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## Real Analysis Exam Spring 2014

**Problem 1.** Suppose that $(X, \mathcal{B}, \mu)$ is a measure space with $\mu(X) < \infty$, and that $\{f_n\}_{n\geq 1}$ and $f$ are measurable functions on $X$ such that $f_n \to f$ almost everywhere.

(a) Suppose that $\int f^2 d\mu < \infty$. Show that $f$ is integrable.

(b) Suppose that there exists $C < \infty$ such that $\int f_n^2 d\mu \leq C$ for all $n \geq 1$. Show that $f_n \to f$ in $L^1$.

(c) Give an example where $\int |f_n| d\mu \leq 1$ for all $n \geq 1$ but $f_n \not\to f$ in $L^1$.

**Solution.**

(a) Suppose that $\int f^2 d\mu < \infty$. Then let

$$E = \{x \,|\, |f(x)| \geq 1\}.$$

Then we note that if $|f(x)| \geq 1$, $|f(x)| \leq f^2(x)$.

Thus,

$$\int |f(x)| d\mu = \int_E |f(x)| d\mu + \int_{E^c} |f(x)| d\mu \leq \int_E f^2(x) d\mu + \int_{E^c} 1 d\mu < \infty$$

since $f^2 \in L^1$ and since $\mu(E^c) \leq \mu(X) < \infty$.

(b) Suppose that there exists $C < \infty$ such that $\int f_n^2 d\mu \leq C$ for all $n \geq 1$. From (a), $f_n \in L^1$ for all $n$.

Now, we note that if $f_n \to f$ a.e., then $f_n^2 \to f^2$ a.e..

This is immediate since

$$|f_n^2(x) - f^2(x)| = |f_n(x) - f(x)||f_n(x) + f(x)| \to 0 \cdot 2|f(x)|.$$

Therefore,

$$\int |f^2| d\mu = \int \liminf |f_n^2| d\mu \leq \liminf \int |f_n^2| d\mu \leq \liminf C = C.$$

Thus, $f^2 \in L^1$ and so by (a), $f \in L^1$.

We now can apply DCT to $f_n(x) - f(x)$.

- $f_n(x) - f(x)$ is measurable by assumption
- $f_n(x) \to f(x)$ a.e. by assumption so $f_n - f \to 0$ a.e.
- $|f_n(x) - f(x)| \leq 2|f(x)|$ a.e. which is in $L^1$

Finally, by DCT,

$$\lim_{n\to\infty} \int |f_n - f| d\mu = \int \lim_{n\to\infty} |f_n - f| d\mu = \int 0 d\mu = 0$$

and so $f_n \to f$ in $L^1$.

(c) Let $f_n(x) = n\chi_{[0,\frac{1}{n}]}$ with the Lebesgue measure. Then

$$\int |f_n(x)| dm = nm\left(\left[0, \frac{1}{n}\right]\right) = 1 \qquad \text{for all } n.$$

However, $\lim\limits_{n\to\infty} f_n(x) = 0$ a.e. and from the computation above, $\int f_n \to 1 \neq 0$. Thus, $f_n \not\to f$ in $L^1$.

✌

**Problem 2.** For what non-negative integer $n$ and positive real $c$ does the integral

$$\int_1^\infty \ln\left(1+\frac{(\sin x)^n}{x^c}\right)dx$$

(a) exist as a (finite) Lebesgue integral?

(b) converge as an improper Riemann integral?

**Solution.**

(a) First,

$$\ln\left(1-\frac{1}{x^c}\right) \leq \ln\left(1+\frac{(\sin x)^n}{x^c}\right) \leq \ln\left(1+\frac{1}{x^c}\right)$$

for all $n$ since ln is an increasing function and $|\sin x| \leq 1$.

Now, we consider two cases.

$\boxed{c \geq 1}$ then $x \leq x^c$ on $[1,\infty)$ and so $\frac{1}{x} \geq \frac{1}{x^c}$. Therefore, $\ln\left(1+\frac{1}{x^c}\right) \leq \ln\left(1+\frac{1}{x}\right)$.

$$\begin{aligned}\int_1^\infty \ln\left(1+\frac{(\sin x)^n}{x^c}\right)dx &\leq \int_1^\infty \ln\left(\frac{x+1}{x}\right)dx\\ &= \int_1^\infty \ln(x+1) - \ln x dx\\ &= x\ln(x+1) - x - [x\ln x - x]\Big|_1^\infty\\ &= x\ln\left(\frac{x+1}{x}\right)\Big|_1^\infty\\ &= 1-\ln(2) < \infty\end{aligned}$$

Note that

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\frac{-1}{x^2}}{\frac{-1}{x^2}} = 1$$

by L'Hopital's Rule.

Thus, this integral exists for all $n$ and for all $c \geq 1$.

$\boxed{0 < c < 1}$ Now, $x^c \leq x$ for all $x \geq 1$ and so

$$\begin{aligned}\int_1^\infty \ln\left(1+\frac{(\sin x)^n}{x^c}\right)dx &\geq \int_1^\infty \ln\left(\frac{x^c-1}{x^c}\right)dx\\ &= \int_1^\infty \ln(x^c-1) - c\ln x dx\end{aligned}$$

Now,

$$\begin{aligned}
\int_1^\infty \ln(x^c-1)dx &= \int_1^\infty \frac{cx^{c-1}}{cx^{c-1}}\ln(x^c-1)dx \\
&= \int_1^\infty \frac{1}{c}u^{1/c-1}\ln(u-1)du \qquad \begin{matrix} u = x^c & u^{1/c}=x \\ du = cx^{c-1}dx & u^{1-1/c} = x^{c-1}\end{matrix} \\
&\geq \int_1^\infty \frac{1}{c}\ln(u-1)du \qquad \text{since } x^c \leq x \implies 1 \leq x^{1-c} = u^{1/c-1} \\
&= \frac{1}{c}(u\ln(u-1)-u)\Big|_1^\infty \\
&= \frac{1}{c}(x^c\ln(x^c-1)-x^c)\Big|_1^\infty
\end{aligned}$$

Now, since $1-c>1$, there exists $x$ sufficiently large such that

$$\frac{1}{x^{1-c}} \leq c^2 < c \implies \frac{x^c}{c} \leq cx$$

Thus, even after subtracting the $\int_1^\infty c\ln x dx$ we still get

$$\frac{1}{c}(x^c\ln(x^c-1)-x^c) - [cx\ln x - cx]\Big|_1^\infty = \text{ positive ln term } + (cx - \frac{x^c}{c}) \to \infty.$$

Thus, the integral diverges for all $0<c<1$ and all $n$.

(b) For all $c \geq 1$, we showed that the Lebesgue integral existed by bounding a Riemann integrable function. Thus, the two integrals coinside.

For $c<1$, the Riemann integral will not exist by the same computation as for the Lebesgue integral.

✌

**Problem 3.** Suppose $f$ is Lebesgue integrable on $\mathbb{R}$. Show that

$$\lim_{t\to 0}\int_{-\infty}^{\infty} |f(x+t)-f(x)|dx = 0.$$

**Solution.** For all $\varepsilon > 0$, there exists a continuous function $g$ which vanishes outside a bounded interval such that $\int |f-g|dx < \varepsilon$.

Thus,

$$\begin{aligned}\int |f(x+t)-f(x)|dx &= \int |f(x+t)-g(x+t)+g(x)-f(x)+g(x+t)-g(x)|dx \\ &\leq \int |f(x+t)-g(x+t)|dx + \int |g(x)-f(x)|dx + \int |g(x+t)-g(x)|dx \\ &< 2\varepsilon + \int |g(x+t)-g(x)|dx.\end{aligned}$$

Now, since $g$ is continuous and vanishes outside a bounded interval, $g \in L^1$. Thus,

1. $\{g(x+t)\} \in L^1$
2. $g(x+t) \to g(x)$ for all $x$ by continuity.
3. Since $g(x)$ is continuous and non-zero only on some interval $[a,b]$ (which we may take to be closed because we can always extend either end by $\varepsilon$), $g(x)$ is bounded and so $|g(x)| \leq M\chi_{[a,b]}$ for some $M < \infty$.
   Thus, $g(x+t) \leq M\chi[a+t, b+t] \in L^1$.

Therefore, by the Dominated Convergence Theorem, and the calculation above,

$$\lim_{t\to 0}\int |f(x+t)-f(x)|dx < 2\varepsilon + \lim_{t\to 0}\int |g(x+t)-g(x)|dx = 2\varepsilon + \int \lim_{t\to 0}|g(x+t)-g(x)|dx = 2\varepsilon.$$

Since $\varepsilon$ was arbitrary, it must be that $\lim_{t\to 0}\int |f(x+t)-f(x)|dx = 0$. ✌

**Problem 4.** Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces such that $\mu(X) > 0$ and $\nu(Y) > 0$. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be measurable functions (with respect to $\mathcal{A}$ and $\mathcal{B}$ respectively) such that

$$f(x) = g(x) \quad \mu \times \nu \text{ -almost everywhere on } X \times Y.$$

Show that there exists a constant $\lambda$ such that $f(x) = \lambda$ for $\mu$-a.e. $x$ and $g(y) = \lambda$ for $\nu$-a.e.

**Solution.** Let $h(x, y) = f(x) - g(y)$. Then $h = 0$ $\mu \times \nu$-a.e. and so $h \in L^1(\mu \times \nu)$. It is clear that $h$ is measurable since $h(x, y) = f \circ \pi_x - g \circ \pi_y$ with $\pi_x(x, y) = x$ and $\pi_y(x, y) = y$ which is a composition of measurable functions in $\mu \times \nu$.

Now, let $X'$ and $Y'$ be any $\sigma$-finite subsets of $X$ and $Y$ respectively.

On these subsets, we may apply Tonelli's Theorem and so

$$\begin{aligned} 0 &= \int |h| d(\mu \times \nu) \\ &= \int \int |f - g| d\mu d\nu \\ \implies &\int |f - g| d\mu = 0 \ \ \nu\text{-a.e.} \end{aligned}$$

$$|f(x) - g(y)| = 0 \ \ \mu\text{-a.e.}$$

However, then $f(x) = g(y)$ $\mu$-a.e. and since $g(y)$ is a constant with respect to $\mu$, this implies that $f(x) = \lambda = g(y_0)$ some fixed $y_0 \in Y'$ $\mu$-a.e. on $X'$.

Similarly, applying Tonelli again,

$$\begin{aligned} 0 &= \int |h| d(\mu \times \nu) \\ &= \int \int |\lambda - g| d\nu d\mu \\ \implies &\int |\lambda - g| d\nu = 0 \ \ \mu\text{-a.e.} \end{aligned}$$

$$|\lambda - g(y)| = 0 \ \ \nu\text{-a.e.}$$

so $g(y) = \lambda$ $\nu$-a.e. on $Y'$.

Now, since $h \in L^1$, $\{(x, y) \,|\, h(x, y) \neq 0\}$ is $\sigma$-finite and is null with respect to $\mu \times \nu$.

Thus, if

$$E = \{x \,|\, f(x) = \lambda\} \qquad F = \{y \,|\, g(y) = \lambda\}$$

then

$$(\mu \times \nu)(E^c \times F) = \mu(E^c)\nu(F) = 0 \qquad \text{and} \qquad (\mu \times \nu)(E \times F^c) = \mu(E)\nu(F^c) = 0$$

since if $f(x) = \lambda \neq g(y) \implies h(x, y) \neq 0$.

However, $E^c \times F$ is a subset of a $\sigma$-finite set and so it is $\sigma$-finite and since we have already showed that $f(x) = \lambda$ $\mu$-a.e. on all $\sigma$-finite sets, $E^c$ must be $\mu$-null. Then, since $\mu(X) > 0$, $\mu(E) > 0$ and so $\nu(F^c) = 0$.

Finally, this shows that $f(x) = \lambda = g(y)$ $\mu$-a.e. and $\nu$-a.e. ✌