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Real Analysis Exam Fall 2018

Problem 1. Let $f : [0, 1] \to \mathbb{R}$ be an absolutely continuous function. Let

$$g(x) = \int_0^1 f(xt)dt, \qquad x \in [0, 1].$$

Show that g is an absolutely continuous function.

Solution. Let $M \in \mathbb{R}$ and

$$F_M = \{x \in [0, 1] : |f(x)| \geq M\}.$$

Now, on F_M^c, $|f(x)| < M$. However, since $f(x) > 0$ a.e., we have that for all $\varepsilon > 0$, there exists $\delta > 0$, so

$$S = \{x \in [0, 1] : |f(x)| > \delta\}$$

has measure $m(S) > 1 - \frac{\varepsilon}{2}$. So $m(S^c \cap [0, 1]) < \frac{\varepsilon}{2}$

Then

$$\int_{E_k} f(x)dx \geq \int_{E_k \cap F_M^c \cap S} f(x)dx \geq \int_{E_k \cap F_M^c \cap S} \delta dx = \delta m(E_k \cap F_M^c S).$$

Therefore, since the left tends 0, we get that

$$\lim_{k \to \infty} m(E_k \cap F_M^c \cap S) \to 0.$$

Now, take K so that for all $k \geq K$, $m(E_k \cap F_M^c \cap S) < \frac{\varepsilon}{2}$, then

$$\begin{aligned} m(E_k \cap F_M^c) &= m(E_k \cap F_M^c \cap S) + m(E_k \cap F_M^c \cap S^c) \\ &< m(E_k \cap F_M^c \cap S) + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Thus, $m(E_k \cap F_M^c) \to 0$ as $k \to \infty$.

Finally,

$$\begin{aligned}\int_{E_k} f(x)dx &= \int_{E_k\cap F_M} f(x)dx + \int_{E_k\cap F_M^c} f(x)dx\\ &\geq \int_{E_k\cap F_M} f(x)dx + \int_{E_k\cap F_M^c} f(x)dx\\ &\geq \left|\int_{E_k\cap F_M} Mdx\right| + \int_{E_k\cap F_M^c} f(x)dx\\ &= Mm(E_k\cap F_M) + \int_{E_k\cap F_M^c} f(x)dx\end{aligned}$$

and since $\int_{E_k} f(x)dx \to 0$ and $m(E_k \cap F_M^c) \to 0$ as $k \to \infty$, and since $f(x) < \infty$ on $E_k \cap F_M^c$, $\int_{E_k\cap F_M^c} f(x)dx \to 0$ as $k \to \infty$ and so at last, $m(E_k \cap F_M) \to 0$ as $k \to \infty$. ✌

Problem 2. Let $f \in L^1(\mathbb{R})$, and let

$$S_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} f\left(x + \frac{j}{n}\right), \qquad x \in \mathbb{R},$$

$$S(x) = \int_x^{x+1} f(y)dy, \qquad x \in \mathbb{R}.$$

Show that $f_n \to f$ in $L^1(\mathbb{R})$.

***We assume that the question meant to say "Show that $S_n \to S$ in $L^1(\mathbb{R})$" since no f_n is ever defined.

Solution. First, we note that

$$\lim_{n\to\infty} S_n(x) = S(x)$$

as a Riemann integral. Now, since f is Lebesgue integrable, it is bounded almost everywhere and since it is Lebesgue measurable, it is continuous almost everywhere.

Namely, we have that $S(x)$ the Riemann integral, will equal the Lebesgue integral for a.e. x.

We will apply DCC to $|S_n(x) - S(x)|$

1. $|S_n(x) - S(x)|$ is measurable.
2. $$\lim_{n\to\infty} |S_n(x) - S(x)| = |\lim_{n\to\infty} S_n(x) - S(x)| = 0$$
   by definition of the Riemann integral. And as stated before, the Riemann integral and Lebesgue integral agree a.e.
3. $|S_n(x) - S(x)| \le 2|S(x)| \in L^1$ for all n and $S \in L^1$ since $f \in L^1$ so by Tonelli

$$\begin{aligned}
\int_{\mathbb{R}} |S(x)|dx &\le \int_{\mathbb{R}} \int_x^{x+1} |f(y)|dydx \\
&= \int_{\mathbb{R}} \int_0^1 |f(u-x)|dudx \qquad u = y - x \\
&= \int_0^1 \int_{\mathbb{R}} |f(u-x)|dxdu \qquad \text{Tonelli} \\
&= \int_0^1 M du \\
&= M < \infty
\end{aligned}$$

Where $M = \int |f(x)|dx < \infty$ since $f \in L^1$.

Therefore, by DCC,

$$\lim_{n\to\infty}\int |S_n(x) - S(x)|dx = \int \lim_{n\to\infty} |S_n(x) - S(x)|dx = 0$$

So $S_n \to S$ in L^1. ✌

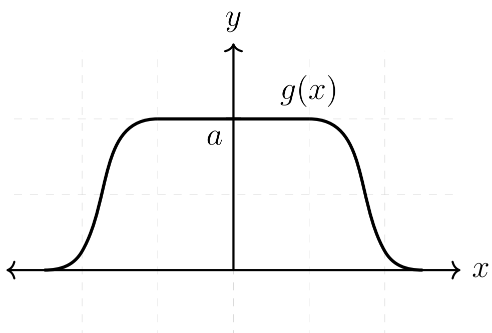
Problem 3. Assume that f_n is a sequence of integrable functions on $\mathbb{R}$ such that

$$\lim_{n\to\infty} \int f_n(x)g(x)dx = g(0)$$

for all g continuous with compact support.

Prove that f_n is not a Cauchy sequence in $L^1(\mathbb{R})$.

Solution. Let $g(x)$ be a smoothed out characteristic function on $[-1, 1]$ times some constant a.



Now, because $\int f_n g dx \to g(0)$ regardless of what g is doing away from 0, we claim that the $\int f_n dx$ is 1 near 0 and 0 everywhere else.

First, let $\varepsilon > 0$ and let $a = 1$. Then

$$\lim_{n\to\infty} \int_{[-\varepsilon,\varepsilon]} f_n(x)dx = \lim_{n\to\infty} \int_{[-\varepsilon,\varepsilon]} f_n(x)g(x)dx = g(0) = 1.$$

Now, shift and stretch g so that $g(0) = 0$ and $g(x) = 1$ is nonzero on some $[-M, -\varepsilon]$ large negative interval.

Then

$$\lim_{n\to\infty} \int_{[-M,-\varepsilon]} f_n(x)dx = \lim_{n\to\infty} \int_{[-M,-\varepsilon]} f_n(x)g(x)dx = g(0) = 0.$$

Similarly, we get that $_{[\varepsilon,M]} f_n(x)dx \to 0$. Namely, for large n, the integrals have value only near 0.

Thus, for all $\varepsilon > 0$, there exists $N > 0$ such that

$$\left| \int f_n(x)dx - 1 \right| < \varepsilon$$

for all $n \geq N$.

However, for all $\delta < 0$, we have that

$$\left| \int_{[-\delta,\delta]} f_n(x)dx - 1 \right| < \varepsilon$$

for all $n \geq N$.

Now, L^1 is complete, so if $f_n \not\to f$ for any measurable function f in L^1, then f_n cannot be cauchy in L^1.

Let f be any measurable function, and $f_n \to f$ in L^1, then taking n large enough we get that

$$\begin{aligned} \varepsilon > \int_{[-\delta,\delta]} |f_n(x) - f(x)|dx &\geq \left| \int_{[-\delta,\delta]} f_n(x) - f(x)dx \right| \\ &\geq \left| \int_{[-\delta,\delta]} f_n(x)dx \right| - \int_{[-\delta,\delta]} |f(x)|dx \\ &> 1 - \varepsilon - \int_{[-\delta,\delta]} |f(x)|dx \end{aligned}$$

Namely,

$$\int_{[-\delta,\delta]} |f(x)|dx = 1 \qquad \text{for all } \delta$$

However, since $f_n \in L^1$, it must be that $f \in L^1$ and this is clearly a contradiction since taking $\delta \to 0$ we cannot have that $\{x \,|\, |f(x)| = \infty\}$ is null.

Thus, f_n does not converge to any L^1 function in L^1, so it cannot be a Cauchy sequence in L^1. ✌