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Real Analysis Exam Fall 2014

Problem 1. Assume that f is integrable on $(0,1)$. Prove that

$$\lim_{a\to\infty}\int_0^1 f(x)x\sin(ax^2)dx = 0.$$

Solution. First, let $f(x)=\chi_E(x)$ for some measurable set $E\subset[0,1]$. Then

$$\begin{aligned}\int_0^1 \chi_E(x)x\sin(ax^2)dx &= \int_E x\sin(ax^2)dx\\ &\le \int_0^1 x\sin(ax^2)dx\\ &= \int_0^a \frac{\sin(u)}{2a}du \qquad \begin{matrix} u=ax^2 & x:[0,1]\\ du=2axdx & u:[0,a]\end{matrix} \qquad = \left.\frac{-\cos(u)}{2a}\right|_0^a\\ &= \frac{-\cos(a)}{2a}+\frac{1}{2a}\to 0 \qquad \text{as } a\to\infty.\end{aligned}$$

Therefore, by linearity of the integral, the statement holds for simple functions.

Now, since $f\in L^1$, there exists some ϕ simple function such that for all $\varepsilon>0$,

$$\int_0^1 |f-\phi|dx<\varepsilon.$$

Thus,

$$\begin{aligned}\left|\int_0^1 f(x)x\sin(ax^2)dx-\int_0^1\phi(x)x\sin(ax^2)dx\right| &\le \int_0^1 |(f(x)-\phi(x))x\sin(ax^2)|dx\\ &\le \int_0^1 |f(x)-\phi(x)|dx<\varepsilon\end{aligned}$$

since $x\sin(ax^2)\le 1$ for all $x\in(0,1)$ and all a.

Thus, since we already showed that

$$\int_0^1 \phi(x)x\sin(ax^2)dx\to 0$$

we are done. ✌

Problem 2. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f and $f_1, f_2, f_3, ...$ be real valued measurable functions on X. If $f_n \to f$ in measure and if $F : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, prove that $F \circ f_n \to F \circ f$ in measure.

Solution. Let $\varepsilon > 0$ be given. Then, since F is uniformly continuous, there exists a $\delta > 0$ such that whenever $|x - y| < \delta$, $|F(x) - F(y)| < \varepsilon$.

Let

$$E = \{x \,|\, |f_n(x) - f(x)| \geq \delta\}$$
$$F = \{x \,|\, |(F \circ f_n)(x) - (F \circ f)(x)| \geq \varepsilon\}.$$

Now, we note that if $x \in E^c$, then $|f_n(x) - f(x)| < \delta$ and so $|F(f_n(x)) - F(f(x))| < \varepsilon$ which implies that $x \in F^c$.

Thus, $E^c \subset F^c$ and so $F \subset E$.

Finally, this implies that since $\mu(E) \to 0$ as $n \to \infty$ (since $f_n \to f$ in measure, then $\mu(F) \to 0$ as $n \to \infty$ as well. ✌

Problem 3. Let f_n be **nonnegative** measurable functions on a measure space $(X, \mathcal{M}, \mu)$ which satisfy $\int f_n d\mu = 1$ for all $n = 1, 2, ...$ Prove that

$$\limsup_n (f_n(x))^{1/n} \leq 1$$

for μ-a.e. x.

Solution. First, for all n, let

$$E_n = \{x \mid f_n(x) > n\}.$$

Then

$$\begin{aligned} 1 &= \int f_n d\mu \\ &= \int_{E_n} f_n d\mu + \int_{E_n^c} f_n d\mu \\ &\geq \int_{E_n} n d\mu + \int_{E_n^c} f_n d\mu \\ &\geq n\mu(E_n) + 0 \qquad \text{since } f_n \geq 0 \\ \implies \mu(E_n) &\leq \frac{1}{n} \end{aligned}$$

Thus, except on a set of shrinking measure, $f_n(x) \leq n$, and so (again except on a set of shrinking measure) $f_n(x)^{1/n} \leq n^{1/n}$.

Finally, since

$$\begin{aligned} y &= \lim_{n\to\infty} n^{1/n} \\ \implies \ln y &= \lim_{n\to\infty} \frac{\ln n}{n} \\ &= \lim_{n\to\infty} \frac{1}{n} \\ &= 0 \\ \implies y &= e^0 = 1 \end{aligned}$$

we have that

$$\limsup_n (f_n(x))^{1/n} \leq \limsup_n (n^{1/n}) = \lim_{n\to\infty} n^{1/n} = 1.$$

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Problem 4. Let $-\infty < a < b < \infty$. Suppose $F : [a, b] \to \mathbb{C}$.

(a) Define what it means for F to be absolutely continuous on $[a, b]$.

(b) Give an example of a function which is uniformly continuous but not absolutely continuous. (Remember to justify your answer.)

(c) Prove that if there exists M such that $|F(x) - F(y)| \leq M|x - y|$ for all $x, y \in [a, b]$, then F is absolutely continuous. Is the converse true? (Again, remember to justify your answer).

Solution.

(a)

$F(x)$ is absolutely continuous on [a,b] if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for any finite collection $\{(a_i, b_i)\}_1^n$ of disjoint subintervals of $[a, b]$ satisfying $\sum_{i=1}^{n} |b_i - a_i| < \delta$ implies that $\sum_{i=1}^{n} |F(b_i) - F(a_i)| < \varepsilon$

(b) Let $f(x)$ be the Cantor Function on $[0, 1]$. Since $f(x)$ is continuous on a closed interval, it is uniformly continuous.

Now, assume that $f(x)$ is absolutely continuous. We already know that $f'(x) = 0$ a.e. since it is only non-constant on the Cantor set (which has Lebesgue Measure 0).

Thus, if $f(x)$ is absolutely continuous on $[0, 1]$, then by the Fundamental Theorem of Lebesgue Integrals,

$$1 = f(1) - f(0) = \int_0^1 f'(x)dx = \int_0^1 0dx = 0.$$

Thus, $f(x)$ cannot be absolutely continuous on $[0, 1]$.

(c) Clearly if such an M exists, it must be nonnegative. If $M = 0$, then F is constant on $[a, b]$ and so it is clearly absolutely continuous.

If $M > 0$, then for all $\varepsilon > 0$, we can let $\delta = \frac{\varepsilon}{M}$. Then, for any finite collection of disjoint intervals $\{(a_i, b_i)\}_1^n$ satisfying

$$\sum_{i=1}^{n} (b_i - a_i) < \delta,$$

we have that

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| \leq \sum_{i=1}^{n} M|b_i - a_i| = M \sum_{i=1}^{n} |b_i - a_i| < M\delta = M\frac{\varepsilon}{M} = \varepsilon.$$

Thus, F is absolutely continuous.

Now, let $F(x) = \sqrt{x}$ on $[0, 1]$. Then $F'(x) = \frac{1}{2\sqrt{x}}$ exists except at 0 (a null set) and since

$$\lim_{b\to 0} \int_b^1 \frac{1}{2\sqrt{x}} dx = \lim_{b\to 0} \sqrt{x}\Big|_b^1 = \lim_{b\to 0}(1 - \sqrt{b}) = 1$$

so $F'(x) \in L^1$ and by the same computation

$$F(x) - F(0) = \int_0^x F'(x)dx.$$

Thus, again by the Fundamental Theorem of Lebesgue Integrals, F is absolutely continuous on $[0, 1]$.

However, for all $y > x$, we have

$$|F(y) - F(x)| = \sqrt{y} - \sqrt{x} = \frac{y - x}{\sqrt{y} + \sqrt{x}}.$$

If there exists some $\infty > M > 0$ such that $\frac{y-x}{\sqrt{y}+\sqrt{x}} \leq M(y - x)$, then $M \geq \frac{1}{\sqrt{y}+\sqrt{x}}$. However, for x, y very small near 0, we can force M to grow as large as we like and so no such finite M exists.

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