Kayla Orlinsky

Real Analysis Exam Spring 2017

> **Problem 1.** Assume that f is a positive absolutely continuous function on $[0, 1]$. Prove that $1/f$ is also absolutely continuous on $[0, 1]$.

Solution. Since f is absolutely continuous and strictly positive, it has a minimum $f(x_0) = \delta > 0$ for some $x_0 \in [0, 1]$. Namely, for all $x \in [0, 1]$,

$$f(x) \geq \alpha \implies \frac{1}{f(x)} \leq \frac{1}{\alpha}$$

Now, let $\varepsilon > 0$ and $\delta > 0$ such that for all $\{(a_i, b_i)\}_{i=1}^n$ finite collections of disjoint subintervals of $[0, 1]$,

$$\sum_{i=1}^n (b_i - a_i) < \delta \implies \sum_{i=1}^n |f(b_i) - f(a_i)| < \alpha^2 \varepsilon.$$

Then,

$$\begin{aligned}
\sum_{i=1}^n \left| \frac{1}{f(b_i)} - \frac{1}{f(a_i)} \right| &= \sum_{i=1}^n \left| \frac{f(a_i) - f(b_i)}{f(b_i) f(a_i)} \right| \\
&\leq \sum_{i=1}^n \frac{|f(b_i) - f(a_i)|}{\alpha^2} \\
&< \frac{1}{\alpha^2} \alpha^2 \varepsilon = \varepsilon
\end{aligned}$$

so $\frac{1}{f}$ is absolutely continuous on $[0, 1]$. ✌

Problem 2. Assume that E is Lebesgue measurable.

(a) Suppose $m(E) < \infty$, where m is the Lebesgue measure. Show that

$$f(x) = \int \chi_E(y)\chi_E(y-x)dm(y)$$

is continuous. (Here, χ_A denotes the characteristic function of a set $A \subset \mathbb{R}$).

(b) Suppose $0 < m(E) \leq \infty$. Show that $S = E - E = \{x - y : x, y \in E\}$ contains an open interval $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

Solution.

(a) Suppose $m(E) < \infty$. First, we note that $y - x \in E$ if and only if $y \in E + x$. Therefore,

$$\begin{aligned} f(x) &= \int \chi_E(y)\chi_E(y-x)dm(y) \\ &= \int \chi_E(y)\chi_{E+x}(y)dm(y) \\ &= m(E \cap (E+x)). \end{aligned}$$

Now, since $m(E) < \infty$, then for all $\varepsilon > 0$, there exists

$$A = \bigcup_{i=1}^{n} (a_i, b_i)$$

a finite union of disjoint open intervals such that $m(E\Delta A) < \varepsilon$.

Thus, if $g(x) = m(A \cap (A+x))$ is continuous, then because

$$\begin{aligned} |g(x) - f(x)| &= |m(A \cap (A+x)) - m(E \cap (E+x))| \\ &= |m(A \cap (A+x) \cap E) + m(A \cap (A+x) \cap E^c) \\ &\qquad - m(E \cap (E+x) \cap A) - m(E \cap (E+x) \cap A^c)| \\ &< |m(A \cap (A+x) \cap E) - m(E \cap (E+x) \cap A)| + \varepsilon \\ &= |m(A \cap (A+x) \cap E \cap (E+x)) + m(A \cap (A+x) \cap E \cap (E+x)^c) \\ &\qquad - m(E \cap (E+x) \cap A \cap (A+x)) - m(E \cap (E+x) \cap A \cap (A+t)^c)| + \varepsilon \\ &< 4\varepsilon \end{aligned}$$

so $f(x)$ must also be continuous.

Now, g is clearly continuous since A is a finite union of intervals so

$$\lim_{x \to y} g(x) = g(y)$$

for all $x, y \in \mathbb{R}$ since we can make the difference $|g(x) - g(y)|$ as small as we like by making $|x - y|$ small.

Namely, f is continuous.

(b) Suppose $0 < m(E) \leq \infty$.

If $m(E) < \infty$ then $g(x) = m(E \cap (E + x))$ is continuous by (a).

Now, $g(0) = m(E) > 0$ by assumption. Therefore, by continuity, there exists $\varepsilon > 0$ small enough that $g(z) > 0$ for all $z \in (-\varepsilon, \varepsilon)$.

Let $z \in (-\varepsilon, \varepsilon)$ and $z \notin E - E$. Then there does not exist $x, y \in E$ such that $z = x - y$.

Namely, for all $y \in E$, $x = z + y \notin E$. Therefore, $E \cap (E + z) = \varnothing$ so $m(E \cap (E + z)) = 0$.

This is a contradiction of $g(z) > 0$ so no such z can exist. Namely, $(-\varepsilon, \varepsilon) \in E - E$.

Now, if $m(E) = \infty$, because $\mathbb{R}$ is σ-finite, $E \subset \mathbb{R}$ is also σ-finite. Namely, there exists $\{E_k\}_{k=1}^{\infty}$ where E_k are disjoint and $m(E_k) < \infty$ for all k and

$$E = \bigcup_{k=1}^{\infty} E_k.$$

Therefore,

$$(-\varepsilon, \varepsilon) \in E_k - E_k \subset E - E.$$

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Problem 3. Assume that f is a continuous function on $[0,1]$. Prove that

$$\lim_{n\to\infty}\int_0^1 nx^{n-1}f(x)dx = f(1).$$

Solution. Since f is continuous on a closed interval, it is bounded and so the Lebesgue integral and Riemann integrals are the same.

Therefore, letting $u = x^n$, we get that $du = nx^{n-1}dx$ and so

$$\lim_{n\to\infty}\int_0^1 nx^{n-1}f(x)dx = \lim_{n\to\infty}\int_0^1 f(u^{1/n})du.$$

Now, we apply DCT.

1. $f(u^{1/n})$ is measurable.

2. $$\lim_{n\to\infty} f(u^{1/n}) = f\left(\lim_{n\to\infty} u^{1/n}\right) = f(1)$$
   for a.e. $u \in [0,1]$ since f is continuous.

3. $f(u^{1/n}) \leq M \in L^1([0,1])$ where $M = \max|f(x)|$ for a.e. $u \in [0,1]$.

Therefore,

$$\lim_{n\to\infty}\int_0^1 nx^{n-1}f(x)dx = \lim_{n\to\infty}\int_0^1 f(u^{1/n})du = \int_0^1 \lim_{n\to\infty} f(u^{1/n})du = \int_0^1 f(1)du = f(1).$$

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Problem 4. Let $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. Let f, g be measurable real valued functions. Show that

$$\int |f-g| d\mu = \int_{-\infty}^{\infty}\int \left|\chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x))\right| d\mu(x) dt.$$

Solution. Note that for fixed t,

$$f(x) > t, g(x) \leq t \implies \chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x)) = 1$$

and

$$f(x) \leq t, g(x) > t \implies \chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x)) = -1$$

all other cases make the integrand 0.

Let

$$E = \{x \,|\, f(x) > t \geq g(x)\}.$$

Therefore, by linearity of the integral we get that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int \left|\chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x))\right| d\mu(x) dt \\
&\quad= \int_{-\infty}^{\infty}\int_E \chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x)) d\mu(x) dt \\
&\qquad\qquad - \int_{-\infty}^{\infty}\int_{E^c} \chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x)) d\mu(x) dt \\
&\quad= \int_{-\infty}^{\infty}\int_E 1 d\mu(x) dt + \int_{-\infty}^{\infty}\int_{E^c} 1 d\mu(x) dt \\
&\quad= \int_{-\infty}^{\infty}\int_E \chi_{(g(x),f(x))}(x) d\mu(x) dt + \int_{-\infty}^{\infty}\int_{E^c} \chi_{(f(x),g(x))}(x) d\mu(x) dt \\
&\quad= \int_E\int_{-\infty}^{\infty} \chi_{(g(x),f(x))}(x) dt d\mu(x) + \int_{E^c}\int_{-\infty}^{\infty} \chi_{(f(x),g(x))}(x) dt d\mu(x) \\
&\quad= \int_E\int_{g(x)}^{f(x)} dt d\mu(x) + \int_{E^c}\int_{f(x)}^{g(x)} dt d\mu(x) \\
&\quad= \int_E f(x) - g(x) d\mu(x) + \int_{E^c} g(x) - f(x) d\mu(x) \\
&\quad= \int_X |f(x) - g(x)| \mu(x)
\end{aligned}$$

with (1) since Ω is σ-finite and the integrand is clearly in $L^+(\mu \times m)$ so the integrals can be swapped. ✌