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Real Analysis Exam Spring 2013

Problem 1. Suppose that $\{f_n\}$ is a sequence of real valued continuously differentiable functions on $[0,1]$ such that

$$\lim_{n\to\infty}\int_0^1 |f_n(x)|dx = 0 \text{ and } \lim_{n\to\infty}\int_0^1 |f_n'(x)|dx = 0.$$

Show that $\{f_n\}$ converges to 0 uniformly on $[0,1]$.

Solution. Since $f'(x)$ exists and is continuous on $[0,1]$,

$$0 = \lim_{n\to\infty}\int_0^1 |f_n'(x)|dx \geq \lim_{n\to\infty}\left|\int_0^1 f_n'(x)dx\right| = \lim_{n\to\infty}|f_n(1) - f_n(0)|.$$

Similarly, since $|f_n(x)| \geq 0$ for all x, we have that $|f_n(b) - f_n(a)| \to 0$ for all $(a,b) \subset [0,1]$ since

$$\int_0^1 |f_n(x)|dx \geq \int_a^b |f_n(x)|dx \qquad 0 \leq a < b \leq 1.$$

Thus, $f_n(x) \to c$ for some constant as $n \to \infty$. Now, we'd like to use the Dominated Convergence Theorem.

1. $\{f_n\} \in L^1$ since each f_n is continuous on $[0,1]$, it is bounded so $|f(x)| \leq M < \infty$ on $[0,1]$.
2. $f_n \to c$ for all x.
3. $|f_n(x)| \leq \sup_n M_n < \infty$ with M_n the upper bound of f_n on $[0,1]$. If $\sup_n M_n = \infty$ then the M_n grow arbitrarily large which contradicts the continuity of f_n on $[0,1]$.

Thus,

$$0 = \lim_{n\to\infty}\int_0^1 |f_n(x)|dx = \int_0^1 \lim_{n\to\infty}|f_n(x)|dx = \int_0^1 cdx = c.$$

Thus, $f_n \to 0$ for all x.

Now, letting

$$M_n = \sup_{x\in[0,1]}|f_n(x)|,$$

then there exists some $x \in [0,1]$ such that $|f_n(x)| \geq M_n - \varepsilon$ and so

$$\lim_{n\to\infty}|f_n(x)| \geq \lim_{n\to\infty} M_n - \varepsilon \quad\Longrightarrow\quad \varepsilon \geq \lim_{n\to\infty} M_n.$$

Thus, $M_n \to 0$ as $n \to \infty$.

Finally, for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that $M_n < \varepsilon$ for all $n > N$ and so, for all $n > N$,

$$|f_n(x)| \leq M_n < \varepsilon.$$

Thus, $f \to 0$ uniformly on $[0, 1]$. ✌

Problem 2. Investigate the convergence of $\sum_{n=0}^{\infty} a_n$ where

$$a_n = \int_0^1 \frac{x^n}{1-x}\sin(\pi x)dx.$$

Solution. First, $\sin(\pi x) \geq 0$ for all $0 \leq x \leq 1$. Now, using a quick sketch we see that $y = -2(x-1)$ seems to be below $\sin(\pi x)$. A quick check shows that

$$\frac{d}{dx}(\sin(\pi x) + 2(x-1)) = \pi\cos(\pi x) + 2 \qquad \text{changes sign once for } \frac{1}{2} \leq x \leq 1$$

which is verified since

$$\frac{d}{dx}(\pi\cos(\pi x) + 2) = -\pi^2\sin(\pi x) \leq 0 \qquad \frac{1}{2} \leq x \leq 1.$$

Thus, since

$$\sin(\pi x) + 2x - 2 = 0 \qquad \text{when } x = \frac{1}{2}, 1$$

and

$$\frac{\sqrt{2}}{2} + \frac{3}{2} - 2 \geq 0 \qquad \text{for } x = \frac{3}{4}$$

so for all $\frac{1}{2} \leq x \leq 1$. Thus, $\sin(\pi x) \geq -2(x-1)$.

Let $f_n(x) = \frac{x^n}{1-x}\sin(\pi x)$. Let $(\mathbb{N}, \nu)$ be the counting measure. Then, $([0,1], m)$ and $(\mathbb{N}, \nu)$ are σ-finite and $f_n(x) \in L^+(m \times \nu)$. Then by Tonelli, we can swap the order of integration, so

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n &= \int_0^1 \sum_{n=0}^{\infty} \frac{x^n}{1-x}\sin(\pi x)dx \\
&= \int_0^1 \sum_{n=0}^{\infty} \frac{x^n}{1-x}\sin(\pi x)dx \\
&= \int_0^1 \frac{\sum_{n=0}^{\infty} x^n}{1-x}\sin(\pi x)dx \\
&= \int_0^1 \frac{1}{(1-x)^2}\sin(\pi x)dx \qquad \text{since } 0 \leq x \leq 1 \implies \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \\
&\geq \int_{1/2}^1 \frac{-1}{(1-x)^2}2(x-1)dx \\
&= \int_{1/2}^1 \frac{1}{(1-x)^2}2(1-x)dx \\
&= \int_{1/2}^1 \frac{1}{(1-x)}2dx \\
&= -\ln|1-x|\Big|_{1/2}^1 = \infty
\end{aligned}$$

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Problem 3. Let $(X, \mathcal{M}, \mu)$ be a measure space, $f_n, f \in L^1(\mu)$. Show that $\int_X |f_n - f| d\mu \to 0$ as $n \to \infty$ if and only if

$$\sup_{A \in \mathcal{M}} \left| \int_A f_n d\mu - \int_A f d\mu \right| \to 0$$

as $n \to \infty$.

Solution. $\boxed{\Longrightarrow}$

$$\begin{aligned}
0 &= \lim_{n\to\infty} \int |f_n(x) - f(x)| d\mu \\
&\geq \lim_{n\to\infty} \sup_{A \in \mathcal{M}} \int_A |f_n(x) - f(x)| d\mu \qquad \text{since } \int_X |f_n - f| \geq \int_A |f_n - f| \text{ for all } A \subset X \\
&\geq \lim_{n\to\infty} \sup_{A \in \mathcal{M}} \left| \int_A f_n(x) - f(x) d\mu \right| \qquad |f_n - f| \in L^1 \text{ for sufficiently large } n \\
&= \lim_{n\to\infty} \sup_{A \in \mathcal{M}} \left| \int_A f_n(x) d\mu - \int_A f(x) d\mu \right|
\end{aligned}$$

Thus, since we are taking the sup of positive values, the sup must then tend to 0.

$\boxed{\Longleftarrow}$ Let $g_n(x) = f_n(x) - f(x)$, then g_n is measurable since f and f_n are and so

$$A = \{x \,|\, g_n(x) \geq 0\} = g^{-1}([0, \infty)) \in \mathcal{M}$$

and similarly,

$$A^c = \{x \,|\, g_n(x) < 0\} = g^{-1}((-\infty, 0)) \in \mathcal{M}.$$

Then,

$$\begin{aligned}
0 &= \lim_{n\to\infty} \sup_{E \in \mathcal{M}} \left| \int_E f_n(x) d\mu - \int_E f(x) d\mu \right| \\
&= \lim_{n\to\infty} \sup_{E \in \mathcal{M}} \left| \int_E f_n(x) - f(x) d\mu \right| \\
&\geq \lim_{n\to\infty} \left| \int_A f_n(x) - f(x) d\mu \right| \\
&= \lim_{n\to\infty} \int_A f_n(x) - f(x) d\mu \qquad \text{since } f_n - f \geq 0 \text{ on } A
\end{aligned}$$

Similarly,

$$0 \geq \lim_{n\to\infty} \left| \int_{A^c} f_n(x) - f(x) d\mu \right|$$

so

$$\lim_{n\to\infty} \int_X |f_n(x) - f(x)| d\mu = \lim_{n\to\infty} \left[\int_A f_n(x) - f(x) d\mu - \int_{A^c} f_n(x) - f(x) d\mu \right] = 0 - 0 = 0.$$

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Problem 4 (Similar to Folland, 3.2.16, p.92). Let μ and ν be σ-finite positive measures, $\mu \geq \nu$ and assume that $\nu << \mu - \nu$ (ν is absolutely continuous with respect to $\mu - \nu$).

Prove that
$$\mu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = 0.$$

Solution. We make several observations. Note, that in all facts used below, σ-finiteness as well as positivity of the measures is necessary.

1. From $\nu << \mu - \nu$, $(\mu - \nu)(E) = 0 \implies \nu(E) = 0 \implies \mu(E) - \nu(E) = 0 \implies \mu(E) = \nu(E) = 0$. Thus, $\mu << \nu$.
2. Since $\mu \geq \nu$, $\nu << \mu$.

Now, we claim that $\mu = \nu$ only on null sets.

Claim 1. Since $\mu << \nu$ and $\nu << \mu$ and $\mu \geq \nu$, $\mu(E) = \nu(E)$ if and only if $\mu(E) = 0$.

Proof. $\boxed{\Longleftarrow}$ Clearly if $\mu(E) = 0$, then $\mu(E) = \nu(E) = 0$ since $\mu << \nu$ and $\nu << \mu$.

$\boxed{\Longrightarrow}$ Assume $\mu(E) \neq 0$. Then $\nu(E) \neq 0$, else, if $\nu(E) = 0$ then $\mu(E) = 0$ since $\mu << \nu$.

Now, if $\mu(E) = \nu(E)$ then $\mu(E) - \nu(E) = 0$ and so $\nu(E) = 0$ since $\nu << \mu - \nu$. However, this is a contradiction.

Thus, $\mu(E) \neq \nu(E)$.

Namely, μ and ν agree only on null sets. ✌

Now, let $f = \frac{d\nu}{d\mu}$. Then $\frac{1}{f} = \frac{d\mu}{d\nu}$ clearly. We will use the fact that $\mu(E) = \int_E f d\nu$.

$$\begin{aligned}
\mu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) &= \mu\left(\{f = 1\}\right) \\
&= \mu\left(\left\{\frac{1}{f} = 1\right\}\right) \\
&= \mu\left(\left\{\frac{d\mu}{d\nu} = 1\right\}\right) \\
&= \int_{\left\{\frac{d\mu}{d\nu} = 1\right\}} f d\nu \\
&= \int_{\left\{\frac{d\mu}{d\nu} = 1\right\}} f \frac{d\mu}{d\nu} d\nu \\
&= \int_{\left\{\frac{d\mu}{d\nu} = 1\right\}} f \frac{1}{f} d\nu \\
&= \int_{\left\{\frac{d\mu}{d\nu} = 1\right\}} d\nu \\
&= \nu\left(\left\{\frac{d\mu}{d\nu} = 1\right\}\right) \\
&= \nu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right)
\end{aligned}$$

From the claim, since μ and ν agree on $\left\{\frac{d\nu}{d\mu} = 1\right\}$, it must be a null set for both and so

$$\mu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = 0.$$

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