Kayla Orlinsky

Real Analysis Exam Fall 2011

Problem 1. Let $f \geq 0$ and suppose $f \in L^1([0,\infty))$. Find

$$\lim_n \frac{1}{n}\int_0^n xf(x)dx.$$

Solution. Let $f_n(x) = \frac{1}{n}\chi_{[0,n]}xf(x)$. Then

$$\int_0^n \frac{1}{n}xf(x)dx = \int_0^\infty f_n(x)dx.$$

Now, we would like to use Dominated Convergence Theorem.

1. $\lim\limits_{n\to\infty} f_n(x) = \dfrac{xf(x)}{\infty} = 0$ a.e.
2. $f_n(x) \leq f(x) \in L^1$ since for $x \in [0,n]$, $x \leq n$ so $\frac{x}{n} \leq 1$. Thus

$$\int f_n(x)dm \leq \int f(x)dm < \infty$$

so $\{f_n\} \in L^1$ for all n.

Thus, by the dominated convergence theorem,

$$\lim_n \int f_n dx = \int \lim_{n\to\infty} f_n(x)dx = 0.$$

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Problem 2. Suppose $f \geq 0$ is absolutely continuous on $[0,1]$ and $\alpha > 1$. Show that f^α is absolutely continuous.

Solution. We would like to use the Fundamental Theorem of Lebesgue Integrals.

1. f is absolutely continuous so f' exists a.e. Thus,
$$(f^\alpha)' = \alpha f^{\alpha-1} f'$$
exists a.e. (since $\alpha > 1$).

2. Since f is absolutely continuous, by FTOLI, $f' \in L^1([0,1])$ and since f is continuous on a closed interval it is bounded so there is some $\infty > M > 0$ with $f(x) \leq M$ for all $x \in [0,1]$. Thus,
$$(f^\alpha)' = \alpha f^{\alpha-1} f' \leq \alpha M^{\alpha-1} f' \in L^1.$$

3. $$\int_0^x \alpha f^{\alpha-1} f' dx = \int_{f(0)}^{f(x)} \alpha u^{\alpha-1} du = u^\alpha \Big|_{f(0)}^{f(x)} = f^\alpha(x) - f^\alpha(0)$$
$$\begin{array}{ll} u = f(x) & x : [0,1] \\ du = f'(x)dx & u : [f(0), f(1)] \end{array}$$
Thus, by FTOLI, f^α is absolutely continuous.

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Problem 3 (part (b) is Folland, 3.5.30, p.107).

(a) Let $\{\mu_k\}$ be a sequence of finite signed measures. Find a finite positive measure μ such that $\mu_k << \mu$ for all k.

(b) Construct an increasing function whose set of discontinuities is $\mathbb{Q}$. (Prove that it is a valid example).

Solution.

(a) Let

$$\mu = \sum_k \frac{|\mu_k|}{|\mu_k|(X)2^k}.$$

Then, if $\mu(E) = 0$, since $|\mu_k| \geq 0$, $|\mu_k|(E) = 0$ for all k. Since

$$0 = |\mu_k|(E) = \mu_k^+(E)+\mu_k^-(E) \implies \mu_k^+(E) = -\mu_k^-(E) \implies \mu_k^+(E) = \mu_k^-(E) = 0 \implies \mu_k(E) = 0.$$

Thus $\mu_k << \mu$ for all k. Furthermore, $\mu \geq 0$ and finally, $\mu < \infty$ since

$$\mu(X) = \sum_k \frac{|\mu_k|(X)}{|\mu_k|(X)2^k} = \sum_k \frac{1}{2^k} < \infty.$$

(b) Let $(r_i)_1^\infty$ be an enumeration of the rationals. Let $f(x) = \sum_{i=1}^\infty \frac{1}{2^i}\chi_{[r_i,\infty)}(x)$. Then clearly if $y > x$, $f(y) \geq f(x)$ since there are more $r_i \leq y$ than $r_i \leq x$ so $f(x)$ is increasing.

Now, fix x. $\boxed{x \in \mathbb{Q}}$ Then $x = r_{i_0}$ for some i_0. Thus, for all $\delta > 0$, there exists some y such that $0 < y - x < \delta$ (so $y > x$) but

$$f(y) - f(x) = \sum_{\{i \,|\, x < r_i \leq y\}} \frac{1}{2^i} \geq \frac{1}{2^i}.$$

Thus,

$$\lim_{y \to x} f(y) \geq \lim_{y \to x} \left(f(x) + \frac{1}{2^i} \right) > f(x).$$

So f is discontinuous at x.

$\boxed{x \notin \mathbb{Q}}$ Fix $\varepsilon > 0$ and pick N such that

$$\sum_{i=N+1}^\infty \frac{1}{2^i}\chi_{[r_i,\infty)}(x) < \varepsilon.$$

Now, let

$$s = \max_{i \leq N} r_i \in (x - \varepsilon, x) \quad \text{and} \quad t = \min_{i \leq N} r_i \in (x, x + \varepsilon).$$

Then $(x-s, x+t)$ contain no r_i for $i \le N$.

If no such r_i exists we set $s = \varepsilon$ and/or $t = \varepsilon$.

Then, let $\delta = \min\{s, t\}$ for all $|y - x| < \delta$ we have that

$$|f(y) - f(x)| \le \sum_{i=N+1}^{\infty} \frac{1}{2^i} \chi_{[r_i, \infty)} < \varepsilon$$

so f is continuous at x.

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Problem 4 (Folland, part (a) 3.4.22, and part (b) 3.4.23, p.100). Let m be the Lebesgue measure on $\mathbb{R}$. For $f \in L^1_{loc}$ and $x \in \mathbb{R}^n$, define the function $A_r f$ by

$$A_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)dy,$$

which is the average value of f on the ball $B(x,r)$ of radius r centered at x, and define the function Hf by $Hf(x) = A_r|f|(x), x \in \mathbb{R}^d$.

(a) Show that for $f \in L^1(\mathbb{R}^n)$, $f \neq 0$, there exist $C, C', R > 0$ such that $Hf(x) \geq C|x|^{-n}$ for all $|x| > R$ and

$$m\left(\{x \,|\, Hf(x) > \alpha\}\right) \geq \frac{C'}{\alpha} \qquad \text{for all sufficiently small } \alpha.$$

(b) Define the function H^*f by

$$H^*f(x) = \sup\left\{ \frac{1}{m(B)} \int_B |f(y)|dy \,\middle|\, B \text{ is a ball containing} x \right\}.$$

Show that $Hf \leq H^*f \leq 2^n Hf$. (Note that unlike Hf, in the definition of H^*f the ball B need not be centered at x.)

Solution.

(a) Let $f \in L^1$, $f \not0$. Let

$$M = \int_{B(r,0)} |f(x)|dx < \infty.$$

Then there exists some $R > 0$ such that

$$\int_{B(R,0)} |f(x)|dx \geq \frac{M}{2}.$$

Since $|x| > R$, $B(R,0) \subset B(2|x|, x)$,

$$\begin{aligned}
\sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)|dy &\geq \frac{1}{m(B(2|x|,x)} \int_{B(2|x|,x)} |f(y)|dy \\
&= \frac{1}{2^n|x|^n} \int_{B(2|x|,x)} |f(y)|dy \\
&\geq \frac{1}{2^n|x|^n} \int_{B(R,0)} |f(y)|dy \\
&\geq \frac{M}{2^{n+1}|x|^n}.
\end{aligned}$$

Letting $C = \frac{M}{2^{n+1}}$ we are done.

Now, for all $\alpha \in (0, \frac{C}{2R^n})$ and for all $x \in \mathbb{R}^n$ with $R < |x| < \left(\frac{C}{\alpha}\right)^{1/n}$,

$$Hf(x) \geq C|x|^{-n} > C\frac{\alpha}{C} > \alpha.$$

Thus

$$\begin{aligned} m(\{x \mid Hf(x) > \alpha\}) &\geq m(\{x \mid C|x|^{-n} > \alpha\}) \\ &= m(\{x \mid \frac{C}{\alpha} > |x|^n\}) \\ &= m(\{x \mid \left(\frac{C}{\alpha}\right)^{1/n} > |x| > R\}) \\ & m(B((C/\alpha)^{1/n}, x)) - m(B(R, x)) \\ &= \left(\frac{C}{\alpha} - R^n\right) m(B(1, x)) > \frac{Cm(B(1, x))}{2\alpha} \end{aligned}$$

Letting $C' = \frac{Cm(B(1,x))}{2}$ we are done.

(b) Let B_r be a ball containing x of radius r. Then certainly $B_r \subset B(2r, x)$ and so

$$\frac{1}{m(B_r)}\int_{B_r} |f(y)|dy \leq \frac{2^n}{m(B(2r,x))}\int_{B(2r,x)} |f(y)|dy \leq 2^n Hf(x)$$

so

$$H^*f \leq 2^n Hf.$$

It is immediate that $Hf \leq H^*f$ since any ball centered at x is also a ball containing x. Thus,

$$Hf \leq H^*f \leq 2^n Hf.$$

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