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Real Analysis Exam Fall 2013

Problem 1. Let μ be a finite Borel Measure on $\mathbb{R}$, which is absolutely continuous with respect to the Lebesgue measure m. Prove that $x \mapsto \mu(A+x)$ is continuous for every Borel set $A \subset \mathbb{R}$.

Solution. First, we will denote $A+x = A_x$. Now, because A is Borel and $\mu(A) < \infty$, for all $\varepsilon > 0$ there exists some set E which is a finite union of open intervals such that $\mu(A\Delta E) < \varepsilon$.

Now, because $\mu << m$, and

$$\begin{aligned} m(A_x \Delta E_x) &= m(A_x \cup E_x) - m(A_x \cap E_x) \\ &= m((A\cup E)_x) - m((A\cap E)_x) \\ &= m(A \cup E)) - m(A \cap E) \\ &= m(A\Delta E) \end{aligned}$$

we have that $\mu(A_x\Delta E_x) < \varepsilon$ for all x (using the fact that we can ensure E satisfies that $m(A_x\Delta E_x) < \delta$ and so $\mu(A_x \Delta E_x) < \varepsilon$ by absolute continuity).

Thus, since $|\mu(A_x) - \mu(E_x)| < \varepsilon$ for all x, it suffices to show that

$$\begin{aligned} f :&\mathbb{R} \to \mathbb{R} \\ &x \mapsto \mu(E_x) \end{aligned}$$

is continuous.

WLOG, let $E = \bigcup_{i=1}^N (a_i, b_i)$.

Let $\varepsilon > 0$ be given and x be fixed. Then let δ be such that

$$m(F) < \delta \implies \mu(F) < \varepsilon$$

by absolute continuity.

Then, whenever $|x-y| < \frac{\delta}{N}$, we have that

$$\begin{aligned} |f(x) - f(y)| &= |\mu(E_x) - \mu(E_y)| \\ &= |\mu(E_x) - (\mu(E_y \backslash E_x) + \mu(E_y \cap E_x))| \\ &= |\mu(E_x) - \mu(E_x \cap E_y) - \mu(E_y\backslash E_x))| \\ &= |\mu(E_x\backslash E_y) - \mu(E_y \backslash E_x)| \\ &\le \mu(E_x \backslash E_y). \end{aligned}$$

Now, since for each interval of $E_x \backslash E_y$ we have $(a_i + x, b_i + x) \backslash (a_i + y, b_i + y) = I_{x \backslash y_i}$ has length at most $\frac{\delta}{N}$, so

$$m(E_x \backslash E_y) \leq \sum_{i=1}^{N} m(I_{x \backslash y_i}) < N \frac{\delta}{N} = \delta$$

so $\mu(E_x \backslash E_y) < \varepsilon$.

Thus, f is continuous. ✌

Problem 2. Let f be a Lebesgue integrable function on $\mathbb{R}$, and assume that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty.$$

Prove that $g(x) = \sum_{n=1}^{\infty} f(a_n x)$ converges almost everywhere and is integrable on $\mathbb{R}$. Also, find an example of a Lebesgue integrable function f on $\mathbb{R}$ such that $g(x) = \sum_{n=1}^{\infty} f(nx)$ converges almost everywhere but is not integrable.

Solution. Let $(\mathbb{N}, \nu)$ be the counting measure. Then since the σ-algebra of ν is the powerset of $\mathbb{N}$, f is $m \times \nu$ measurable.

Let $M = \int |f(x)| dx$.

Furthermore, since m and ν are σ-finite measure spaces, and $|f(a_n x)| \in L^+(m \times \nu)$, by Tonelli,

$$\begin{aligned}
\int |g(x)| dx &= \int \left| \sum_{n=1}^{\infty} f(a_n x) \right| dx \\
&\leq \int \sum_{n=1}^{\infty} |f(a_n x)| dx \\
&= \sum_{n=1}^{\infty} \int |f(a_n x)| dx \\
&= \sum_{n=1}^{\infty} \int \frac{|f(u)|}{a_n} du \qquad \begin{array}{c} u = a_n x \\ du = a_n dx \end{array} \\
&= \sum_{n=1}^{\infty} \frac{M}{a_n} < \infty
\end{aligned}$$

since

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \leq \sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty.$$

Now, let $f(x) = \chi_{[0,1]}(x)$. Then $f \in L^1$. Furthermore,

$$f(nx) = \chi_{[0,1]}(nx) = \chi_{[0,\frac{1}{n}]}(x) \qquad \text{since } 0 \leq nx \leq 1 \implies 0 \leq x \leq \frac{1}{n}.$$

$$\sum_{n=1}^{\infty} \chi_{[0,\frac{1}{n}]}(x) < \infty$$

for all $x \neq 0$ since for all $x \in [0, \frac{1}{n}]$, there exists some N such that $\frac{1}{N} < x$ so $\sum_{n=N}^{\infty} f(nx) = 0$. Thus, $g(x)$ converges a.e..

Now, since $f(nx) \in L^+(m \times \nu)$ by the same reasoning as before,

$$\int \sum_{n=1}^{\infty} \chi_{[0,\frac{1}{n}]}(x) dx = \sum_{n=1}^{\infty} \int \chi_{[0,\frac{1}{n}]}(x) dx = \sum_{n=1}^{\infty} m([0, \frac{1}{n}]) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

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Problem 3. Assume $b > 0$. Show that the Lebesgue Integral

$$\int_1^\infty x^{-b} e^{\sin x} \sin(2x) dx$$

exists if and only if $b > 1$.

Solution. $\boxed{b > 1}$

$$\int_1^\infty x^{-b} e^{\sin x} \sin(2x) dx \leq \int_1^\infty e x^{-b} dx < \infty$$

by p-test and since $|\sin a| \leq 1$ for all a.

$\boxed{0 < b \leq 1}$

$$\int_1^\infty x^{-b} e^{\sin x} \sin(2x) dx \geq \sum_{n=1}^\infty \int_{n\pi/12}^{(4n+1)\pi/12} e^{-1} x^{-b} \frac{1}{2} dx = \infty.$$

In other words, we restrict the domain to where $\sin(2x) \geq \frac{1}{2}$ and this integral is still infinite. This implies that $\int f^+ dx = \infty$. Similarly, restricting to where $\sin(2x) \leq -\frac{1}{2}$ gives that $\int f^- dx = \infty$.

Thus, the integral does not exist. ✌

Problem 4. Suppose that F is the distribution function of a Borel measure μ on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$. Prove that

$$\int_{-\infty}^{\infty} (F(x+a) - F(x))dx = a$$

for all $a > 0$.

Solution. First, since $x + a > x$ because $a > 0$

$$\int_{\mathbb{R}} (F(x+a) - F(x))dx = \int_{\mathbb{R}} \mu([x, x+a))dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{[x,x+a)} d\mu dm$$

Now, since μ is finite and m is σ-finite, and $\chi_{[x,x+a)} \in L^+(m \times \mu)$ because μ is a Borel measure, the order of integration can be swapped and so we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{[x,x+a)} d\mu dm = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{[x,x+a)} dm d\mu = \int_{\mathbb{R}} a d\mu = a\mu(\mathbb{R}) = a.$$

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