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Real Analysis Exam Spring 2016

Problem 1. Let

$$f(y) = \sum_n \frac{x}{x^2 + yn^2}.$$

Show that $g(y) = \lim_{x\to\infty} f(x,y)$ exists for all $y > 0$. Find $g(y)$.

Solution. Let $(\mathbb{N}, \nu)$ be the counting measure space. Fix $y > 0$ and since we care to examine $x \to \infty$, we may take $x > 0$ as well.

We would like to use **Dominated Convergence Theorem**. Let $f_n(x,y) = \frac{x}{x^2+yn^2}$.

1. $\{f_n\}$ measurable for all n.
2. $\lim_{x\to\infty} f_n(x,y) = 0$ for all $y > 0$.
3. Now, using calculus,

$$\frac{\partial}{\partial x} f_n(x,y) = \frac{x^2 + yn^2 - x(2x)}{(x^2+yn^2)^2} = \frac{yn^2 - x^2}{(x^2+yn^2)^2} = 0 \implies x = \sqrt{yn^2} = \sqrt{y}n.$$

Clearly this is a maximum for $f_n(x,y)$, and so we see that

$$f_n(x,y) \leq \frac{\sqrt{y}n}{yn^2 + yn^2} = \frac{1}{2\sqrt{y}n}.$$

Now, for every fixed $y > 0$, let

$$h(n) = \begin{cases} \frac{1}{2y^{1/3}n^{4/3}} & \text{if } yn^2 \geq 1 \\ \frac{1}{2\sqrt{y}n} & \text{if } yn^2 < 1 \end{cases}$$

Note that since we are working over the counting measure on $\mathbb{N}$, our variable of integration is n and so h must be a function of n independent of x (the variable over which we are taking the limit).

Then, for all y,

$$\sum_n h(n) = \sum_{n<1/\sqrt{y}} \frac{1}{2\sqrt{y}n} + \sum_{n\geq 1/\sqrt{y}} \frac{1}{2y^{1/3}n^{4/3}} < \infty$$

since $4/3 > 1$ and the first sum is over finitely many n.

Furthermore, for $yn^2 \geq 1$, $1/\sqrt{y}n \leq 1$ and so

$$f_n(x,y) \leq \frac{1}{2\sqrt{y}n} \leq \left(\frac{1}{2\sqrt{y}n}\right)^{2/3} = h(n).$$

When $yn^2 < 1$, $h(n)$ is exactly the upper bound for $f_n(x,y)$ calculated previously.
Thus, $f_n(x,y) \leq h(n) \in L^1(\nu)$ for all $x > 0$ and all $y > 0$.

Finally, by DCT,

$$\lim_{x\to\infty} \sum_n f_n(x,y) = \sum_n \lim_{x\to\infty} f_n(x,y) = \sum_n 0 = 0$$

for all $x, y > 0$. ✌

Problem 2. Let $A \subset \mathbb{R}$ be Lebesgue measurable. Show that $n(\chi_A * \chi_{[0,\frac{1}{n}]}) \to \chi_A$ pointwise a.e. as $n \to \infty$. (Recall that $(f * g)(x) = \int f(x-y)g(y)dy$ for $x \in \mathbb{R}$).

Solution. First,

$$n(\chi_A * \chi_{[0,\frac{1}{n}]}) = n\int_{\mathbb{R}} \chi_A(x-y)\chi_{[0,1/n]}(y)dy = n\int_{[0,1/n]} \chi_A(x-y)dy = \frac{1}{m([0,1/n])}\int_{[0,1/n]} \chi_A(x-y)dy$$

Now, we do the following changes, letting $r = \frac{1}{n}$ and noticing that if $x - y \in A$, then $x - y = a \in A$ and so $y = x - a \in x - A = \{x - a \,|\, a \in A\}$. Finally, since $0 \leq y \leq 1/n$, $x - 1/n \leq x - y \leq x$.

Now, we would like to apply the Lebesgue Differntiation Theorem.

1. $\chi_A \in L^1_{\text{loc}}$
2. $[x-r, x]$ shrinks nicely to x

Thus,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{m([0,1/n])}\int_{[0,1/n]} \chi_A(x-y)dy &= \lim_{r\to 0} \frac{1}{m([0,r])}\int_{[0,r]} \chi_A(x-y)dy \\ &= \lim_{r\to 0} \frac{1}{m([x-r,x])}\int_{[x-r,x]} \chi_{x-A}(y)dy \\ &= \lim_{r\to 0} \frac{1}{m([x-r,x])}\int_{[x-r,x]} \chi_A(y)dy \qquad m(A) = m(x-A) \\ &= \chi_A(x) \qquad \text{a.e. by LDT.}\end{aligned}$$

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Problem 3.

(a) Prove that if a sequence of integrable functions f_n on $[0,1]$ satisfies $\int_0^1 |f_n(x)|dx \leq \frac{1}{n^2}$ for $n \in \mathbb{N}$, then $f_n \to 0$ a.e. on $[0,1]$ as $n \to \infty$.

(b) Show that the above fact is not true if $1/n^2$ is replaced by $1/\sqrt{n}$.

Solution.

(a) First, let $(\mathbb{N}, \nu)$ be the counting measure space. We would like to use Tonelli.

  (a) Both $(\mathbb{N}, \nu)$ and $([0,1], m)$ are σ-finite.

  (b) Since any function is measurable with respect to the counting measure, and $|f_n| \in L^+([0,1])$, so $|f_n| \in L^+(\mathbb{N} \times [0,1])$.

Thus,

$$\sum_n^\infty \int_0^1 |f_n(x)|dx = \int_0^1 \sum_n^\infty |f_n(x)|dx \leq \sum_n^\infty \frac{1}{n^2} < \infty.$$

Therefore, $\sum_n^\infty |f_n(x)| < \infty$ m-a.e. which implies $|f_n(x)| \to 0$ a.e. and so $f_n \to 0$ a.e. as $n \to \infty$.

Note that Dominated Convergence would be difficult in this case since we do not know that the f_n are bounded.

(b) Let $f_n(x)$ be the moving box on $[0,1]$. Then, $f_n(x) = \chi_{[\frac{j-1}{2^k}, \frac{j}{2^k}]}$ with $n = 2^k + j$ and $0 \leq j < 2^k$.

Then,

$$\int_0^1 |f_n(x)|dx = \frac{1}{2^k} \leq \frac{1}{\sqrt{2^{k+1}}} = \frac{1}{\sqrt{2^k + 2^k}} \leq \frac{1}{\sqrt{n}}$$

for all n since $j < 2^k$ and so $n = 2^k + j \leq 2^k + 2^k$.

However, the moving box does not converge to anything a.e.. In fact, it does not converge for any x.

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Problem 4 (Folland, 2.3.25, p.59). Let

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Also, let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rationals. Define

$$g_n(x) = \frac{1}{2^n} f(x - r_n), \qquad x \in \mathbb{R}$$

and let

$$g(x) = \sum_{n=1}^{\infty} g_n(x) \qquad x \in \mathbb{R}$$

(a) Prove that g is integrable on $\mathbb{R}$

(b) Prove that g is discontinuous at every point in $\mathbb{R}$.

Solution.

(a) Let $(\mathbb{N}, \nu)$ be the counting measure space.

Then

(i) Both $(\mathbb{N}, \nu)$ and $(\mathbb{R}, m)$ are σ-finite.

(ii) f is continuous everywhere and so $g_n(x)$ is continuous everywhere, thus because the σ-algebra for ν is defined as the $\mathcal{P}(\mathbb{N})$, $_n$ is measurable and positive so $g_n \in L^+(\nu \times m)$.

Thus, by Tonelli,

$$\begin{aligned} \int_{\mathbb{R}} \sum_n g_n(x) dx &= \sum_n \int_{\mathbb{R}} g_n(x) dx \\ &= \sum_n \int_{(r_n, 1+r_n)} \frac{1}{2^n \sqrt{x - r_n}} dx \\ &= \sum_n \frac{1}{2^n} 2\sqrt{x - r_n} \Big|_{r_n}^{1+r_n} \\ &= \sum_n \frac{1}{2^{n-1}} < \infty \end{aligned}$$

Thus, $g \in L^1(\mathbb{R})$.

(b) For any $M > 0$ and $x \in \mathbb{R}$, there exists some $N \in \mathbb{N}$ such that $1 > x - r_N \geq \frac{1}{2^{2N} M^2} > 0$. Then

$$g(x) \geq \frac{1}{2^N} f(x - r_N) \geq \frac{1}{2^N} 2^N M = M.$$

Thus, on any interval containing x, $g(x)$ can be made arbitrarily large on that interval and so it cannot be continuous.

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