n$ implies it cannot intersect any elements of this set by definition.

Second, $\mathcal{B}'$ is countable since it comes from $\mathcal{B}$ which is countable.

Finally, let $I \in \mathcal{B}$. Then there exists n so $I \in F_n$.

Now, if $I \notin H_n$, then $n > 0$ since $H_0 = F_0$. Also, I must intersect some $J \in G_0 \cup \cdots \cup G_{n-1}$.

In this case, the radius of I is in $(\frac{R}{2^{n+1}}, \frac{R}{2^n}]$ and since $J \in G_0 \cup \cdots \cup G_{n-1}$, the radius of J is in $(\frac{R}{2^n}, R]$. Since I intersects J, the worst possible case is that $I = [x - \frac{R}{2^n}, x + \frac{R}{2^n}]$ intersects J at an end point, I has maximal length of $2\frac{R}{2^n} = \frac{R}{2^{n-1}}$ and J has minimal length of $2\frac{R}{2^n} = \frac{R}{n^{n-1}}$. However, even in this case, $I \subset \tilde{J}$. To see this, WLOG, take

$$J = \left[\left(x + \frac{2R}{2^n}\right) - \frac{R}{2^n}, \left(x + \frac{2R}{2^n}\right) + \frac{R}{2^n}\right]$$

so J intersects I at its right endpoint.

Then

$$\tilde{J} = \left[\left(x + \frac{2R}{2^n}\right) - \frac{5R}{2^n}, \left(x + \frac{2R}{2^n}\right) + \frac{5R}{2^n}\right] = \left[x - \frac{3R}{2^n}, x + \frac{7R}{2^n}\right]$$

and since $x - \frac{3R}{2^n} < x - \frac{R}{2^n}$ and $x + \frac{7R}{2^n} > x + \frac{R}{2^n}$, we have that $I \subset \tilde{J}$, and similarly for J intersecting I at the right end point.

Now, if $I \in H_n$, then by maximality of G_n, we have that I intersects some J in G_n. Again, $I \in F_n$ so the radius of I is in $(\frac{R}{2^{n+1}}, \frac{R}{2^n}]$ and similarly for J.

As in the previous case, even if $I \cap J$ is a single endpoint, so (in the case of the right end point of I being the intersection point)

$$I = \left[x - \frac{R}{2^n}, x + \frac{R}{2^n}\right], J = \left[\left(x + \frac{3R}{2^{n+1}}\right) - \frac{R}{2^{n+1}}, \left(x + \frac{3R}{2^{n+1}}\right) + \frac{R}{2^n}\right]$$

then

$$\tilde{J} = \left[\left(x + \frac{3R}{2^{n+1}}\right) - \frac{5R}{2^{n+1}}, \left(x + \frac{3R}{2^{n+1}}\right) + \frac{5R}{2^n}\right] = \left[x - \frac{R}{2^n}, x + \frac{4R}{2^n}\right]$$

so $I \subset \tilde{J}$, and similarly for intersection at a left endpoint of I.

Therefore, we have that $\mathcal{B}'$ is indeed a pair-wise disjoint countable subcollection of $\mathcal{B}$, with

$$\bigcup_{I \in \mathcal{B}} I \subset \bigcup_{I \in \mathcal{B}'} \tilde{I}.$$

✌

Problem 2. Assume that f is absolutely continuous on $[0,1]$, and assume that $f' = g$ a.e., where g is a continuous function. Prove that f is continuously differentiable on $[0,1]$.

Solution. Because f is absolutely continuous, by **Fundamental Theorem of Lebesgue Integration**

$$f(x) = f(0) + \int_0^x f'(t)dt = f(0) + \int_0^x g(t)dt \qquad \text{since } f' = g \text{ a.e..}$$

Since g is continuous on $[0,1]$, by the Fundamental Theorem of Calculus part (II), $\int_0^x g(t)dt$ is differentiable on $[0,1]$ and $\frac{d}{dx}\int_0^x g(t)dt = g(x)$ is continuous. Therefore, since

$$f(x) - f(0) = \int_0^x g(t)dt$$

and

$$\frac{d}{dx}\int_0^x g(t)dt = \frac{d}{dx}(f(x) - f(0)) = f'(x)$$

we have that f is continuously differntiable. ✌

Problem 3. Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) = 1$. Let $A_1, A_2, ..., A_{50} \in \mathcal{M}$. Assume that almost every point in X belongs to at least 10 of these sets. Prove that at least one of the sets has measure greater than or equal to 1/5.

Solution. Let

$$f(x) = \sum_{i=1}^{50} \chi_{A_i}(x).$$

Then $f(x)$ counts the number of A_i that x is in.

Let

$$B = \{x \mid f(x) < 10\}.$$

Since $\mu(B^c) = 1$,

$$\sum_{i=1}^{n} \mu(A_i) = \int_X f(x) d\mu = \int_{B^c} f(x) d\mu \geq \int_{B^c} 10 d\mu = 10.$$

Let i be such that $\mu(A_i)$ is maximal (possible since $\mu(A_i) \leq 1$ for $i = 1, ..., 50$).

Then

$$50\mu(A_i) \geq \sum_{i=1}^{n} \mu(A_i) \geq 10 \implies \mu(A_i) \geq \frac{1}{5}.$$

✌

Problem 4. Let $f : [0, \infty) \to \mathbb{R}$ be absolutely continuous on every closed subinterval of $[0, \infty)$ and

$$f(x) = f(0) - \int_0^x g(t)dt, \qquad \text{for } x \geq 0,$$

where $g \in \mathcal{L}^1([0, \infty))$. Show that

$$\int_0^\infty \frac{f(2x) - f(x)}{x} dx = (\log 2) \int_0^\infty g(t)dt.$$

Solution. Because f is absolutely continuous on ever closed subinterval, it is also integrable on every closed subinterval.

Thus, (because scaling does not affect Lebesgue integration) we may write

$$\begin{aligned}
\int_0^\infty \frac{f(2x) - f(x)}{x} dx &= \lim_{\varepsilon \to 0} \lim_{R \to \infty} \left[\int_\varepsilon^R \frac{f(2x)}{x} dx - \int_\varepsilon^R \frac{f(x)}{x} dx \right] \\
&= \lim_{\varepsilon \to 0} \lim_{R \to \infty} \left[\int_{2\varepsilon}^{2R} \frac{f(u)}{2u/2} du - \int_\varepsilon^R \frac{f(u)}{u} du \right] \qquad u = 2x \\
&= \lim_{\varepsilon \to 0} \lim_{R \to \infty} \left[\int_{2\varepsilon}^{R} \frac{f(u)}{u} du + \int_R^{2R} \frac{f(u)}{u} du - \int_\varepsilon^{2\varepsilon} \frac{f(u)}{u} du - \int_{2\varepsilon}^R \frac{f(u)}{u} du \right] \\
&= \lim_{\varepsilon \to 0} \lim_{R \to \infty} \left[\int_R^{2R} \frac{f(u)}{u} du - \int_\varepsilon^{2\varepsilon} \frac{f(u)}{u} du \right] \\
&= \lim_{\varepsilon \to 0} \lim_{R \to \infty} \left[\int_1^2 \frac{Rf(Rx)}{Rx} dx - \int_1^2 \frac{\varepsilon f(\varepsilon x)}{\varepsilon x} dx \right] \qquad x = \frac{u}{R}, \frac{u}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \lim_{R \to \infty} \left[\int_1^2 \frac{f(Rx) - f(\varepsilon x)}{x} dx \right] \\
&= \lim_{\varepsilon \to 0} \lim_{R \to \infty} \left[\int_1^2 \frac{\int_{\varepsilon x}^{Rx} g(t)dt}{x} dx \right] \\
&= \int_1^2 \left(\int_0^\infty g(t)dt \right) \frac{1}{x} dx \qquad (1) \\
&= \log(2) \int_0^\infty g(t)dt
\end{aligned}$$

Now, we note that (1) follows by the dominated convergence theorem.

Namely, g is in L^1 so $\displaystyle\int_{\varepsilon x}^{Rx} g(t)dt < M \in L^1([1, 2])$ with $M < \infty$ and real. Since both limits converge almost everywhere, by dominated convergence we may bring the limits in one at a time. ✌