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## Real Analysis Exam Spring 2011

**Problem 1.** Let $A \subset \mathbb{R}$ and suppose that for each $\varepsilon > 0$ there are Lebesgue-measurable sets $E, F$ with $E \subset A \subset F$ and $m(F \backslash E) < \varepsilon$. Show that $A$ is Lebesgue measurable.

**Solution.** First,

$$A = E \cup (A \cap E^c \cap F) = E \cup (A \cap (F \backslash E)).$$

Now, for all $n$, there exists $F_n$ and $E_n$ Lebesgue measurable with $m(F_n \backslash E_n) < \frac{1}{n}$ and $E_n \subset A \subset F_n$.

Let

$$E = \bigcup^{\infty} E_n \subset A \quad \text{and} \quad F = \bigcap^{\infty} F_n \supset A.$$

Then, let

$$E'_i = \bigcup_{n=1}^{i} E_n$$

so $E'_1 \subset E'_2 \subset \cdots$ and so

$$A \backslash E'_1 \supset A \backslash E'_2 \supset \cdots$$

Furthermore, since

$$m(A \backslash E'_1) \leq m(F_1 \backslash E'_1) = m(F_1 \backslash E_1) < 1 < \infty,$$

by continuity from below,

$$\begin{aligned} m(\bigcap^{\infty}(A \backslash E'_n)) &= m(A \backslash \bigcap E'_n) \\ &= \lim_{n\to\infty} m(A \backslash E'_n) \\ &\leq \lim_{n\to\infty} m(F_n \backslash E'_n) \\ &\leq \lim_{n\to\infty} m(F_n \backslash E_n) \\ &\leq \lim_{n\to\infty} \frac{1}{n} = 0. \end{aligned}$$

Thus, since $m(A \backslash E) \leq m(A \backslash E_n)$ for all $n$, we have that $m(A \backslash E) = 0$.

By a similar argument, $m(F \backslash A) = 0$.

Now, since

$$m(F \backslash E) = m(F \backslash A) + m(A \backslash E) = 0 + 0 = 0$$

we have that

$$A = E \bigsqcup (A \cap (F \backslash E))$$

however, since $A \cap (F \backslash E) \subset F \backslash E$ which is null, we have that $A \cap (F \backslash E) \in \mathcal{L}$ since $m$ is complete and since $E \in \mathcal{L}$ by assumption, we have that $A \in \mathcal{L}$ since it is the union of two measurable sets. ✌

**Problem 2.** Let $f > 0$ be a Lebesgue-integrable function on $[0,1]$. Show that

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_{[0,1]} (f^\varepsilon - 1) dm = \int_{[0,1]} \log f dm.$$

Here $m$ denotes Lebesgue measure. HINT: Decompose $f$ (or $\log f$) into two parts.

**Solution.** First,

$$\lim_{\varepsilon \to 0^+} \frac{f^\varepsilon - 1}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{f^\varepsilon \log f}{1} = \log f$$

for all $x$ by L'Hopital's Rule.

Now, let

$$E = \{x \in [0,1] \mid f(x) > 1\}.$$

Then,

$$\int_{[0,1]} \frac{f^\varepsilon - 1}{\varepsilon} dm = \int_E \frac{f^\varepsilon - 1}{\varepsilon} dm + \int_{E^c} \frac{f^\varepsilon - 1}{\varepsilon} dm.$$

Now, we let $\varepsilon = \frac{1}{n}$ and

$$f_n(x) = \frac{f^{1/n}(x) - 1}{\frac{1}{n}} = n(f^{1/n}(x) - 1).$$

$\boxed{\text{on } E}$ We would like to use Monotone Convergence Theorem.

1. $\{f_n\} \in L^+$. Since $f > 1$ on $E$, $f^{1/n} > 1$ and so $n(f^{1/n} - 1) > 0$ for all $n$. Furthermore, $f_n$ is measurable since $f$ is.

2. 
$$\begin{aligned} \frac{d}{dn} f_n(x) &= f^{1/n} - 1 + n(f^{1/n} \log f)(-1/n) \\ &= f^{1/n} - 1 - \frac{f^{1/n} \log f}{n} \\ &= \frac{n(f^{1/n} - 1) - f^{1/n} \log f}{n} \end{aligned}$$

   Since, $f > 1$, $f^{1/n} > 1$ for all $n$ and so for large enough $n$, the derivative is eventually positive and so, after perhaps removing the first $N$ terms, $f_n \le f_{n+1}$.

3. $f_n \to \log f$ for all $x \in E$.

Thus, by the monotone convergence theorem, we may bring the limit inside the integral on $E$.

$\boxed{\text{on } E^c}$ Now, $f_n \le 0$. However, by the same argument as above, for sufficiently large $n$, $f_n' \le 0$. Thus, $-f_n(x) \ge 0$ and $-f_n' \ge 0$.

1. for $f \leq 1$, $f_n$ is negative, so $\{-f_n\} \in L^+$.

2. Using the same derivative as above, since $f^{1/n} - 1 < 0$ and $f^{1/n} \log f < 0$ for all $n$ on $E^c$ we have that $-f_n' \geq 0$ for sufficently large $n$. So $-f_n$ is eventually increasing.

3. $-f_n \to -\log f$ for all $x \in E^c$

Thus, by the monotone convergence theorem we can bring the limit inside the intergral on $E$.

Finally,

$$\begin{aligned}
\lim_{\varepsilon \to 0^+} \int_{[0,1]} \frac{f^\varepsilon - 1}{\varepsilon} dm &= \lim_{n\to\infty} \left[ \int_E f_n - \int_{E^c} -f_n \right] \\
&= \int_E \lim_{n\to\infty} f_n dm - \int_{E^c} \lim_{n\to\infty} -f_n dm \\
&= \int_E \log f dm - \int_{E^c} -\log f dm \\
&= \int \log f dm.
\end{aligned}$$

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**Problem 3.** Suppose $f \in L^1$ is absolutely continuous, and

$$\lim_{h \searrow 0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right| dx = 0.$$

Show that $f = 0$ a.e.

**Solution.** Let $h = \frac{1}{n}$ and $f_n(x) = n(f(x + \frac{1}{n}) - f(x))$.

$$\lim_{n \to \infty} \int |f_n(x)| dx = 0$$

implies that $|f_n(x)| \to 0$ in $L^1$.

Now, since $f$ is absolutely continuous, $f'$ exists a.e. and $f_n \to f'$ where it exists.

Thus, since $f_n \to 0$ in $L^1$, there exists a subsequence $\{f_{n_k}\}$ which converges to 0 a.e. However, since $f_{n_k} \to f'$ a.e., this implies that $f' = 0$ a.e

Finally, since $f$ is absolutely continuous, by the Fundamental Theorem of Lebesgue Integrals, on any closed interval $[a, b]$, we have that

$$f(x) - f(a) = \int_a^x f'(t) dt = 0 \implies f(x) = f(a) \text{ for all } x > a.$$

Similarly, we have that $f(x) = f(a)$ for all $x < a$. Thus, $f(x) = c$ for some constant a.e.

Since $f \in L^1$, and

$$\int |c| dx = \infty \qquad \text{for all } c \neq 0$$

we have that $f = 0$ a.e. ✌

**Problem 4.**

(a) Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$, and suppose $F_1, ..., F_7$ are 7 measurable sets with $\mu(F_j) \geq \frac{1}{2}$ for all $j$. Show that there exists indices $i_1 < i_2 < i_3 < i_4$ for which $F_{i_1} \cap F_{i_2} \cap F_{i_3} \cap F_{i_4} \neq \varnothing$.

(b) Let $m$ denote the Lebesgue measure on $[0,1]$ and let $f_n \in L^1(m)$ be nonnegative and measurable with

$$\int_{[0,1/n]} f_n dm \geq 1/2$$

for all $n \geq 1$. Show that $\int_{[0,1]}[\sup_n f_n(x)]m(dx) = \infty$. HINT: Part (b) does not necessarily use part (a).

**Solution.**

(a) Assume not. Then

$$\sum_{i=1}^{7} \chi_{F_i}(x) \leq 3$$

for all $x \in X$. (Else there exists $x \in F_i$ for at least 4 of the $F_i$).

Now,

$$\int \sum_{i=1}^{7} \chi_{F_i}(x) d\mu \leq \int 3 = 3\mu(X) = 3$$

however,

$$\int \sum_{i=1}^{7} \chi_{F_i}(x) d\mu = \sum_{i=1}^{7} \int_X \chi_{F_i}(x) d\mu = \sum_{i=1}^{7} \mu(F_i) \geq \sum_{i=1}^{7} \frac{1}{2} = \frac{7}{2} > 3.$$

Thus, there exists some $x$ such that for some indices $i_1 < i_2 < i_3 < i_4$, $x \in F_{i_1} \cap F_{i_2} \cap F_{i_3} \cap F_{i_4}$.

(b) Let $M > 0$ be large and

$$E = \left\{ x \in \left[0, \frac{1}{n}\right] \mid f_n(x) > M \right\}.$$

Now, for all $n > 2M$, we have that

$$\begin{aligned}
\frac{1}{2} \leq \int_{[1,\frac{1}{n}]} f_n(x)dm \\
&= \int_E f_n(x)dm + \int_{E^c} f_n(x)dm \\
&\leq \int_E f_n(x)dm + \int_{E^c} Mdm \\
&\leq \int_E f_n(x)dm + \frac{M}{n} \qquad m\left(\left[1, \frac{1}{n}\right]\right) = \frac{1}{n} \\
&< \int_E f_n(x)dm + \frac{1}{2}
\end{aligned}$$

Thus,

$$\int_E f_n(x)dm > 0$$

for all $n > 2M$ so $f_n(x) > M$ on a set of positive measure for arbitrary $M$.

Thus,

$$\int_{[0,1]} [\sup_n f_n(x)] m(dx) = \infty$$

since $f_n$ grow arbitrarily large near 0.

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