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Real Analysis Exam Fall 2016

Problem 1. Let $(X, \mathcal{F}, \mu)$ be a *finite* measure space, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of *nonnegative* measurable functions. Prove that $f_n \to 0$ in measure if and only if

$$\lim_{n\to\infty} \int \frac{f_n}{f_n+1} d\mu = 0.$$

Solution.

$\boxed{\Longrightarrow}$ Assume $f_n \to 0$ in measure. Let $\delta > 0$ and $\varepsilon = \frac{\delta}{2\mu(X)}$, and N large enough that such that

$$\mu(\{x \mid f_n(x) \geq \varepsilon\}) < \frac{\delta}{2} \qquad \forall n \geq N.$$

Let $E_n = \{x \mid f_n(x) \geq \varepsilon\}$. Then since $f_n \geq 0$ for all n, $\frac{f_n}{f_n+1} \leq 1$.

Thus, for all $n \geq N$,

$$\begin{aligned}
\int \frac{f_n}{f_n+1} d\mu &= \int_{E_n} \frac{f_n}{f_n+1} d\mu + \int_{E_n^c} \frac{f_n}{f_n+1} d\mu \\
&\leq \int_{E_n} 1 d\mu + \int_{E_n^c} \frac{\varepsilon}{f_n+1} d\mu \\
&\leq \mu(E_n) + \int_{E_n^c} \varepsilon d\mu \\
&= \mu(E_n) + \varepsilon\mu(E_n^c) \\
&< \frac{\delta}{2} + \varepsilon\mu(E_n^c) \\
&\leq \delta.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \int \frac{f_n}{f_n+1} d\mu = 0.$$

$\boxed{\Longleftarrow}$ Assume

$$\lim_{n\to\infty} \int \frac{f_n}{f_n+1} d\mu = 0.$$

Note that

$$\begin{aligned} f_n(x) &\geq \varepsilon \\ f_n(x) + \varepsilon f_n(x) &\geq \varepsilon + \varepsilon f_n(x) \\ f_n(x)(1+\varepsilon) &\geq \varepsilon(1 + f_n(x)) \\ \frac{f_n(x)}{f_n(x)+1} &\geq \frac{\varepsilon}{\varepsilon+1} \end{aligned}$$

since of course $f_n \geq 0$ so $f_n + 1 > 0$.

Therefore,

Since

$$\int \frac{f_n}{f_n+1} d\mu \geq \int_{E_n} \frac{f_n}{f_n+1} d\mu \geq \int_{E_n} \frac{\varepsilon}{\varepsilon+1} d\mu = \frac{\varepsilon}{\varepsilon+1}\mu(E_n)$$

and the left hand side goes 0 as $n \to \infty$, we have that $\mu(E_n) \to 0$ as $n \to \infty$.

Thus, $f_n \to 0$ in measure. ✌

Problem 2. Let $(X, \mathcal{F}, \mu)$ be a *finite* measure space, and let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ be a sequence of sets. Assume that $\mu(A_n) \geq \delta$ for all $n \in \mathbb{N}$, where $\delta > 0$. Prove that there exists a set $S \in \mathcal{F}$ of positive measure such that for every $x \in S$, is in A_j for infinitely many j.

Solution. We are looking for exactly the limit superior: the set of x such that x is in infinitely many A_j.

Let

$$S = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j.$$

The certainly, if $x \in S$, then x is in infinitely many A_j. Furthermore, $S \in \mathcal{F}$ since it is a countable intersection of countable unions of elements of $\mathcal{F}$ which is a σ-algebra.

We would like to show that $\mu(S) > 0$.

Now, we note that

$$\int \sum_{n=1}^{\infty} \chi_{A_n}(x) d\mu = \sum_{n=1}^{\infty} \int \chi_{A_n}(x) d\mu = \sum_{n=1}^{\infty} \mu(A_n) \geq \sum_{n=1}^{\infty} \delta = \infty.$$

Therefore, since $\mu(X) < \infty$,

$$\left\{ x \,\middle|\, \sum_{n=1}^{\infty} \chi_{A_n}(x) = \infty \right\}$$

must have strictly positive measure. Namely, $\{x \,|\, x \in A_n \text{ for infinitely many } n\}$ has strictly positive measure and since this set is a subset of S, we have that $\mu(S) > 0$.

✌

Problem 3. Let $f_n : [0,1] \to [0,\infty)$ be Lebesgue measurable and such that $f_n(x) \to 0$ for almost every x. Assume that

$$\sup_n \int_0^1 \varphi(f_n(x))dx \leq 1$$

for some continuous $\varphi : [0,\infty) \to [0,\infty)$ which satisfies $\varphi(t)/t \to \infty$ as $t \to \infty$. Prove that $\int_0^1 f_n(x)dx \to 0$ as $n \to \infty$. (Provide a detailed proof).

Solution. First, since $\varphi(t)/t \to \infty$, we have that for all $M > 0$ there exists a T such that $\varphi(x)/x \geq M$ for all $x \geq T$.

Thus,

$$\frac{x}{\varphi(x)} \leq \frac{1}{M} \qquad x \geq T.$$

Let

$$E_n^T = \{x \in [0,1] \,:\, f_n(x) \geq T\}.$$

Then,

$$\begin{aligned}
\left|\int_0^1 f_n(x)dx\right| &= \int_0^1 f_n(x)dx \\
&= \int_{E_n^T} f_n(x)dx + \int_{(E_n^T)^c} f_n(x)dx \\
&= \int_{E_n^T} \frac{\varphi(f_n(x))}{\varphi(f_n(x))} f_n(x)dx + \int_{(E_n^T)^c} f_n(x)dx \\
&\leq \int_{E_n^T} \varphi(f_n(x))\frac{1}{M}dx + \int_{(E_n^T)^c} f_n(x)dx \\
&\leq \frac{1}{M}1 + \int_{(E_n^T)^c} f_n(x)dx \to \frac{1}{M} \qquad (1)
\end{aligned}$$

with (1) by DCT.

1. f_n is measurable for all n.
2. $f_n \to 0$ a.e. $x \in [0,1]$.
3. on $(E_n^T)^c$, $f_n \leq T \in L^1([0,1])$.

Therefore, since M was arbitrary, we have that

$$\int_0^1 f_n(x)dx \to 0 \qquad n \to \infty.$$

✌

Problem 4. Let $h : [0, \infty) \to \mathbb{R}$ be continuous with compact support. Prove that

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \frac{h(\alpha x) - h(\beta x)}{x} dx = h(0) \log \frac{\alpha}{\beta}$$

for every $\alpha, \beta > 0$.

Solution. WLOG, take $\beta > \alpha$.

$$\begin{aligned}
\lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \frac{h(\alpha x)}{x} dx &= \lim_{\varepsilon \to 0^+} \left[\int_\varepsilon^\infty \frac{h(\alpha x)}{x} dx - \int_\varepsilon^\infty \frac{h(\beta x)}{x} dx \right] \\
&= \lim_{\varepsilon \to 0^+} \lim_{R \to \infty} \left[\int_\varepsilon^R \frac{\alpha h(\alpha x)}{\alpha x} dx - \int_\varepsilon^R \frac{\beta h(\beta x)}{\beta x} dx \right] \\
&= \lim_{\varepsilon \to 0^+} \lim_{R \to \infty} \left[\int_{\alpha\varepsilon}^{\alpha R} \frac{h(u)}{u} du - \int_{\beta\varepsilon}^{\beta R} \frac{h(u)}{u} du \right] \qquad u = \alpha x, \beta x \qquad (1) \\
&= \lim_{\varepsilon \to 0^+} \lim_{R \to \infty} \left[\int_{\alpha\varepsilon}^{\beta\varepsilon} \frac{h(u)}{u} du + \int_{\beta\varepsilon}^{\alpha R} \frac{h(u)}{u} du - \int_{\beta\varepsilon}^{\alpha R} \frac{h(u)}{u} du - \int_{\alpha R}^{\beta R} \frac{h(u)}{u} du \right] \\
&= \lim_{\varepsilon \to 0^+} \lim_{R \to \infty} \left[\int_{\alpha\varepsilon}^{\beta\varepsilon} \frac{h(u)}{u} du - \int_{\alpha R}^{\beta R} \frac{h(u)}{u} du \right] \\
&= \lim_{\varepsilon \to 0^+} \lim_{R \to \infty} \left[\int_{\alpha\varepsilon}^{\beta\varepsilon} \frac{\frac{1}{\varepsilon} h(u)}{\frac{1}{\varepsilon} u} du - \int_{\alpha R}^{\beta R} \frac{\frac{1}{R} h(u)}{\frac{1}{R} u} du \right] \\
&= \lim_{\varepsilon \to 0^+} \lim_{R \to \infty} \left[\int_\alpha^\beta \frac{h(x/\varepsilon)}{x} dx - \int_\alpha^\beta \frac{h(x/R)}{x} dx \right] \qquad x = u/\varepsilon, u/R \\
&= \lim_{\varepsilon \to 0^+} \lim_{R \to \infty} \left[\int_\alpha^\beta \frac{h(x/\varepsilon) - h(x/R)}{x} dx \right] \\
&= \int_\alpha^\beta \lim_{\varepsilon \to 0^+} \lim_{R \to \infty} \frac{h(x/\varepsilon) - h(x/R)}{x} dx \qquad (2) \\
&= \int_\alpha^\beta \frac{-h(0)}{x} dx \\
&= -h(0) \log x \Big|_\alpha^\beta \\
&= -h(0) \log \frac{\beta}{\alpha} \\
&= h(0) \log \frac{\alpha}{\beta}
\end{aligned}$$

with (1) because u-sub is supported by Lebesgue integration because of the scaling and shifting properties of the Lebesgue measure, and (2) because of DCT.

Namely,

1. $\frac{h(x/\varepsilon)}{x}$ is measurable since h is continuous and so measurable, and $\frac{1}{x}$ is continuous on $[\alpha, \beta]$ and so measurable.

2. The various limits exist, specifically, because h is continuous $h(x/R) \to h(0)$ as $R \to \infty$ and since h has compact support, $h(x/\varepsilon) = 0$ for ε small. Namely

$$\lim_{\varepsilon \to 0^+} \lim_{R \to \infty} \frac{h(x/\varepsilon) - h(x/R)}{x} = \frac{0 - h(0)}{x} = \frac{-h(0)}{x}.$$

3. Because h is continuous and has compact support, it has a maximum value on $[\alpha, \beta]$ which is a finite. Therefore,

$$\frac{h(x/\varepsilon) - h(x/R)}{x} \leq \frac{2M}{\alpha} \in L^1([\alpha, \beta])$$

for a.e. $x \in [\alpha, \beta]$ where M is the maximum of h on $[\alpha/R, \beta/\varepsilon]$.

Therefore, by DCT, we can bring both limits inside the integral. ✌