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## Real Analysis Exam Spring 2012

Note! There are a multitude of typos in the questions and hints for this exam (mainly in questions 3 and 4). To stay true to the exam, the typos in the question statements have not been rectified. My solutions are to what I perceived each question to mean.

**Problem 1.** Let $f$ and $g$ be real integrable functions on a $\sigma$-finite measure space $(X, \mathcal{M}, \mu)$, and for $t \in \mathbb{R}$ let

$$F_t = \{x \in E \mid f(x) > t\} \quad \text{and} \quad G_t = \{x \in E \mid g(x) > t\}.$$

Show that

$$\int_X |f - g| d\mu = \int_{-\infty}^{\infty} \mu((F_t \backslash G_t) \cup (G_t \backslash F_t)) dt.$$

**Solution.** Since $f, g$ are integrable, they are measurable and so $F_t$ and $G_t$ are $\mu$-measurable (because $F_t = f^{-1}(t, \infty)$ and $G_t = g^{-1}(t, \infty)$).

Thus, $\chi_{F_t \Delta G_t} \in L^+(\mu \times m)$ and since $m$ and $\mu$ are $\sigma$-finite, by Tonelli,

$$\begin{aligned}
\int_{\mathbb{R}} \mu(F_t \Delta G_t) dt &= \int_{\mathbb{R}} \int_X \chi_{F_t \Delta G_t} d\mu dt \\
&= \int_X \int_{\mathbb{R}} \chi_{F_t \Delta G_t} dt d\mu \\
&= \int_X \int_{\mathbb{R}} \chi_{F_t \backslash G_t} + \chi_{G_t \backslash F_t} dt d\mu \\
&= \int_X \left[ \int_{g(x)}^{f(x)} dt + \int_{f(x)}^{g(x)} dt \right] d\mu \qquad \text{on } F_t \cap G_t^c, g(t) \leq t < f(t) \\
&= \int_{\{x \mid f \geq g\}} f(x) - g(x) d\mu + \int_{\{x \mid f \leq g\}} g(x) - f(x) d\mu \\
&= \int_X |f(x) - g(x)| d\mu
\end{aligned}$$

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**Problem 2.** Show that
$$\int_{\pi}^{\infty} \frac{dx}{x^2(\sin^2 x)^{1/3}}$$
is finite.

**Solution.** Let $\varepsilon > 0$. Then since
$$\sin^{2/3}(k\pi + \varepsilon) = \sin^{2/3}(\varepsilon) \geq \varepsilon,$$
(and similarly for $\sin^{2/3}(k\pi - \varepsilon) = \sin^{2/3}(\varepsilon)$ since we are squaring sin which is an odd function) we have that
$$\frac{1}{\sin^{2/3} x} \leq \frac{1}{\varepsilon}$$
near $k\pi$. This is easily verified since
$$\frac{d}{dx}(\sin^{2/3} x - x) = \frac{2}{3}\sin^{-1/3} x \cos x - 1 = \frac{2\cos x}{3\sin^{1/3} x} \geq 0 \qquad \text{near } 0^+$$
and since
$$\sin^{2/3} x - x = 0 \qquad \text{at } x = 0$$
we have that $\sin^{2/3} x - x$ is increasing and positive near $0^+$ so in that region, $\sin^{2/3} x \geq x$.

Now, since
$$\int_{\pi}^{\infty} \frac{1}{x^2 \sin^{2/3} x} dx = \sum_{k=1}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{1}{x^2 \sin^{2/3} x} dx$$
it suffices to show that the integral near any $k\pi$ is small.

However, using the above, we have that
$$\begin{aligned}
\int_{k\pi-\varepsilon}^{k\pi+\varepsilon} \frac{1}{x^2 \sin^{2/3} x} dx &\leq \int_{k\pi-\varepsilon}^{k\pi+\varepsilon} \frac{1}{x^2 \varepsilon} dx \\
&= \left.\frac{-1}{x\varepsilon}\right|_{k\pi-\varepsilon}^{k\pi+\varepsilon} \\
&= \frac{1}{\varepsilon}\left(\frac{-1}{k\pi+\varepsilon} + \frac{1}{k\pi-\varepsilon}\right) \\
&= \frac{1}{\varepsilon}\left(\frac{-(k\pi-\varepsilon)}{(k\pi-\varepsilon)(k\pi+\varepsilon)} + \frac{k\pi+\varepsilon}{(k\pi+\varepsilon)(k\pi-\varepsilon)}\right) \\
&= \frac{1}{\varepsilon}\left(\frac{2\varepsilon}{k^2\pi^2 - \varepsilon^2}\right) \\
&= \frac{2}{k^2\pi^2 - \varepsilon^2}
\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\pi}^{\infty} \frac{1}{x^2 \sin^{2/3} x} dx &= \sum_{k=1}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{1}{x^2 \sin^{2/3} x} dx \\ &= \sum_{k=1}^{\infty} \int_{k\pi+\varepsilon}^{(k+1)\pi-\varepsilon} \frac{1}{x^2 \sin^{2/3} x} dx + \sum_{k=1}^{\infty} \int_{k\pi-\varepsilon}^{k\pi+\varepsilon} \frac{1}{x^2 \sin^{2/3} x} dx \\ &\leq \int_{\pi}^{\infty} \frac{1}{x^2 \varepsilon} dx + \sum_{k=1}^{\infty} \frac{2}{k^2\pi^2 - \varepsilon^2} < \infty\end{aligned}$$

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**Problem 3.** A collection of functions $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1$ on the measure space $(X, \mathcal{M}, \mu)$ is said to be *uniformly integrable* if

$$\lim_{M\to\infty} \sup_{\alpha\in\mathcal{A}} \int_{\{x\,|\,|f_\alpha(x)>M\}} |f_\alpha| = 0.$$

(a) Prove that if $f \in L^1$ then $\{f\}$ is uniformly integrable.

(b) Prove that if $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{f_\beta\}_{\beta\in\mathcal{B}}$ are two collections of uniformly integrable functions then $\{f_\gamma\}_{\gamma\in\mathcal{A}\cup\mathcal{B}}$ is uniformly integrable.

(c) Show that if $\mu(X) < \infty$, and $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1$ is uniformly integrable then

$$\sup_{\alpha\in\mathcal{A}} \int |f| d\mu < \infty.$$

Give an example to show that the conclusion fails without the condition $\mu(X) < \infty$.

(d) Again, let $\mu(X) < \infty$ and suppose $\{f_n\}_{n=0}^\infty \subset L^1(\mu)$ such that $f_n \to f_0$ a.e. and $\int |f_n| d\mu \to \int |f_0| d\mu$. Prove that $\{f_n\}_{n=0}^\infty$ is uniformly integrable. Hint: Consider some $\phi_M$, a continuous bounded function on $[0, \infty)$, equal to 0 on $[M, \infty)$, for which $|t|\mathbf{1}\{|t| > M\} \le |t| - \phi_M(|t|)$.

**Solution.**

(a) Since $f \in L^1$, $\{x\,|\,f(x) = \infty\}$ is $\mu$-null and so $\mu(\{x\,|\,|f(x)| > M\}) \to 0$ as $M \to \infty$. Thus, if $\int |f| d\mu = N$

$$\lim_{M\to\infty} \int_{\{x\,|\,f(x)>M\}} |f(x)| d\mu \le \lim_{M\to\infty} N\mu(\{x\,|\,|f(x)| > M\}) = 0.$$

(b)

$$\begin{aligned}
\lim_{M\to\infty} \sup_{\alpha\in\mathcal{A}\cup\mathcal{B}} \int_{\{x\,|\,f_\alpha(x)>M\}} |f_\alpha| &= \lim_{M\to\infty} \max\left\{ \sup_{\alpha\in\mathcal{A}} \int_{\{x\,|\,f_\alpha(x)>M\}} |f_\alpha|, \sup_{\beta\in\mathcal{B}} \int_{\{x\,|\,f_\beta(x)>M\}} |f_\beta| \right\} \\
&= \lim_{M\to\infty} \sup_{\alpha\in\mathcal{A}} \int_{\{x\,|\,f_\alpha(x)>M\}} |f_\alpha| \qquad \text{WLOG take } \sup_{\alpha\in\mathcal{A}} \ge \sup_{\beta\in\mathcal{B}} \\
&= 0 \qquad \text{since } \{f_\alpha\} \text{ are uniformly integrable.}
\end{aligned}$$

(c)

$$\begin{aligned}
\sup_{\alpha\in\mathcal{A}} \int |f_\alpha| d\mu &= \sup_{\alpha\in\mathcal{A}} \left[ \int_{\{x\,|\,f_\alpha(x)>M\}} |f_\alpha| d\mu + \int_{\{x\,|\,f_\alpha(x)\le M\}} |f_\alpha| d\mu \right] \\
&\le \sup_{\alpha\in\mathcal{A}} \left[ \int_{\{x\,|\,f_\alpha(x)>M\}} |f_\alpha| d\mu + M\mu(X) \right] < \infty
\end{aligned}$$

Now, let $f_n(x) = \frac{1}{nx}$ on $[0, \infty)$ with the Lebesgue measure. Then $\{x \mid f_n(x) > M\}$ is $m$-null for $M \geq 1$ so

$$\lim_{M\to\infty} \sup_{n\in\mathbb{N}} \int_{\{x \mid f_n(x) > M\}} |f_n| dm = 0$$

but $\frac{1}{nx}$ diverges as an integral for all $n$, so

$$\sup_{n\in\mathbb{N}} \int_1^\infty \frac{1}{nx} dx = \infty.$$

(d) From (c),

$$\infty > \sup_{n\in\mathbb{N}} \int |f_n| d\mu \geq \limsup_{n\in\mathbb{N}} \int |f_n| d\mu = \lim_{n\to\infty} \int |f_n| d\mu = \int |f_0| d\mu.$$

So $f_0 \in L^1(\mu)$.

Again using (c), we note that since $f_n \in L^1$, for all $n$ and for all $\varepsilon > 0$, there exists some $M_n$ such that

$$\int_{\{x \mid |f_n(x)| > M_n\}} |f_n(x)| d\mu < \varepsilon$$

and

$$\infty > \sup_{n\in\mathbb{N}} \int |f_n| d\mu \geq \sup_{n\in\mathbb{N}} \int_{\{x \mid |f_n(x)| > M_n\}} |f_n(x)| d\mu.$$

So, because the sup is finite, and since each $f_n \in L^1$, for all $\varepsilon > 0$ there exists some $M > 0$ such that

$$\sup_n \int_{\{x \mid |f_n(x)| > M\}} |f_n(x)| d\mu < \varepsilon \tag{1}$$

Finally, this implies that

$$\lim_{M\to\infty} \sup_n \int_{\{x \mid |f_n(x)| > M\}} |f_n(x)| d\mu = 0$$

and so $\{f_n\}$ are uniformly integrable.

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(1) Note that if no such $M$ exists, then for all $M$

$$\sup_n \int_{\{x \mid |f_n(x)| > M\}} |f_n(x)| d\mu \geq \varepsilon$$

and so there is some $\int |f_n| d\mu$ such that

$$\int_{\{x \mid |f_n(x)| > M\}} |f_n(x)| d\mu + \frac{\varepsilon}{2} \geq \sup_n \int_{\{x \mid |f_n(x)| > M\}} |f_n(x)| d\mu \geq \varepsilon$$

which implies that for all $M$

$$\int_{\{x \mid |f_n(x)| > M\}} |f_n(x)| d\mu \geq \frac{\varepsilon}{2}$$

which is a contradiction OF $|f_n(x)| \in L^1$.

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**Problem 4.** Let $\mathbb{M}$ be the collection of all finte measures on the measure space $(X, \mathcal{M})$.

(a) Show that

$$d(\nu,\lambda) = 2\sup_{E\in\mathcal{M}} |\nu(E) - \lambda(E)|$$

defines a metric on $\mathbb{M}$.

(b) For any $\mu \in \mathbb{M}$, that dominates measures $\nu$ and $\lambda$ with $\nu(X) = \lambda(X) = 1$, let

$$p = \frac{d\nu}{d\mu} \qquad \text{and} \qquad q = \frac{d\lambda}{d\mu}.$$

Prove that

$$d(\nu,\lambda) = \int |p(x) - q(x)|d\mu = 2\left(1 - \int(\min\{p(x), q(x)\})d\mu\right).$$

Hint: notice that $\mu(E) - \lambda(E) = \lambda(E^c) - \nu(E^c)$.

**Solution.**

(a)
- $d(\nu,\lambda) \geq 0$ for all $\lambda, \nu \in \mathbb{M}$.
- $d(\nu,\lambda) = 0 \iff \nu = \lambda$ for all $\lambda, \nu \in \mathbb{M}$ is immediate from the definition.
- $d(\nu,\lambda) = 2\sup_E |\nu(E) - \lambda(E)| = 2\sup_E |\lambda(E) - \nu(E)| = d(\lambda,\nu)$ for all $\lambda, \nu \in \mathbb{M}$.
- $$\begin{aligned} d(\nu,\mu) + d(\mu,\lambda) &= 2\sup_E |\nu(E) - \mu(E)| + 2\sup_E |\mu(E) - \lambda(E)| \\ &\geq 2\sup_E (|\nu(E) - \mu(E)| + |\mu(E) - \lambda(E)|) \qquad \sup A + \sup B \geq \sup(A+B) \\ &\geq 2\sup_E |\nu(E) - \lambda(E)| \qquad \text{Triangle Inequality} \\ &= d(\nu,\lambda) \end{aligned}$$

So $d$ is a metric on $\mathbb{M}$.

(b) Note that since $\nu(E) = \lambda(E) = 1$, for all $E$, we have that

$$\nu(E) + \nu(E^c) = \lambda(E) + \lambda(E^c) \implies \nu(E) - \lambda(E) = \lambda(E^c) - \nu(E^c).$$

**Claim 1.** $2\sup_E \left|\int_E (p-q)d\mu\right| = \int |p-q|d\mu$

*Proof.* $\boxed{\leq}$

$$\begin{aligned}
2\sup_E |\nu(E)-\lambda(E)| &= \sup_E |2(\nu(E)-\lambda(E))| \\
&= \sup_E |\nu(E)-\lambda(E)+\lambda(E^c)-\nu(E^c)| \\
&\leq \sup_E (|\nu(E)-\lambda(E)| + |\nu(E^c)-\lambda(E^c)|) \\
&= \sup_E \left( \left|\int_E (p-q)d\mu\right| + \left|\int_{E^c}(p-q)d\mu\right| \right) \\
&\leq \int |p-q| d\mu
\end{aligned}$$

$\boxed{\geq}$ Let $E = \{x \mid p(x) - q(x) \geq 0\}$. Then

$$\begin{aligned}
\int_X |p-q|d\mu &= \int_E (p-q)d\mu + \int_{E^c}(q-p)d\mu \\
&= \nu(E)-\lambda(E)+\lambda(E^c)-\nu(E^c) \\
&= 2(\nu(E)-\lambda(E)) \\
&\leq 2\sup|\nu(E)-\lambda(E)| \\
&= 2\sup_E \left|\int_E (p-q)d\mu\right|
\end{aligned}$$

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Now, assuming that "dominates" implies that $\nu << \mu$ and $\lambda << \mu$, we have that

$$\nu(X) = \int p d\mu = 1 \quad \text{and} \quad \lambda(X) = \int q d\mu = 1.$$

Finally, this gives

$$\begin{aligned}
d(\nu,\lambda) &= 2\sup_{E\in\mathcal{M}} |\nu(E)-\lambda(E)| \\
&= 2\sup_{E\in\mathcal{M}} \left|\int_E p d\mu - \int_E q d\mu\right| \\
&= 2\sup_E \left|\int_E (p-q)d\mu\right| \\
&= \int |p-q|d\mu \qquad \text{from the claim} \\
&= \int_E (p-q)d\mu + \int_{E^c}(q-p)d\mu \\
&= \int_X (p-\min\{p,q\})d\mu + \int_X (q-\min\{p,q\})d\mu \\
&= 2\left(1 - \int \min\{q,p\})d\mu\right)
\end{aligned}$$

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