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Real Analysis Exam Fall 2017

Problem 1. Let $(X, \mathcal{A}, \mu)$ be a measure space and f, g, f_n, g_n measurable so that $f_n \to f$ and $g_n \to g$ in measure. Is it true that $f_n^3 + g_n \to f^3 + g$ in measure if

(a) $\mu(X) = 1$

(b) $\mu(X) = \infty$

In both cases prove the statement or provide a counter example.

Solution.

(a) True. Since $|f_n^3 + g_n - f^3 - g| \leq |f_n^3 - f| + |g_n - g| = |f_n - f||f_n^2 + f_n f + f^2| + |g_n - g|$, the finiteness of the measure guarentees that $\mu(\{x : |f_n^2 + f_n f + f^2| \geq \varepsilon\})$ is bounded and since the other two measures are shrinking to zero, we can guarentee that the entire set shrinks to a null set.

Specifically, for n large, $M > 0$ large, and $\delta > 0$ small,

$$\begin{aligned}
\mu(\{x : |f_n^3 + g_n - f^3 - g| \geq \varepsilon\}) &\leq \mu(\{x : |f_n - f||f_n^2 + f_n f + f^2| + |g_n - g| \geq \varepsilon\}) \\
&\leq \mu(\{x : |f_n - f||f_n^2 + f_n f + f^2| \geq \varepsilon\}) + \mu(\{x : |g_n - g| \geq \varepsilon\} \\
&\leq \mu(\{x : |f_n - f| \geq \frac{\varepsilon}{M}, |f_n^2 + f_n f + f^2| \geq M\}) \\
&\qquad + \mu(\{x : |f_n - f||f_n^2 + f_n f + f^2| \geq \varepsilon, |f_n^2 + f_n f + f^2| < M\}) \\
&\qquad\qquad + \mu(\{x : |g_n - g| \geq \varepsilon\} \\
&= \mu(\{x : |f_n - f| \geq \frac{\varepsilon}{M}, |f_n^2 + f_n f + f^2| \geq M\}) \\
&\qquad + \mu(\{x : |f_n - f| \geq \frac{\varepsilon}{\delta}, \delta < |f_n^2 + f_n f + f^2| < M\}) \\
&\qquad\qquad + \mu(\{x : |g_n - g| \geq \varepsilon\} \\
&\to 0 \qquad n \to \infty, M \to \infty, \delta \to 0
\end{aligned}$$

(b) False. Let $X = \mathbb{R}$ and $\mu = m$ the Lebesgue measure. Let $f_n(x) = x + \frac{1}{n}$. Then $f_n^3(x) = x^3 + \frac{3}{n}x^2 + \frac{3}{n^2}x + \frac{1}{n^3}$. Let $g_n(x) = 0$. Then $g(x) = 0$ and $f(x) = x$.

Then

$$\mu(\{x \,|\, |f_n(x) - f(x)| \geq \varepsilon\}) = \mu(\{x \,|\, |\frac{1}{n}| \geq \varepsilon\}) = 0$$

for all $n \geq N$ where $\frac{1}{N} < \varepsilon$.

However,

$$\mu(\{x \,|\, |f_n(x) - f(x)| \geq \varepsilon\}) = \mu(\{x \,|\, |\frac{3}{n}x^2 + \frac{3}{n^2}x + \frac{1}{n^3}| \geq \varepsilon\}) = \infty$$

since for all n, on the interval $[\varepsilon\frac{n^2}{3}, \infty)$, $|f_n - f| \geq \varepsilon$.

✌

Problem 2. Let $f \in L^1(\mathbb{R})$. Show that the series

$$\sum_{n=1}^{\infty} f(x+n)$$

converges absolutely for Lebesgue almost every $x \in \mathbb{R}$.

Solution. Fix $k \in \mathbb{Z}$. Then,

$$\begin{aligned}
\int_k^{k+1} \sum_{n=1}^{\infty} |f(x+n)|dx &= \sum_{n=1}^{\infty} \int_k^{k+1} |f(x+n)|dx && (1)\\
&= \sum_{n=1}^{\infty} \int_{k+n}^{k+n+1} |f(u)|du \qquad u = x+n && (2)\\
&= \int_k^{\infty} |f(u)|du < \infty
\end{aligned}$$

With (1) because $|f(x+n)| \in L^+$ so the sum and integral can be swapped and (2) because linear u-sub preserves the Lebesgue integral thanks to the shifting and scaling properties of the Lebesgue measure.

Finally, since the integral is finite, the sum must be finite a.e. Namely, $\sum_{n=1}^{\infty} |f(x+n)|dx < \infty$ for a.e. $x \in [k, \infty)$. Since $k \in \mathbb{Z}$ was arbitrary, we have that the sum is finite for a.e. $x \in \mathbb{R}$ and so the sum converges absolutely. ✌

Problem 3. Assume that $E \subset \mathbb{R}$ is such that $m(E \cap (E+t)) = 0$ for all $t \neq 0$, where m is the Lebesgue measure on $\mathbb{R}$. Prove that $m(E) = 0$.

Solution. First, since $\mathbb{R}$ is σ-finite, E is σ-finite so there exists $\{E_k\}_{k=1}^\infty$ such that

$$E = \bigcup_{k=1}^\infty E_k \qquad m(E_k) < \infty.$$

Furthermore,

$$m(E_k \cap (E_k + t)) \leq m(E \cap (E_k + t)) \leq m(E \cap (E+t)) = 0$$

so it suffices to check that $m(E_k) = 0$ for all k.

If $m(E_k) < \infty$, then for all $\varepsilon > 0$, there exists

$$A = \bigcup_{i=1}^n (a_i, b_i)$$

a finite union of disjoint open intervals such that $m(E_k \Delta A) < \varepsilon$.

Now, for all $t \neq 0$

$$\begin{aligned}
m(A \cap (A+t)) &= m(A \cap (A+t) \cap E) + m(A \cap (A+t) \cap E^c) \\
&= m(A \cap (A+t) \cap E \cap (E+t)) + m(A \cap (A+t) \cap E^c \cap (E+t)) \\
&\qquad + m(A \cap (A+t) \cap E \cap (E+t)^c) + m(A \cap (A+t) \cap E^c \cap (E+t)^c) \\
&= 0 + m([A \backslash E] \cap (A+t) \cap (E+t)) + m(A \cap E \cap [(A+t) \backslash (E+t)]) \\
&\qquad + m([A \backslash E] \cap (A+t) \cap (E+t)^c) \\
&< 2\varepsilon + m(A \cap E \cap [(A \backslash E) + t]) \\
&\leq 2\varepsilon + m((A \backslash E) + t) \\
&= 2\varepsilon + m(A \backslash E) \\
&< 3\varepsilon
\end{aligned}$$

Namely, $m(A \cap (A+t)) < 3\varepsilon$ for all t. However, for $t > 0$

$$m(A \cap (A+t)) = m\left(\bigcup_{i=1}^n (a_i, b_i) \cap \bigcup_{i=1}^n (a_i + t, b_i + t)\right) = \sum_{i=1}^n b_i - (a_i + t) = m(A) - nt.$$

Letting $t = \frac{\varepsilon}{n}$ we see that

$$m(A) < 4\varepsilon.$$

Therefore,

$$m(E_k) = m(E_k \cap A) + m(E_k \cap A^c) < 4\varepsilon + \varepsilon = 5\varepsilon.$$

Since ε was arbitrary, $m(E_k) = 0$ for all k.

Namely, $m(E) = 0$. ✌

Problem 4. Let $(X, \mathcal{A}, \mu)$ be a measure space and f_n a sequence of non-negative measurable functions. Prove that if $\sup_n f_n$ is integrable, then

$$\limsup_n \int_X f_n d\mu \leq \int_X \limsup_n f_n d\mu.$$

Also show that

(a) the inequality may be strict and

(b) that the inequality may fail unless $\sup_n f_n \in L^1$.

Solution. Let

$$g_k(x) = \sup_{n \geq k} f_n(x)$$

and $g(x) = \sup_n f_n(x)$.

Now, since for all n, $f_n(x) \leq \sup_n f_n(x)$, we have that

$$\int_X f_n d\mu \leq \int_X \sup_n f_n d\mu.$$

Namely, $\int_X \sup_n f_n d\mu$ is an upper bound for $\int_X f_n d\mu$ and so

$$\sup_n \int_X f_n d\mu \leq \int_X \sup_n f_n d\mu.$$

Now, we claim that

$$\lim_{k\to\infty} \int_X g_k(x) d\mu = \int_X \lim_{k\to\infty} g_k(x) d\mu.$$

We will use DCT.

1. $g_k(x)$ is measurable for all k.
2. $$\lim_{k\to\infty} g_k(x) = \limsup_n f_n(x)$$
   so the limit exists a.e..
3. $g_k(x) \leq g(x) \in L^1$ for all k and for a.e. x.

Therefore, by DCT,

$$\limsup_n \int_X f_n d\mu \leq \lim_{k\to\infty} \int_X \sup_{n\geq k} f_n(x) d\mu = \int_X \limsup_n f_n(x) d\mu.$$

(a) Let$X = [0,1]$ $\mu = m$ the Lebesgue measure and $f_n(x)$ be the moving box, or typewriter sequence. Namely, for each n, there exists k so $2^k \leq n < 2^{k+1}$, let

$$f_n(x) = \chi_{[\frac{n}{2^k}-1, \frac{n+1}{2^k}-1]}(x).$$

Namely,

$$\begin{aligned} f_1(x) &= \chi_{[0,1]} \qquad 2^0 = 1 \leq 1 \\ f_2(x) &= \chi_{[0,\frac{1}{2}]} \qquad 2^1 = 2 \leq 2 \\ f_3(x) &= \chi_{[\frac{1}{2},1]} \qquad 2^1 = 2 \leq 3 \\ &\vdots \end{aligned}$$

Now, by nature of the moving box, for each $x \in [0,1]$ there exists an infinite number of n so that $f_n(x) = 1$. Therefore,

$$\limsup_n f_n(x) = 1.$$

Namely,

$$\int_{[0,1]} \limsup_n f_n(x)dx = \int_{[0,1]} 1dx = 1.$$

Now,

$$\limsup_n \int_{[0,1]} f_n(x)dx = \limsup_n \frac{1}{2^k} = 0$$

since k grows with n,

Thus, the inequality may be strict.

(b) Let $X = (0,1]$ $\mu = m$ the Lebesgue measure and $f_n(x) = n\chi_{(0,\frac{1}{n}]}(x)$. Then for all x, since for all $x \in (0,1]$, there exists N so $\frac{1}{N+1} < x \leq \frac{1}{N}$ so $\sup_n f_n(x) = N$.

$$\limsup_n f_n(x) = \lim_{k\to\infty} \sup_{n\geq k} f_n(x) = 0.$$

Thus,

$$\int_{(0,1]} \limsup_n f_n(x)dx = 0$$

and

$$\limsup_n \int_{(0,1]} f_n(x)dx = \limsup_n nm\left(\left(0, \frac{1}{n}\right]\right) = \limsup_n 1 = 1.$$

Namely, the inequality does not hold. Note that $\sup_n f_n(x) \notin L^1$ since it explodes near 0.

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