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Real Analysis Exam Spring 2015

Problem 1. Consider the sequence

$$f_n(x) = \left(1+\frac{x}{n}\right)^{-n}\cos\left(\frac{x}{n}\right), \qquad n = 1, 2, \dots .$$

Evaluate

$$\lim_n \int_0^\infty f_n(x)dx,$$

being careful to justify your answer.

Solution. We would like to use Dominated Convergence Theorem.

1. $\{f_n\}$ is measurable for all n.
2.

$$\begin{aligned}
y &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{-n} \\
\implies \ln(y) &= \lim_{n\to\infty} -n\ln\left(1+\frac{x}{n}\right) \\
&= \lim_{n\to\infty}\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{-1}{n}} \\
&= \lim_{n\to\infty}\frac{\frac{1}{1+\frac{x}{n}}\frac{-x}{n^2}}{\frac{1}{n^2}} \qquad \text{L'Hopital's Rule.} \\
&= \lim_{n\to\infty}\frac{-x}{1+\frac{x}{n}} \\
&= -x \\
\implies y &= e^{-x}
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} f_n(x) = \left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{-n}\right)\left(\lim_{n\to\infty}\cos\left(\frac{x}{n}\right)\right) = e^{-x}\cos(0) = e^{-x}$$

since both limits exist separately. Furthermore, this limit holds for all x.

3. Now, for all $n > 1$,

$$\left(1+\frac{x}{n}\right)^{-n} \leq (1+x)^{-n} \leq (1+x)^{-2} \in L^1.$$

Note that since

$$\frac{1}{(1+x)^n} = \left(\frac{1}{1+x}\right)^n$$

and $\frac{1}{1+x} \leq 1$ for all $x \geq 0$, we have that $(1+x)^{-n} \leq (1+x)^{-n+1}$ for all n.

Thus, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty \lim_{n\to\infty} f_n(x)dx = -e^{-x}\Big|_0^\infty = 1.$$

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Problem 2. Suppose that $f : [0, \infty) \to \mathbb{R}$ is Lebesgue integrable.

(a) Show that there exists a sequence $x_n \to \infty$ such that $f(x_n) \to 0$.

(b) Is it true that $f(x)$ must converge to 0 as $x \to \infty$? Give a proof or a counter example.

(c) Suppose additionally that f is differentiable and $f'(x) \to 0$ as $x \to \infty$. Is it true that $f(x)$ must converge to 0 as $x \to \infty$? Give a proof or counter example.

Solution.

(a) Let $\{x_n\}$ be such that for all n, $x_n > n$ and $f(x_n) < \frac{1}{n}$.

If no such sequence exists, then for all sequences with $x_n > n$, $f(x_n) \geq \frac{1}{n}$. However, then

$$\int_n^\infty f(x)dx \geq \int_n^\infty \frac{1}{n} = \infty$$

which contradicts that $f \in L^1$. Thus, the sequence given exists.

(b) No. Let $f(x) = \chi_{\mathbb{Q}}$. Then $f \in L^1$ since $m(\mathbb{Q}) = 0$, however $\lim_{x\to\infty} f(x)$ does not exist.

(c) Assume that $f \not\to 0$ as $x \to \infty$. Then there exists some $\{x_n\}$ tending to infinity with $f(x_n) \geq \varepsilon$ for all n. (WLOG we take $f(x_n) \geq 0$, however if f is everywhere negative, then $-f(x_n) \geq \varepsilon$ and the rest of the proof is similar).

Since $|f'(x_n)| \leq \frac{\varepsilon}{2}$ for large enough n, and since differentiability implies continuity, we may apply the Fundamental Theorem of Calculus. (Note that on any closed interval $[x_n, x_n + 1]$, f must be bounded) so for all $x_n \leq x \leq x_n + 1$

$$|f(x) - f(x_n)| = \left| \int_x^{x_n+1} f'(t)dt \right| \leq \left| \int_{x_n}^{x_n+1} f'(t)dt \right| \leq \int_{x_n}^{x_n+1} \frac{\varepsilon}{2} dt = \frac{\varepsilon}{2}.$$

However,

$$\begin{aligned} |f(x) - f(x_n)| &\leq \frac{\varepsilon}{2} \\ -\frac{\varepsilon}{2} &\leq f(x) - f(x_n) \\ \varepsilon - \frac{\varepsilon}{2} \leq f(x_n) - \frac{\varepsilon}{2} &\leq f(x) \end{aligned}$$

However, then

$$\int f(t)dt \geq \sum_{n=N}^{\infty} \int_{x_n}^{x_n+1} f(t)dt \geq \sum_{n=N}^{\infty} \frac{\varepsilon}{2} = \infty$$

Again, this contradicts $f \in L^1$ and so no such sequence can exist. Namely, $f \to 0$ as $x \to \infty$.

Problem 3. Define $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$.

(a) Show that

$$\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)dx = 0$$

and

$$\int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)dx = \log(b/a).$$

(b) What can you deduce about the value of

$$\int_0^{\infty} \sum_{n=1}^{\infty} |f_n(x)|dx?$$

Solution.

(a)

$$\begin{aligned}
\int_0^{\infty} f_n(x)dx &= \int_0^{\infty} ae^{-nax} - be^{-nbx} dx \\
&= \frac{ae^{-nax}}{-na} - \frac{be^{-nbx}}{-nb}\Big|_0^{\infty} \\
&= 0 - \left(\frac{-1}{n} + \frac{1}{n}\right) = 0.
\end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)dx = 0.$$

Now, using the convergence of Geometric Series (because $e^{ax} \geq 1$ for all $a > 0$ and all $x \geq 0$), we have that

$$\sum_{n=1}^{\infty} f_n(x) = a \sum_{n=1}^{\infty} \left(\frac{1}{e^{ax}}\right)^n - b \sum_{n=1}^{\infty} \left(\frac{1}{e^{bx}}\right)^n = \frac{ae^{-ax}}{1 - e^{-ax}} - \frac{be^{-bx}}{1 - e^{-bx}}.$$

Thus,

$$
\begin{aligned}
\int_0^\infty \sum_{n=1}^\infty f_n(x)dx &= \int_0^\infty \frac{ae^{-ax}}{1-e^{-ax}}dx - \int_0^\infty \frac{be^{-bx}}{1-e^{-bx}}dx \\
&= \int_0^1 \frac{du}{u} - \int_0^1 \frac{dw}{w} \qquad \begin{array}{cc} u = 1-e^{-ax} & x:[0,\infty] \\ du = ae^{-ax} & u:[0,1] \end{array} \quad \text{similarly for } w \\
&= \ln(u) - \ln(w) \\
&= \ln|1-e^{ax}| - \ln|1-e^{-bx}|\Big|_0^\infty \\
&= \ln(1) - \lim_{x\to 0}\ln\left(\frac{1-e^{-ax}}{1-e^{-bx}}\right) \\
&= \lim_{x\to 0}\ln\left(\frac{1-e^{-bx}}{1-e^{-ax}}\right) \qquad \text{absorbing the negative} \\
&= \ln\left(\lim_{x\to 0}\frac{1-e^{-bx}}{1-e^{-ax}}\right) \qquad \ln \text{ is continuous} \\
&= \ln\left(\lim_{x\to 0}\frac{be^{-bx}}{ae^{-ax}}\right) \qquad \text{L'Hopital's Rule} \\
&= \ln\left(\frac{b}{a}\right).
\end{aligned}
$$

Note that it was necessary for $b > a > 0$.

(b) $f_n(x)$ is certainly a continuous function for all x and all n, thus f_n is measurable. Furthermore, if $(\mathbb{N}, \nu)$ is the counting measure space, then $f_n(x)$ will certainly be measurable with respect to $m \times \nu$.

Since both $([0,\infty), m)$ and $(\mathbb{N}, \nu)$ are σ-finite measure spaces, and $|f_n(x)| \in L^+(m\times\nu)$, by Tonelli, the integral and summation of $|f_n(x)|$ can be swapped.

However, from (a), we saw that swapping the order for $f_n(x)$ gave different results. It must then be the case that Fubini does not apply to $f_n(x)$ and so $f_n(x) \notin L^1(m\times\nu)$. Thus,

$$\int |f_n(x)| d(m\times\nu) = \int_0^\infty \sum_{n=1}^\infty |f_n(x)|dx = \infty$$

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Problem 4. Assume that f is integrable on $[0,1]$ with respect to the Lebesgue measure m, and let $F(x) = \int_0^x f(t)dt$. Assume that $\phi : \mathbb{R} \to \mathbb{R}$ is Lipschitz, i.e., there exists a constant $C \geq 0$ such that

$$|\phi(x_1) - \phi(x_2)| \leq C|x_1 - x_2|, \qquad x_1, x_2 \in \mathbb{R}.$$

Prove that there exists a function g which is integrable on $[0,1]$ such that $\phi(F(x)) = \int_0^x g(t)dt$ for $x \in [0,1]$.

Solution. First, since $F : [0,1] \to \mathbb{R}$ and $F(x) - F(0) = F(x) = \int_0^x f(t)dt$ with $f \in L^1([0,1])$, by the Fundamental Theorem of Lebesgue Integrals, F is absolutely continuous.

Furthermore, we may replace f with F' (as the two are equal a.e.).

Now, ϕ is certainly absolutely continuous. If $C = 0$, then ϕ is contant and absolute continuity is immediate. If $C > 0$, then for all $\varepsilon > 0$, letting $\delta = \frac{\varepsilon}{C}$, for all finite disjoint collections of intervals $\{(a_i, b_i)\}_1^n$ satisfying that

$$\sum_{i=1}^n (b_i - a_i) < \delta$$

we have that

$$\sum_{i=1}^n |\phi(b_i) - \phi(a_i)| \leq \sum_{i=1}^n C|b_i - a_i| = C\sum_{i=1}^n (b_i - a_i) < C\delta = C\frac{\varepsilon}{C} = \varepsilon.$$

Thus, ϕ is absolutely continuous. Finally, let $\varepsilon > 0$ be given. Let δ_F and δ_ϕ be the associated constants for the definition of absolute continuity of F and ϕ respectively.

Then let

$$\delta = \min\{\delta_F, \delta_\phi\}.$$

Then, for any finite collection of disjoint intervals $\{(a_i, b_i)\}_1^n$ satisfying

$$\sum_{i=1}^n (b_i - a_i) < \delta,$$

we have that

$$\sum_{i=1}^n |\phi(F(b_i)) - \phi(F(a_i))| \leq \sum_{i=1}^n C|F(b_i) - F(a_i)| = C\sum_{i=1}^n |F(b_i) - F(a_i)| < C\varepsilon.$$

Thus, $\phi(F(x))$ is absolutely continuous and since $\phi(F(x)) : [0,1] \to \mathbb{R}$, by the Fundamental Theorem of Lebesgue Integrals, there must exist a function $g \in L^1([0,1])$ such that

$$\phi(F(x)) - \phi(F(0)) = \phi(F(x)) - \phi(0) = \int_0^x g(t)dt.$$

With possibly shifting g by a constant we obtain our result. ✌