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Real Analysis Exam Fall 2010

Problem 1. Let $\mathcal{A}$ be a collection of pairwise disjoint subsets of a σ-finite measure space, and suppose each set in $\mathcal{A}$ has strictly positive measure. Show that $\mathcal{A}$ is at most countable.

Solution. Because X is σ-finite, let $X = \bigcup_{i=1}^{\infty} E_i$ with $\mu(E_i) < \infty$ for all i. Furthermore, we can let E_i be disjoint by letting $F_1 = E_1$, $F_2 = E_2 \backslash E_1$, ... , $F_i = E_i \backslash \bigcup_{j=1}^{i-1}$.

Now, let $\mathcal{A} = \{A_\alpha\}_{\alpha \in I}$ for some indexing set I.

We now prove a claim:

Claim 1. The an uncountable sum of strictly positive numbers is infinite.

Proof. Let $\mathcal{K} = \{K_\kappa\}_{\kappa \in P}$ be an uncountable collection of strictly positive real numbers. Then

$$\sum_{\kappa \in P} K_\kappa = \sup \left\{ \sum_{i=1}^{N} K_{\kappa_i} \mid \text{ all finite subcollections of } P \right\}.$$

Now, let

$$S_n = \{\kappa_i \mid K_{\kappa_i} > \frac{1}{n}\}.$$

Then

$$\sum_{\kappa \in P} K_\kappa \geq \sup_{\kappa_i \in S_n} K_{\kappa_i} > \sum_{\kappa_i \in S_n} \frac{1}{n}.$$

Thus, if the sum is to be finite, it must be the case that S_n is finite for all n since $\frac{1}{n}$ is a positive constant.

Therefore, for the sum to be finite

$$S = \bigcup_{n \in \mathbb{N}} S_n = \{\kappa_i \mid K_{\kappa_i} > 0\}$$

is at most countable.

However, by assumption, all of the $K_\kappa > 0$ and so it must be that the sum is infinite. ✌

Assume that A is uncountable. Then since the $A_\alpha \in \mathcal{A}$ are uncountable and we can write

$$\mu(A_\alpha) = \mu(\bigcup_{i=1}^{\infty}(A_\alpha \cap E_i)) = \sum_{i=1}^{\infty}\mu(A_\alpha \cap E_i) > 0,$$

it must be that there exist i such that $\mu(A_\alpha \cap E_i) > 0$ for an uncountable number of α.

Index this set of α as $\{A_\beta\}_{\beta\in J}$ with J uncountable.

Then,

$$\begin{aligned}\sum_{\beta\in J}\mu(A_\beta \cap E_i) &= \sup_{\beta\in J}\left\{\sum_{j=1}^{n}\mu(A_{\beta_j} \cap E_i) \,|\, \text{ finite subcollections of } J\right\}\\ &\leq \mu(E_i)\\ &< \infty \text{ since } A_{\beta_j} \text{ are disjoint.}\end{aligned}$$

However, from the claim and since $\mu(A_{\beta_j} \cap E_i) > 0$ for all $\beta_j \in J$, this sup must be infinite, which contradicts that it is bounded by $\mu(E_i)$.

Thus, $\mathcal{A}$ is at most countable. ✌

Problem 2.

(a) Let m denote the Lebesgue measure on $\mathbb{R}$ and let f be an integrable function. Show that for $a > 0$,

$$\int f(ax)m(dx) = \frac{1}{a}\int f(x)m(dx).$$

HINT: Consider a restricted class of functions f first.

(b) Let F be a measurable function on $\mathbb{R}$ satisfying $|F(x)| \leq C|x|$ for all x, and suppose F is differentiable at 0. Show that

$$\lim_{n\to\infty}\int_{\mathbb{R}} \frac{nF(x)}{x(1+n^2x^2)}m(dx) = \pi F'(0).$$

HINT: Use (a).

Solution.

(a) Let $f(x) = \chi_E(x)$ for E measurable. Now, if $ax \in E$ then $x \in \frac{E}{a}\{\frac{e}{a}\,|\, e \in E\}$. Thus, $\chi_E(ax) = \chi_{E/a}(x)$ and so

$$\int \chi_E(x)dm = \int \chi_{E/a}(x)dm = m(E/a) = \frac{1}{a}m(E)$$

since $a > 0$ by the scaling property of Lebesgue measure.

Thus, by linearity, the above property holds for simple functions.

Now, for all $\varepsilon > 0$ there exists some ϕ simple function such that $\int |f - \phi|dm < \varepsilon$. Thus,

$$\begin{aligned}
\left|\int f(ax)dm - \int \frac{1}{a}f(x)dm\right| &= \left|\int f(ax)dm - \int \frac{1}{a}f(x)dm + \int \phi(ax)dm - \int \phi(ax)dm\right| \\
&\leq \left|\int f(ax) - \phi(ax)dm\right| + \left|\frac{1}{a}\int \phi(x) - f(x)dm\right| \\
&\leq \int |f(ax) - \phi(ax)|dm + \frac{1}{a}\int |f(x) - \phi(x)|dm \\
&< \varepsilon + \frac{\varepsilon}{a}
\end{aligned}$$

and since ϕ and ε were arbitrary, we are done.

(b) Note that

$$F'(0) = \lim_{x\to 0}\frac{F(x) - F(0)}{x} = \lim_{x\to 0}\frac{F(X)}{X} = \lim_{n\to\infty}\frac{F(u/n)}{u/n} = \lim_{n\to\infty}\frac{nF(u/n)}{u}$$

for fixed u and $x = \frac{u}{n}$.

Note that $F(0) = 0$ since $|F(0)| \leq C|0| = 0$.

Now, from (a), we use u-substitution $u = nx$, $du = ndx$.

Then

$$\int \frac{nF(x)}{x(1+n^2x^2)}dx = \int \frac{F(u/n)}{(u/n)(1+u^2)}du = \int \frac{nF(u/n)}{u}\frac{1}{1+u^2}du.$$

Now, we apply Dominated Convergence Theorem.

(i)

$$\lim_{n\to\infty} \frac{nF(u/n)}{u}\frac{1}{1+u^2} = \frac{F'(0)}{1+u^2}$$

for a.e. u.

(ii)

$$\left|\frac{nF(u/n)}{u}\frac{1}{1+u^2}\right| \leq \frac{C|u/n||n/u|}{1+u^2} = \frac{C}{1+u^2} \in L^1$$

so $\frac{nF(u/n)}{u}\frac{1}{1+u^2} \in L^1$.

Thus, by DCT,

$$\lim_{n\to\infty}\int \frac{nF(u/n)}{u}\frac{1}{1+u^2}du = \int_{-\infty}^{\infty} \frac{F'(0)}{1+u^2}du = F'(0)\tan^{-1}(u)\Big|_{-\infty}^{\infty} = F'(0)(\frac{\pi}{2} - \frac{-\pi}{2}) = \pi F'(0).$$

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Problem 3. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$ and let f be a measurable function with $|f| < 1$. Prove that

$$\lim_{n\to\infty} \int_X (1 + f + \cdots + f^n) d\mu$$

exists (it may be ∞). HINT: First consider $f \geq 0$.

Solution. We note that

$$1 + f + \cdots + f^n = \frac{f^{n+1} - 1}{f - 1} \qquad \text{for } |f| < 1.$$

Let

$$f_n(x) = \frac{f^{n+1} - 1}{f - 1}.$$

Note that $f_n \to \frac{1}{1-f}$ a.e. since $|f| < 1$.

$\boxed{f \geq 0}$ Then

$$\lim_{n\to\infty} \int_X \sum_{k=0}^{n} f^k d\mu = \lim_{n\to\infty} \sum_{k=0}^{n} \int_X f^k d\mu = \sum_{k=0}^{\infty} \int_X f^k d\mu = \int_X \sum_{k=0}^{\infty} f^k d\mu = \int_X \frac{1}{1-f} d\mu$$

since $f \in L^+$ and $|f| < 1$.

Alternatively, we could use monotone convergence theorem. Since $0 \leq f < 1$ we have that $f^{n+1} \leq f^n$ for all n so $f_{n+1}(x) \leq f_n(x)$ for all x.

Let

$$g_n(x) = \frac{1}{1-f} - f_n(x).$$

1. g_n is measurable since f is and because $f^n - 1 \leq 1$ for all n, $g_n(x) \geq 0$ for all n. Thus, $\{g_n\} \subset L^+$.
2. $g_n \to 0$ a.e.
3. $g_n \leq g_{n+1}$ for all n.

Thus, by the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int g_n(x) d\mu = 0 \implies \lim_{n\to\infty} \int f_n d\mu = \int \frac{1}{1-f} d\mu.$$

NOTE: This integral depends entirely on f. If $f(x) = 0$ a.e., then the integral is $\int 1 d\mu = \mu(X) < \infty$.

If $\mu = m$, and $X = (0,1)$ and $f(x) = x$, then $x < 1$ and

$$\int_0^1 \frac{1}{1-x} dm = -\log|1-x|\Big|_0^1 = \infty.$$

$\boxed{\text{arbitrary } f}$ We let $f_n(x)$ be as before. now, by the geometric series test, $f_n \to \frac{1}{1-f}$ uniformly.

Thus, for all $\varepsilon > 0$ and a.e. x, there exists an $N \in \mathbb{N}$ such that $|f_n(x) - \frac{1}{1-f}| < \varepsilon$ for all $n \geq N$. Thus,

$$\left|\int \frac{1}{1-f} - f_n d\mu\right| \leq \int \left|\frac{1}{1-f} - f_n\right| d\mu \leq \int \varepsilon d\mu = \varepsilon\mu(X).$$

Therefore,

$$\lim_{n\to\infty} \int f_n d\mu = \int \frac{1}{1-f} d\mu.$$

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Problem 4. Let $\{F_j\}$ be a sequence of nonnegative nondecreasing right-continuous functions on $[a, b]$ and suppose $F(x) = \sum_{j=1}^{\infty} F_j(x)$ is finite for all $x \in [a, b]$. Show that

$$F'(x) = \sum_{j=1}^{\infty} F_j'(x) \qquad \text{for } m\text{-a.e. } x \in [a, b].$$

HINT: Consider the corresponding measures μ_F and μ_{F_j}.

Solution. First, we let ν be the counting measure on $\mathbb{N}$.

1. $F_i \in L^1(\nu)$ since $F(x) < \infty$.
2. $\lim_{x \to y^+} F_i(x) = F_i(y)$ by right continuity.
3. $F_i(x) \leq F_i(b) \in L^1(\nu)$ since $F(b) < \infty$ and since F_i are increasing on $[a, b]$.

Thus, by Dominated Convergence Theorem,

$$\lim_{x \to y^+} F(x) = \lim_{x \to y^+} \sum_{i=1}^{\infty} F_i(x) = \sum_{i=1}^{\infty} F_i(y) = F(y)$$

so F is also right continuous. Furthermore, clearly F is also increasing and nonnegative since the F_i are so μ_F makes sense.

Therefore,

$$\begin{aligned} \mu_F([a, b]) &= F(b) - F(a) \\ &= \sum_{i=1}^{\infty} F_i(b) - \sum_{i=1}^{\infty} F_i(a) \\ &= \sum_{i=1}^{\infty} (F_i(b) - F_i(a)) \\ &= \sum_{i=1}^{\infty} \mu_{F_i}([a, b]). \end{aligned}$$

Thus, clearly $\mu_{F_i} << \mu_F$ for all i.

Now, by Lebesgue-RN, there exists some $f \in L^1(m)$ and λ complex measure with $\lambda \perp m$ and $d\mu_F = d\lambda + f dm$.

We apply the same theorem to the μ_i with λ_i, f_i such that $d\mu_{F_i} = d\lambda_i + f_i dm$.

Thus,

$$\mu_F(E) = \lambda(E) + \int_E f dm$$

and similarly,

$$\mu_{F_i}(E) = \lambda_i(E) + \int_E f_i dm.$$

Now, since $[x, x+h]$ shrinks nicely to x as $h \to 0$ by the Lebesgue Differentiation Theorem

$$\lim_{h\to 0} \frac{\mu_F([x,x+h])}{m([x,x+h])} = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x) \qquad m\text{-.a.e.}$$

Thus, $f(x) = F'(x)$ a.e.. Similarly, $f_i = F_i'(x)$ m-a.e. for all i.

Now, we have already used the Dominated Convergence Theorem to swap the integral with the sum and so using what we have so far,

$$\begin{aligned}
\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} &= \lim_{h\to 0} \frac{\sum F_i(x+h) - \sum F_i(x)}{h} \\
&= \lim_{h\to 0} \frac{\sum (F_i(x+h) - F_i(x))}{h} \\
&= \sum \lim_{h\to 0} \frac{F_i(x+h) - F_i(x)}{h} \\
&= \sum f_i(x) \\
&= \sum_{i=1}^{\infty} F_i'(x) \qquad m\text{-a.e.}
\end{aligned}$$

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