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Real Analysis Exam Spring 2010

Problem 1. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *upper semicontinuous* (or *u.s.c*) if for all $x \in \mathbb{R}$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that $f(y) < f(x) + \varepsilon$ whenever $|y - x| < \delta$.

(a) Show that every u.s.c. function is Borel measurable. HINT: Consider $\{x \mid f(x) < a\}$.

(b) Suppose μ is a finite measure on $\mathbb{R}$ and A is a closed subset of $\mathbb{R}$. Using (a) or otherwise, show that the function $x \mapsto \mu(x + A)$ is measurable. Here $x + A = \{x + y \mid y \in A\}$.

Solution.

(a) Since

$$f^{-1}((-\infty, a)) = \{x \mid f(x) < a\} = A,$$

we check that $f^{-1}((-\infty, a))$ is open. Let $x \in A$. Then since f is usc, for all $\varepsilon > 0$ there exists a δ such that

$$f(y) - \varepsilon < f(x) < a \qquad \text{whenever } |y - x| < \delta.$$

Now, for $\varepsilon = a - f(x)$, there is some δ where, for all $|y - x| < \delta$ we have that

$$f(y) - f(x) < \varepsilon = a - f(x) \implies f(y) < a.$$

Thus, $B(\delta, x) \subset A$.

Thus, A is open and since all open sets are Borel, we have that A is Borel.

(b) Since A is closed, A^c is open and so A is a Borel set. Thus, there exists some E, which is a union of finitely many open intervals such that $\mu(A \Delta E) < \varepsilon$.

Thus, it suffices to check the statement holds for $x \mapsto \mu(x + E)$.

Let

$$\begin{aligned} f(x) :& \mathbb{R} \to \mathbb{R} \\ & x \mapsto \mu(x + E) \end{aligned}$$

We would like to show that f is usc. Let $\varepsilon > 0$ be given and x be fixed. WLOG, let

$$E = \bigcup_{i=1}^{N} (a_i, b_i) \qquad x + E = \bigcup_{i=1}^{N} (a_i + x, b_i + x).$$

Now, fix $b \in \mathbb{R}$. Let $B_n = (b, b + \frac{1}{n})$. Then $B_1 \supset B_2 \supset \cdots$ and since $\mu(X) < \infty$, by continuity

$$0 = \mu(\bigcap^{\infty} B_n) = \lim_{n\to\infty} \mu(b, b + \frac{1}{n}).$$

Since, b was arbitrary, this implies that for all $\varepsilon > 0$ there exists some δ such that for all $x < y < x + \delta$, $\mu(b + x, b + y) < \varepsilon$.

Let δ_i be such that $\mu(a_i + x, b_i + x + \delta_i) < \varepsilon$.

Now, let

$$\delta = \frac{1}{N} \max_i \{\mu(b_i + x, b_i + x + \delta_i)\}.$$

Then, for all $x < y < x + \delta$, we have that

$$\begin{aligned} f(y) = \mu(y + E) &\leq m(x + E) + \mu(\cup_{i=1}^{N} \mu(b_i + x, b_i + y) \\ &\leq \mu(x + E) + \sum_{i=1}^{N} \mu(b_i + x, b_i + y) \\ &< \mu(x + E) + \sum_{i=1}^{N} \delta \\ &< \mu(x + E) + \varepsilon = f(x) + \varepsilon \end{aligned}$$

Thus, f is usc and so it is measurable by (a).

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Problem 2. Suppose $\{f_n\}$ and f are measurable functions on $(X, \mathcal{M}, \mu)$ and $f_n \to f$ in measure. Is it necessarily true that $f_n^2 \to f^2$ in measure if

(a) $\mu(X) < \infty$

(b) $\mu(X) = \infty$

Solution.

(a) Let
$$E_n = \{x \,|\, |f_n^2(x) - f^2(x)| \geq \varepsilon\}$$
Then
$$\begin{aligned}\mu(E_n) &= \mu(\{x \,|\, |f_n(x) - f(x)||f_n(x) + f(x)| \geq \varepsilon\}) \\ &= \mu(\{x \,|\, |f_n(x) - f(x)||f_n(x) + f(x)| \geq \varepsilon \text{ and } |f_n(x) + f(x)| \geq k\}) \\ &+ \mu(\{x \,|\, |f_n(x) - f(x)||f_n(x) + f(x)| \geq \varepsilon \text{ and } |f_n(x) + f(x)| < k\}) \\ &\leq \mu(\{x \,|\, |f_n(x) + f(x)| \geq k\}) + \mu(\{x \,|\, |f_n(x) - f(x)| \geq \frac{\varepsilon}{k})\end{aligned}$$
Now, we assume that $f(x) < \infty$ a.e. which is safe it is not specified that f is defined over the extended reals.

Thus,
$$\mu(E_n) \leq \lim_{k\to\infty} \lim_{n\to\infty} \mu(\{x \,|\, |f_n(x) + f(x)| \geq k\}) + \mu(\{x \,|\, |f_n(x) - f(x)| \geq \frac{\varepsilon}{k}) = 0.$$

(b) Let $f_n(x) = x + \frac{1}{n}$ then $|f_n - x| = |\frac{1}{n}|$ and since for all $\varepsilon > 0$, there exists an N such that
$$\frac{1}{n} \leq \varepsilon$$
for all $n \geq N$, we have that
$$\mu(\{x \,|\, |f_n(x) - x| \geq \varepsilon\}) \to 0 \qquad \text{as } n \to \infty.$$
However, assuming $\mu = m$, and $X = [0, \infty)$, we have that $f_n^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2}$ and
$$|f_n^2 - f^2| \geq \varepsilon \implies \left|\frac{2x}{n} + \frac{1}{n^2}\right| \geq \varepsilon \implies x \geq \frac{n\varepsilon}{2} - \frac{1}{2}.$$
Thus,
$$m(\{x \,|\, |f_n - f| \geq \varepsilon\}) = m\left(\{x \,|\, x \geq \frac{n\varepsilon}{2} - \frac{1}{2}\right) = m\left(\frac{n\varepsilon}{2} - \frac{1}{2}, \infty\right) = \infty \qquad \text{for all } n.$$
Thus, $f_n^2 \not\to f^2$ in measure.

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Problem 3. Suppose $f : [0,1] \to \mathbb{R}$ is as strictly increasing absolutely continuous function. Let m denote the Lebesgue measure. If $m(E) = 0$ show that $m(f(E)) = 0$.

Solution. Let $E \subset [0,1]$ with $m(E) = 0$.

Now, since f is absolutely continuous and strictly increasing we have that f is one-to-one on $[0,1]$. Thus, if $y \in f(E)$, then $y = f(x)$ for exactly one $x \in E$ and similarly, if $x \in E$ then there is one $y = f(x) \in f(E)$. Thus,

$$\chi_E(x) = \chi_{f(E)}(y).$$

Furthermore, f' exists a.e. by the Fundamental Theorem of Lebesgue Integrals.

Thus,

$$m(f(E)) = \int \chi_{f(E)}(y)dy = \int \chi_E(u)f'du = 0$$
$$u = y = f(x) \qquad du = f'(x)dx$$

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Problem 4. For $n \geq 1$ define h_n on $[0,1]$ by

$$h_n = \sum_{j=1}^{n} (-1)^j \chi_{(\frac{j-1}{n}, \frac{j}{n}]}.$$

Here χ_E denotes the characteristic function of E. If f is Lebesgue integrable on $[0,1]$, show that

$$\lim_{n\to\infty} \int_{[0,1]} f h_n dm = 0.$$

HINT: First consider f in a suitably smaller function space.

Solution. Let $f(x) = \chi_E(x)$ with $E \subset [0,1]$. Then for fixed n, $h_n \in L^1([0,1])$ since $|h_n| = (0,1]$ so we can apply Fubini to fh_n on $m \times \nu$ with ν the counting measure on $\mathbb{N}$. Thus,

$$\begin{aligned}
\int_{[0,1]} f h_n dm &= \int_{[0,1]} \sum_{j=1}^{n} (-1)^j \chi_{(\frac{j-1}{n}, \frac{j}{n}] \cap E)} dm \\
&= \sum_{j=1}^{n} (-1)^j \int_{[0,1]} \chi_{(\frac{j-1}{n}, \frac{j}{n}] \cap E)} dm \\
&= \sum_{\text{even j}}^{n} \int_{[0,1]} \chi_{(\frac{j-1}{n}, \frac{j}{n}] \cap E)} dm - \sum_{\text{odd j}}^{n} \int_{[0,1]} \chi_{(\frac{j-1}{n}, \frac{j}{n}] \cap E)} dm \\
&= \sum_{\text{even j}}^{n} m\left(\left(\frac{j-1}{n}, \frac{j}{n}\right] \cap E\right) - \sum_{\text{odd j}}^{n} m\left(\left(\frac{j-1}{n}, \frac{j}{n}\right] \cap E\right) \\
&= m\left(\bigcup_{\text{even j}}^{n} \left(\frac{j-1}{n}, \frac{j}{n}\right] \cap E\right) - m\left(\bigcup_{\text{odd j}}^{n} \left(\frac{j-1}{n}, \frac{j}{n}\right] \cap E\right)
\end{aligned}$$

Let

$$A_n = \bigcup_{\text{even } j}^{n} \left(\frac{j-1}{n}, \frac{j}{n}\right] \quad \text{so} \quad A_n^c = \bigcup_{\text{odd } j}^{n} \left(\frac{j-1}{n}, \frac{j}{n}\right].$$

Let $E = (a,b) \subset [0,1]$. Then for all $\varepsilon > 0$ there exists N such that for some j,k $|j/N - a| < \varepsilon$ and $|k/n - b| < \varepsilon$. Then E will be almost perfectly partitioned. Specifically,

$$|m(A_n \cap E) - m(A^c \cap E)| \leq \frac{1}{N} + 2\varepsilon.$$

Thus,

$$m(A_n \cap E) - m(A^c \cap E) \to 0 \qquad \text{as } n \to \infty.$$

Therefore, the same is true for finite unions of open intervals.

Now, for all E, and for all ε there exists F, a finite union of open intervals such that

$$m(E\Delta F) < \varepsilon.$$

Thus,

$$\begin{aligned}
|m(A_n \cap E) - m(A_n^c \cap E)| &= |m(A_n \cap E) - m(A_n^c \cap E) + m(A_n \cap F) \\
&\qquad - m(A_n^c \cap F) + m(A_n^c \cap F) - m(A_n \cap F)| \\
&\leq |(m(A_n \cap E) - m(A_n \cap F)) - (m(A_n^c \cap E) - m(A_n^c \cap F))| \\
&\qquad + |m(A_n \cap F) - m(A_n^c \cap F)| \\
&= |m(A_n \cap (E\backslash F)) - m(A_n^c \cap (E\backslash F))| + |m(A_n \cap F) - m(A_n^c \cap F)| \\
&\leq 2\varepsilon + |m(A_n \cap F) - m(A_n^c \cap F)|
\end{aligned}$$

And since we have already seen that

$$|m(A_n \cap F) - m(A_n^c \cap F)| \to 0$$

for F, we have that the same holds for E and so

$$\lim_{n\to\infty} \int_{[0,1]} \chi_E h_n dm = 0.$$

Thus, the above holds for all simple functions f by linearity of the integral.

Finally, since for all $\varepsilon > 0$ there exists a simple function ϕ such that $\int |f - \phi| dm < \varepsilon$, we have that

$$\left|\int_{[0,1]} f h_n dm - \int_{[0,1]} \phi h_n dm\right| = \left|\int_{[0,1]} (f - \phi) h_n dm\right| \leq \int_{[0,1]} |f - \phi||h_n| dm = \int_{[0,1]} |f - \phi| dm < \varepsilon$$

so, tending ε to 0 we have our result. ✌