Ob Sp. 1

(1) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and in $L^1(\mathbb{R})$. For each of (i) and (ii) give a proof or a counterexample.

(i) Is it true that $f$ is bounded on $\mathbb{R}$?

(ii) Is it true that $f(x) \to 0$ as $x \to \infty$?

How do the results for (i) and (ii) change under the additional assumption that $f'$ exists everywhere and is bounded?

Counterexample for (i) and (ii):

Let $g = \sum_{n=1}^{\infty} 2^n \chi_{[n, n+1/2^{2n}]}$ For $k = 1, 2, 3, \ldots$
$g(k) = 2^k \to \infty$ as $k \to \infty$. Thus $g$ is not bounded on $\mathbb{R}$ and $g(x) \not\to 0$ as $x \to \infty$.
And $\int |g| = \sum_{n=1}^{\infty} \int 2^n \chi_{[n, n+1/2^{2n}]} = \sum_{n=1}^{\infty} 2^n/2^{2n} = \sum_{n=1}^{\infty} 1/2^n < \infty$.
$g$ is not continuous, but define $f$ to be the function which has a triangle under each rectangle of the indicators in $g$, whose height is equal to that of the rectangle.
Then $\int|f| < \int |g| < \infty$, and $f$ unbounded, $f \not\to 0$ as $x \to \infty$.

$\square$

Now, assume $f'$ exists everywhere and is bounded by $M$.
(ii) Assume not: then for some $\varepsilon > 0$, $\forall N \in \mathbb{N}$, $\exists$ $x > N$ s.t. $|f(x)| > \varepsilon$.
Then we have $\int_x^{x+\varepsilon/M} |f(x)| \geq \varepsilon^2/2M$, since MVT will not allow $|f(x)|$ to dip below the triangle of area $\varepsilon^2/2M$.
We can find infinitely many of these situations that are non overlapping, since $f \not\to 0$. Thus $\int |f| \geq \sum^{\infty} \varepsilon^2/2M = \infty$. $\Rightarrow\Leftarrow$

$\square$

(i) By the same logic, $f(x) \to 0$ as $x \to -\infty$.
Thus, $\exists$ $N \in \mathbb{N}$ s.t. $\forall$ $x \notin [-N, N]$, $|f(x)| < 1$.
And $f$ continuous $\Rightarrow$ $f$ bounded on $[-N, N]$, compact.
Hence, $f$ bounded on $\mathbb{R}$.

$\square$

06 Sp. 2

(2) For $y > 0$ define
$$G(y) = \int_0^\infty \frac{1 - e^{-yx^2}}{x^2}\, dx.$$
(a) Show that this integral is finite for all $y > 0$.
(b) Show that $G$ is differentiable, and find an explicit formula for $G'(y)$ and $G(y)$. HINT: You may take as given that $\int_0^\infty e^{-s^2}\, ds = \sqrt{\pi}/2$.

(a) We see that $0 < e^{-yx^2} \le 1 \Rightarrow \frac{1-e^{-yx^2}}{x^2} \le \frac{1}{x^2}$

Thus for $\varepsilon > 0$, $\int_\varepsilon^\infty \frac{1-e^{-yx^2}}{x^2}\,dx \le \int_\varepsilon^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_\varepsilon^\infty = \frac{1}{\varepsilon} < \infty.$

Now, by L'Hopital, we have
$$\lim_{x\to 0} \frac{1-e^{-yx^2}}{x^2} = \lim_{x\to 0} \frac{2yxe^{-yx^2}}{2x} = \lim_{x\to 0} ye^{-yx^2} = y$$

Thus $\exists\ \varepsilon > 0$ s.t. $\frac{1-e^{-yx^2}}{x^2} < 2y \quad \forall\ x \in (0,\varepsilon)$.

Hence, $\int_0^\infty \frac{1-e^{-yx^2}}{x^2}\,dx = \int_0^\varepsilon \frac{1-e^{-yx^2}}{x^2}\,dx + \int_\varepsilon^\infty \frac{1-e^{-yx^2}}{x^2}\,dx \le 2y\varepsilon + \frac{1}{\varepsilon} < \infty.$

$\square$

(b)
$$\begin{aligned}
G'(y) = \lim_{h\to 0} \frac{G(y+h)-G(y)}{h} &= \lim_{h\to 0} \frac{1}{h}\int_0^\infty \frac{1-e^{-(y+h)x^2}}{x^2} - \frac{1-e^{-yx^2}}{x^2}\,dx \\
&= \lim_{h\to 0} \frac{1}{h}\int_0^\infty \frac{e^{-yx^2} - e^{-yx^2}e^{-hx^2}}{x^2}\,dx \\
&= \lim_{h\to 0} \frac{1}{h}\int_0^\infty e^{-yx^2}\frac{1-e^{-hx^2}}{x^2}\,dx \\
&= \int_0^\infty e^{-yx^2} \lim_{h\to 0}\frac{1-e^{-hx^2}}{hx^2}\,dx && \text{by DCT} \\
&= \int_0^\infty e^{-yx^2}\,dx && \text{by L'Hopital} \\
&= \frac{1}{\sqrt{y}}\int_0^\infty e^{-s^2}\,ds && s = \sqrt{y}\,x \\
&= \sqrt{\pi}/2\sqrt{y} && ds = \sqrt{y}\,dx
\end{aligned}$$

$\Rightarrow\ G(y) = \sqrt{\pi y} + C$. And $G(y) \to 0$ as $y \to 0$, by DCT

$\Rightarrow\ G(y) = \sqrt{\pi y}$.

$e^{-yx^2}\frac{1-e^{-hx^2}}{x^2} \le \frac{1-e^{-hx^2}}{x^2} \le \frac{1-e^{-(\sup h)x^2}}{x^2}$ integrable on $(0,\infty)$ by (a),
$\forall\ h$ in the sequence $\Rightarrow$ DCT applies.

$$\lim_{h\to 0} \frac{1-e^{-hx^2}}{hx^2} = \lim_{h\to 0} \frac{x^2e^{-hx^2}}{x^2} = \lim_{h\to 0} e^{-hx^2} = 1.$$

$\square$

06 Sp. 3

(3) Let $(X, \mathcal{M}, \mu)$ be a $\sigma$-finite measure space. Let $f_n, f$ be real-valued measurable functions and suppose $f_n \to f$ a.e. Then there exists a partition of $X$ into disjoint measurable sets $E_0, E_1, E_2, \ldots$ with $\mu(E_0) = 0$ and with $f_n \to f$ uniformly on $E_i$ for each $i \geq 1$. HINT: Egoroff's Theorem requires a finite measure space.

$(X, M, \mu)$ $\sigma$-finite $\Rightarrow$ $\exists$ countable partition $X = X_1 \sqcup X_2 \sqcup \cdots$ s.t. $\mu(X_i) < \infty$ $\forall i$. Thus Egoroff's Thm can be used on an arbitrary $X_i$. $\Rightarrow \exists F_{i,1} \subset X_i$ s.t. $\mu(X_i \setminus F_{i,1}) < 1$ and $f_n \to f$ uniformly on $F_{i,1}$.
Then we repeat: $\exists F_{i,2} \subset X_i \setminus F_{i,1}$ s.t. $\mu(X_i \setminus \bigcup_{k=1}^{2} F_{i,k}) < \frac{1}{2}$ and $f_n \to f$ uniformly on $F_{i,2}$.
This continues, and for arbitrary $j$ we see $\exists F_{i,j} \subset X_i \setminus \bigcup_{k=1}^{j-1} F_{i,k}$ s.t. $\mu(X_i \setminus \bigcup_{k=1}^{j} F_{i,k}) < \frac{1}{j}$ and $f_n \to f$ uniformly on $F_{i,j}$.

Now, define $F_{i,0} := X_i \setminus \bigcup_{k=1}^{\infty} F_{i,k} = \bigcap_{j=1}^{\infty} (X_i \setminus \bigcup_{k=1}^{j} F_{i,k})$ which is a nested intersection. And since $\mu(X_i \setminus F_{i,1}) < \infty$, we can use continuity from above to deduce
$$\mu(F_{i,0}) = \lim_{j \to \infty} \mu(X_i \setminus \bigcup_{k=1}^{j} F_{i,k}) \leq \lim_{j \to \infty} \frac{1}{j} = 0$$
$\Rightarrow \mu(F_{i,0}) = 0$.

Finally, define $E_0 := \bigcup_{i=1}^{\infty} F_{i,0}$, and define $\{E_i\}_1^{\infty}$ to be an enumeration of the countable $\{F_{i,k}\}_{i,k=1}^{\infty}$. Countable additivity gives $\mu(E_0) = 0$, and we've seen that $f_n \to f$ uniformly on each $E_i$, $i \geq 1$.

$\square$

# 06 Sp. 4

(4) A function $g : \mathbb{R} \to \mathbb{R}$ is said to be *lower semi-continuous* if

$$\liminf g(x_n) \geq g(x) \quad \text{whenever } x_n \to x.$$

(a) Suppose that $f_k, k = 1, 2, 3, \ldots$ is a sequence of continuous functions, and $f(x) = \sup_{k\geq 1} f_k(x)$ is finite for all $x$. Show that $f$ is lower semi-continuous.

(b) Show that a lower semi-continuous function is measurable.

(a) Lemma: If $g^{-1}((a,\infty))$ is open $\forall a \in \mathbb{R}$, then $g$ is l.s.c.

Pf: let $x \in \mathbb{R}$. let $x_n \to x$. Assume $g(x) > \liminf g(x_n)$.
Set $a \in \mathbb{R}$ s.t. $\liminf g(x_n) < a < g(x)$.
$g^{-1}((a,\infty))$ open $\Rightarrow$ $g^{-1}((-\infty, a])$ closed.
$\liminf g(x_n) < a \Rightarrow \exists$ subsequence $\{x_{n_i}\}$ s.t. $\lim g(x_{n_i}) < a$
and s.t. $g(x_{n_i}) < a \quad \forall i$. Then $\{x_{n_i}\} \subset g^{-1}((-\infty, a])$
$\Rightarrow \lim x_{n_i} = x \in g^{-1}((-\infty, a]) \Rightarrow g(x) \leq a \Rightarrow\Leftarrow$.

$\square$

Now, consider $f^{-1}((a,\infty))$. Let $x \in f^{-1}((a,\infty))$.
$\Rightarrow f(x) > a \Rightarrow \sup_k f_k(x) > a \Rightarrow \exists\, K$ s.t. $f_K(x) > a$,
$f_K$ cont. $\Rightarrow \exists\, \delta$ s.t. $f_K(y) > a \quad \forall y \in (x-\delta, x+\delta) \Rightarrow f(y) > a \quad \forall$ such $y$
$\Rightarrow (x-\delta, x+\delta) \subset f^{-1}((a,\infty))$. Thus, $f^{-1}((a,\infty))$ open $\Rightarrow f$ l.s.c. $\square$

(b) Claim: if $g$ is l.s.c. $\Rightarrow g^{-1}((-\infty, a])$ closed for $a \in \mathbb{R}$.

Pf: Let $x_n \to x$ in $\mathbb{R}$, with $\{x_n\} \subset g^{-1}((-\infty, a])$.
$\Rightarrow g(x_n) \leq a \quad \forall n$.
$g$ l.s.c. $\Rightarrow g(x) \leq \liminf g(x_n) \leq a$
$\Rightarrow x \in g^{-1}((-\infty, a])$
Thus $g^{-1}((-\infty, a])$ closed, measurable

Hence, $g$ is measurable

$\square$

06 Fa.1

**1.** Find necessary and sufficient conditions for a subset $X \subset \mathbb{R}$ to belong to the $\sigma$-algebra generated by all one-point subsets of $\mathbb{R}$.

Let $\mathcal{M}$ denote the $\sigma$-algebra generated by all one-point sets.
Let $\mathcal{N} = \{X : X \text{ or } X^c \text{ countable}\}$, a set.

Claim: $\mathcal{M} = \mathcal{N}$.

First, we prove $\mathcal{N} \subset \mathcal{M}$. Indeed, if $X$ countable, then $X = \bigcup_1^\infty \{x_i\} \Rightarrow X \in \mathcal{M}$. And if $X^c$ countable, $X^c \in \mathcal{M}$ $\Rightarrow X \in \mathcal{M}$.

Second, we prove $\mathcal{M} \subset \mathcal{N}$. If $\mathcal{N}$ is a $\sigma$-algebra, then $\mathcal{M} \subset \mathcal{N}$ since $\mathcal{M}$ is the smallest $\sigma$-algebra containing all one-point sets.
$\mathcal{N}$ is closed under complement: Let $X \in \mathcal{N}$. If $X$ countable then $X^c \in \mathcal{N}$ since $X = (X^c)^c$ is countable. Otherwise, $X^c$ countable $\Rightarrow$ $X^c \in \mathcal{N}$.
$\mathcal{N}$ is closed under countable union: Let $\{X_i\}_1^\infty \subset \mathcal{N}$.
If $X_i$ countable $\forall i$ then $\bigcup X_i$ countable $\Rightarrow \bigcup X_i \in \mathcal{N}$.
If not, then for some $i$, $X_i^c$ countable.
$\Rightarrow (\bigcup X_i)^c = \bigcap X_i^c$ countable $\Rightarrow \bigcup X_i \in \mathcal{N}$.

$\square$

06 Fa. 2

**2.** Let $(X, \mu)$ be a measure space. Which of the following implications are true?

**a.** $\mu(X) < \infty$ and $f \in L^2(\mu)$ implies $f \in L^1(\mu)$.

**b.** $\mu(X) = \infty$ and $f \in L^2(\mu)$ implies $f \in L^1(\mu)$.

**c.** $\mu(X) < \infty$ and $f \in L^1(\mu)$ implies $f \in L^2(\mu)$.

**d.** $\mu(X) = \infty$ and $f \in L^1(\mu)$ implies $f \in L^2(\mu)$.

Give proof or counter-example in each case.

06 Fa. 3

**3.** Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} e^{-(x-y)} & \text{if } x > y, \\ 0 & \text{if } x = y, \\ -e^{-(y-x)} & \text{if } x < y. \end{cases}$$

**a.** Is $f$ Lebesgue integrable?

**b.** Is it true that

$$\int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x,y)\,dx \right) dy = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(x,y)\,dy \right) dx \ ?$$

06 Fa. 4

**4.** Let $f \in L^1(\mathbb{R})$. Show that for each $n = 1, 2, 3, \ldots$, the function

$$f_n(x) = f(x)(\sin x)^n$$

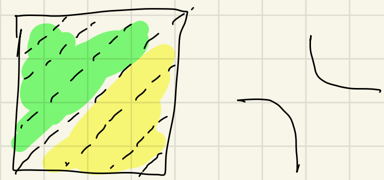
also belongs to $L^1(\mathbb{R})$ and that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = 0.$$

08 sp 1

$E = [0,1] \times [0,1]$

(i) $\displaystyle\int_E \frac{1}{x-y}\, dm(x,y)$



Def $f = \dfrac{1}{x-y}$,

$$\int_E \frac{1}{x-y}\, dm(x,y) = \int_{E,\, x>y} f - \int_{E,\, y>x} -f$$

$$= \infty - \infty$$

$\Rightarrow$ integral does not exist. $\square$

Folland requires at least one of $\int f^+$, $\int f^-$ to be finite.

(ii) $\displaystyle\int_E \frac{1}{x+y}\, dm(x,y)$

Def $f = \dfrac{1}{x+y}$. $f \in L^+(E)$ measurable

$$\Rightarrow \int_E f\, dm(x,y) = \int_0^1 \int_0^1 f\, dx\, dy$$

$$= \int_0^1 \Big[\ln(x+y)\Big]_0^1 dy$$

$$= \int_0^1 \ln(y+1) - \ln(y)\, dy$$

$u = \ln(y+1)$. $dv = dy$

$du = \dfrac{1}{y+1} dy$ $\quad v = y+1$

$$= \Big[(y+1)\ln(y+1) - y\Big]_0^1 - \Big[y\ln(y) - y\Big]_0^1$$

$$= 2\ln(2) \cancel{-1} - \ln(1) \cancel{+0} - \big[\ln(1) \cancel{-1} - \cancel{0\ln(0) + 0}\big]$$

$\lim_{y\to 0} y\ln(y) = 0 \cdot -\infty$ gives $0$

$$= 2(\ln 2 - \ln 1) = 2\ln(2) = \ln(4). \quad \square$$

08 sp. 2

2. Let $f$ be a nonnegative measurable function on [0,1] such that $\int_0^1 f(x)\,dx = 1$. Define a measure $\mu$ on $[0,1]$ by

$$\mu(A) = \int_A f(x)\,dx, \quad A \in \mathcal{B}([0,1]).$$

Let $K$ be the intersection of all compact subsets $E$ of $[0,1]$ such that $\mu(E) = 1$. Find $\mu(K)$.

Folland 4.21: $X$ compact $\iff$ for every family $\{F_\alpha\}_{\alpha \in A}$ of closed sets with the finite intersection property, $\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$.
A family of subsets of $X$ is said to have the finite intersection property if $\bigcap_{\alpha \in B} F_\alpha \neq \emptyset$ $\forall$ finite $B \subset A$.

To see this, look at $[0,1]$ and $(0,1)$.
Family $\{[0, 1/n]\}$ has finite intersection prop. and non empty intersection.
Family $\{(0, 1/n]\}$ in $(0,1)$ has finite intersection prop. but empty intersection.
$[0,1]$ compact, $(0,1)$ not compact.

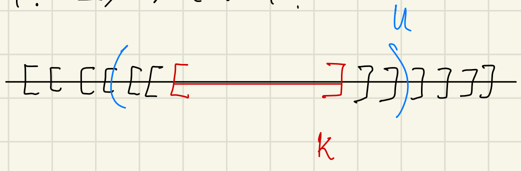
---

$U \supset K$, open. $U^c$ closed $\Rightarrow$ $U^c$ compact.
$\Rightarrow$ any open cover of $U^c$ has a finite subcover.
$U \supset K$ $\Rightarrow$ $U^c \subset K^c = (\bigcap E)^c = \bigcup E^c$, an open cover of $U^c$.
$\Rightarrow$ $\exists$ finite subcover $\{E_i^c\}_1^m$ of $U^c$.
$\bigcup_1^m E_i^c \supset U^c$ $\Rightarrow$ $(\bigcup_1^m E_i^c)^c \subset U$ $\Rightarrow$ $\bigcap_1^m E_i \subset U$.

Now, let $E_1, E_2$ be compact with $\mu(E_1) = \mu(E_2) = 1$.
Then $\mu(E_1 \cap E_2) = \int_{E_1 \cap E_2} f = \int_{X - (E_1^c \cup E_2^c)} f = \int_X f - \int_{E_1^c} f - \int_{E_2^c} f$

$$= 1 - \mu(E_1^c) - \mu(E_2^c) = 1 - 0 - 0 = 1.$$

Induction shows then $\mu(\bigcap_1^m E_i) = 1$. $\Rightarrow$ $\mu(U) = 1$.
$\mu(K) = \inf\{\mu(U) : U \text{ open}, U \supset K\}$
since $\mu$ is a Lebesgue–Stieltjes measure. $\Rightarrow$ $\mu(K) = 1$.

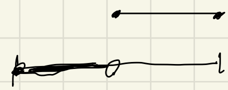



$\Box$

08 sp. 3

$f : [0,1] \to \mathbb{R}$ $\quad$ $f$ cont. a.e. $\overset{?}{\Longleftrightarrow}$ $\exists$ cont. $g$ s.t. $f = g$ a.e.

$\not\Longrightarrow$ Let $f = \begin{cases} 0 & x \in [0, 1/2) \\ 1 & x \in [1/2, 1] \end{cases}$



Let $g : [0,1] \to \mathbb{R}$ be s.t. $f = g$ a.e.
Let $x \in [0, 1/2)$. Assume $g(x) \neq 0$.
$g$ continuous $\Rightarrow$ $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $y \in (x-\delta, x+\delta) \cap [0,1]$
$\Rightarrow$ $g(y) \in (g(x) - \varepsilon, g(x) + \varepsilon)$. Take $\varepsilon = |g(x)|/2$.
Then $g(y) \neq 0$ $\forall\, y \in (x-\delta, x+\delta)$ $\Rightarrow$ $f \neq g$ a.e. $\Rightarrow\Leftarrow$.
So $g(x) = 0$ $\forall\, x \in [0, 1/2)$.
Similarly, $g(x) = 1$ $\forall\, x \in [1/2, 1]$.
That is $g = f$, so $g$ not cont.

$\not\Longleftarrow$ Let $g = 1$. Let $f = \begin{cases} 1 & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$

$m([0,1] \cap \mathbb{Q}) = 0$ $\Rightarrow$ $f = g$ a.e.
But $f$ is continuous nowhere.

Indeed, take $\varepsilon = 1/2$. There is no $\delta > 0$
s.t. $y \in (x-\delta, x+\delta)$ $\Rightarrow$ $f(y) \in (1/2, 3/2)$
or $f(y) \in (-1/2, 1/2)$.

$\square$

# 08 sp. 4

$f \in L^1(\mathbb{R})$. Claim:

$$\lim_{m\to\infty} \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} f(x)dx \right| = \|f\|_{L^1(\mathbb{R})}.$$

$\|f\|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} |f|$, call this value $C$.

e.g. $m=7$ $\quad \left|\int_{-7}^{-48/7} f(x)dx\right| + \left|\int_{-48/7}^{-47/7} f dx\right| + \dots + \left|\int_{7}^{50/7} f dx\right|.$

$$\text{LHS} \le \lim_{m\to\infty} \sum_{k=-m^2}^{m^2} \int_{k/m}^{(k+1)/m} |f(x)|dx$$

$$= \lim_{m\to\infty} \int_{-m}^{(m^2+1)/m} |f(x)|dx = \int_{-\infty}^{\infty} |f| dx = \|f\|_{L^1(\mathbb{R})}.$$

Let $\varepsilon > 0$. Is LHS $>$ RHS $- \varepsilon$?

First, assume $g \in L^1(\mathbb{R})$ cont. and vanishes outside bounded interval.

Define $g_m := \sum_{k=-m^2}^{m^2} m \left| \int_{k/m}^{(k+1)/m} g(x)dx \right| \chi_{[k/m,(k+1)/m)}$

we see $\lim_m \int_{\mathbb{R}} g_m(x)\,dx = \text{LHS}$. (with $f$ replaced by $g$).

Claim: $\lim_m g_m = |g|$.

Let $x \in \mathbb{R}$. $g_m(x) = m \left| \int_{k/m}^{(k+1)/m} g(y)dy \right|$ s.t. $x \in [k/m, (k+1)/m)$.

Assume $g(x) \neq 0$ $\quad g$ cont. $\Rightarrow$ $\exists M$ large enough s.t. $g(y)$ is the same sign as $g(x)$ for all $y \in [k/m, (k+1)/m)$, and all $m > M$.

$\Rightarrow$ $g_m(x) = m \int_{k/m}^{(k+1)/m} |g(y)|dy \quad \forall\, m > M$.

By IVT, for each $m$ here, $\exists\, y_m \in [k/m, (k+1)/m)$ s.t.

$|g(y_m)| = \frac{1}{m}\int_{k/m}^{(k+1)/m} |g(y)|\,dy$. These $y_m \to x$ as $m \to \infty$.

Then $\lim g_m(x) = \lim |g(y_m)| = |g(x)|$ since $g$ cont.

all $g_m \le \sup|g|\, \chi_{\{x : g(x)\neq 0\}} \in L^1(\mathbb{R})$. DCT $\Rightarrow$ result.

Since we have lebesgue measure on $\mathbb{R}$, we can always find such a $g$ s.t. $\int |f-g| < \varepsilon \Rightarrow \int |g| > \int |f| - \varepsilon$.

We know $\exists$ $m$ s.t. $\sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} g(x)\,dx \right| > \int |g| - \varepsilon$.

$$\sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} g(x) \right| - \left| \int_{k/m}^{(k+1)/m} f(x) \right|$$
$$\leq \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} g(x) - f(x)\,dx \right|.$$
$$\leq \sum_{k=-m^2}^{m^2} \int_{k/m}^{(k+1)/m} |g-f|\,dx < \varepsilon.$$
$$\Rightarrow \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} f(x)\,dx \right| > \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} g(x)\,dx \right| - \varepsilon$$

$$\Rightarrow \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} f(x)\,dx \right| > \int |f| - 3\varepsilon.$$

Take $\varepsilon \to 0$, then

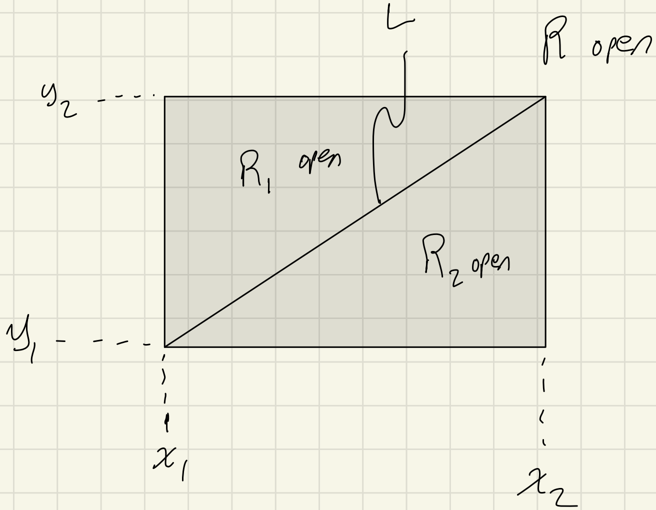
$$\lim_m \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} f(x)\,dx \right| \geq \int |f|$$

$$\Rightarrow \lim_m \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} f(x)\,dx \right| = \int |f|.$$

$\square$

# 08 Fa. 1

(1) Let $\mu, \nu$ be finite Borel measures on $\mathbb{R}^2$ such that $\mu(B) = \nu(B)$ for every open triangular region $B$ in the plane. Show that $\mu(E) = \nu(E)$ for all Borel sets $E$. [Note added later: This is a modified version of the problem actually asked, which was inappropriately difficult, with balls in place of triangles.]



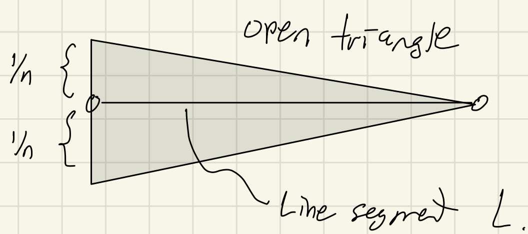

$R = L \sqcup R_1 \sqcup R_2$

$$\mu(R) = \mu(L) + \mu(R_1) + \mu(R_2)$$
$$= \mu(L) + \nu(R_1) + \nu(R_2)$$

So, if $\mu(L) = \nu(L)$ then $\mu(R) = \nu(R)$. And we'd be done since $\mathcal{B}(\mathbb{R}^2)$ generated by rectangles.

So, it's left to show agreement on a line segment.

Let $\{L_n\}$ be a collection of open triangles as here. We see that $L = \bigcap_1^\infty L_n$.




Since, $\mu, \nu$ finite we can use continuity from above to say that $\mu(L) = \lim_n \mu(L_n) = \lim_n \nu(L_n) = \nu(L)$.

$\square$

# 08 F.2

(2) Show that

$$\lim_{n\to\infty} \int_0^\infty \frac{1 + nx^2 + n^2x^4}{(1+x^2)^n}\, dx$$

exists, and determine its value.

Consider $\displaystyle\lim_{n\to\infty} \frac{1+nx^2+n^2x^4}{(1+x^2)^n} = \frac{\infty}{\infty}$, so use L'Hopital.

$$= \lim_{n\to\infty} \frac{x^2 + 2nx^4}{(1+x^2)^n \ln(1+x^2)} = \frac{\infty}{\infty} \quad \text{use L'Hopital again}$$

$$= \lim_{n\to\infty} \frac{x^4}{(1+x^2)^n (\ln(1+x^2))^2} = 0.$$

So if we can bring the limit inside, the value is 0.

Q: is the integrand dominated by some $g \in L^1((0,\infty))$?
If so, DCT tells us we can bring lim inside.

## 08 F. 3

$f : \mathbb{R} \to \mathbb{R}$ bounded (by $P$, say), measurable function, $m$ Lebesgue measure.
Suppose $\exists$ $M > 0$ and $c \in (0,1)$ s.t.

$$m(\{x : |f(x)| \geq t\}) \leq \frac{M}{t^c} \quad \forall t > 0.$$

Claim: $f$ Lebesgue integrable, i.e. $\int_{\mathbb{R}} |f| < \infty$.

Define $f_n = |f| \chi_{\{x : |f(x)| \geq \frac{1}{n}\}}$

$$\int_{\mathbb{R}} f_n(x)\,dx = \int_{\{x : |f(x)| \geq 1/n\}} |f| \leq M n^c P.$$

$$a = m(\{x : |f(x)| \geq 1/3\}) \leq M 3^c$$
$$b = m(\{x : |f(x)| \geq 1/2\}) \leq M 2^c$$

$$m(\{x : 1/3 \leq |f(x)| \not\leq 1/2\}) = \underset{\leq M3^c}{a} - \underset{\leq M2^c}{b} \leq M 3^c - b$$
also $\geq a - M2^c$

e.g. take $M = P = 1$. ($c = 1/2$) Define $f = \begin{cases} 1 & x \in [0,1) \\ 1/2 & [1, \sqrt{2}) \\ 1/3 & [\sqrt{2}, \sqrt{3}) \\ \vdots & \end{cases}$ 0, else

$$\int f = 1 + \frac{\sqrt{2} - 1}{2} + \frac{\sqrt{3} - \sqrt{2}}{3} + \frac{\sqrt{4} - \sqrt{3}}{4} \cdots$$

$$\int_{\{x : |f(x)| \geq 1/n\}} |f| = \int_{\{x : 1/n \leq |f(x)| \not\leq 1/(n-1)\}} |f| + \int_{\{x : |f(x)| \geq 1/(n-1)\}} |f|$$
$$\leq \frac{1}{n-1}\left(M n^c - M(n-1)^c\right) + M(n-1)^c P$$
$$= \frac{M n^c}{n-1} - \frac{M}{(n-1)^{1-c}} + M(n-1)^c P$$

$$\int_{|f|\geq 1} |f| \leq MP$$ max out on $\geq 1$

$$\Rightarrow \int_{|f|\geq 1/2} |f| \leq MP + (M2^c - M)1 = M(P+2^c-1)$$

max out on each layer.

$$\Rightarrow \int_{|f|\geq 1/3} |f| \leq M(P+2^c-1) + (M3^c - M2^c)\tfrac{1}{2}$$
$$= M(P+2^c-1+3^c/2-2^c/2)$$

$$\Rightarrow \int_{|f|\geq 1/4} |f| \leq M(P+2^c-1+3^c/2-2^c/2) + (M4^c - M3^c)\tfrac{1}{3}$$
$$= M(P+2^c-1+3^c/2-2^c/2+4^c/3-3^c/3)$$

Does converge?

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$$\int_{\mathbb{R}} |f(x)|\,dx = \int_{\mathbb{R}} \underbrace{\int_0^{|f(x)|} dt}_{= t\big]_0^{|f(x)|} = |f(x)|}\,dx \quad ?$$ ✓

# 08Fa.4

Define $\quad T_0^1(g) = \sup\limits_{0=x_0\leq x_1\leq\cdots\leq x_n=1} \sum_{i=1}^{n} |g(x_i)-g(x_{i-1})|$

for $g:[0,1]\to\mathbb{R}$. $\quad f_n\to f \quad \forall\, x\in[0,1]$, real-valued.

(i) Claim: $\quad T_0^1(f) \leq \liminf_n T_0^1(f_n)$.

That is, $\quad \sup_{\{x_i\}} \sum_{i=1}^{n} |f(x_i)-f(x_{i-1})| \leq \lim\limits_{K} \inf\limits_{n\geq K} \sup_{\{x_i\}} \sum_{i=1}^{n} |f_n(x_i)-f_n(x_{i-1})|$

09 Sp. 1

a) $(X, \mathcal{B}, \mu)$ $\mu$ finite. $\mathcal{A} \subset \mathcal{B}$ an algebra.

$E \in \mathcal{B}$ approx. from inside by $\mathcal{A}$ if $\forall \varepsilon > 0$, $\exists A \in \mathcal{A}$ with $A \subset E$, $\mu(E \setminus A) < \varepsilon$.

Claim: $\mathcal{C} = \{E \in \mathcal{B} ; E \text{ approx. from inside by } \mathcal{A}\}$ closed under countable unions.

Let $\{E_i\}_1^\infty$ be approx. from inside by $\mathcal{A}$.

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists A_i \in \mathcal{A}$ with $A_i \subset E_i$, $\mu(E_i \setminus A_i) < \varepsilon/2^i$ $\forall i$.

Is $\bigcup_1^\infty E_i$ approx. from inside by $\mathcal{A}$? Fix $\varepsilon > 0$.

Def $F_i = E_i \setminus \bigcup_{j=1}^{i-1} E_j$ so that $\bigcup_1^\infty F_i = \bigcup_1^\infty E_i$ and $F_i$ disjoint.

$$\mu\left(\bigcup_1^\infty F_i\right) = \underbrace{\sum_1^\infty \mu(F_i)}_{\text{call this } M} \leq \mu(X) < \infty.$$

$$\sum_1^\infty \mu(F_i) = \lim_{K \to \infty} \sum_{i=1}^K \mu(F_i) = M$$

$\Rightarrow$ $\exists$ $K \in \mathbb{N}$ s.t. $\sum_{i=1}^K \mu(F_i) > M - \varepsilon$.

Define $A := \bigcup_1^K A_i \in \mathcal{A}$ since $\mathcal{A}$ is an Algebra.

$$\begin{aligned} \mu\left(\bigcup_1^\infty E_i \setminus A\right) &= \sum_1^K \mu(F_i \setminus A_i) + \mu\left(\bigcup_{K+1}^\infty F_i \setminus A\right) \\ &\leq \quad \varepsilon \quad + \quad \varepsilon \\ &= 2\varepsilon \end{aligned}$$

And $A \subset \bigcup_1^\infty E_i$ $\Rightarrow$ $\bigcup_1^\infty E_i$ approx. from inside by $\mathcal{A}$. $\square$

09Sp. 1

b) Find an example which shows $C$ need not be closed under complements.

$C = \{E \in \mathcal{B} ; E \text{ approx from inside by } A\}$

Let $X = [0,1]$, $E = [0,1] \cap \mathbb{Q}$.
With Lebesgue measure, we have $m(E) = 0$.
Let $A := \{\emptyset, X\}$, which is an algebra.
$\forall \varepsilon > 0$, $\emptyset \subset E$ and $m(E \setminus \emptyset) = 0 < \varepsilon$, so
$E$ is approx. from the inside by $A$.
But $E^c = [0,1] \setminus \mathbb{Q}$ has measure $m(E^c) = 1$.
And $X \not\subset E^c$ and $m(E^c \setminus \emptyset) = 1 \not< \varepsilon$ for some $\varepsilon > 0$.
$\Rightarrow$ $E^c$ not approx. from inside by $A$.

$\square$

# 09Sp. 2

$f, g$ abs. cont. on $[a,b]$
$\Rightarrow$ $\forall \varepsilon > 0$, $\exists\ \delta$ s.t. $\forall$ finite unions of disjoint intervals $(a_1, b_1), \dots, (a_N, b_N)$ on $[a,b]$, we have

$$\Sigma_1^N b_i - a_i < \delta \Rightarrow \Sigma_1^N |f(b_i) - f(a_i)| < \varepsilon.$$

This holds for $g$ too. Since $[a,b]$ compact, $f, g$ bounded, say by $M$.

a) Claim: $(fg)$ abs. cont. Induct on $N$.
First, for $N = 1$, $\varepsilon > 0$, we want to show $\exists\ \delta$ s.t. for any interval $(a_0, b_0) \subset [a,b]$, we have

$$b_0 - a_0 < \delta \Rightarrow |(fg)(b_0) - (fg)(a_0)| < \varepsilon.$$

$f, g$ abs. cont. $\Rightarrow$ $\exists\ \delta_f, \delta_g$ s.t. for any interval $(a_0, b_0) \subset [a,b]$

$$b_0 - a_0 < \delta_f \Rightarrow |f(b_0) - f(a_0)| < \varepsilon/2M$$
$$b_0 - a_0 < \delta_g \Rightarrow |g(b_0) - g(a_0)| < \varepsilon/2M$$

Take $\delta = \min\{\delta_f, \delta_g\}$. Then, $b_0 - a_0 < \delta$ gives

$$\begin{aligned}
|(fg)(b_0) - (fg)(a_0)| &= |f(b_0)g(b_0) - f(a_0)g(a_0)| \\
&= |f(b_0)g(b_0) - [f(b_0) - c]\, g(a_0)| \quad \text{for some } |c| < \varepsilon/2M \\
&= |f(b_0)(g(b_0) - g(a_0)) + c\, g(a_0)| \\
&\le |f(b_0)|\,|g(b_0) - g(a_0)| + |c|\,|g(a_0)| \\
&< M \quad \varepsilon/2M \quad\quad \varepsilon/2M \quad M \\
&= \varepsilon.
\end{aligned}$$

Now, assume the result is true for unions of at most $N$ disjoint intervals. Claim: also true for $N+1$.
for $\varepsilon > 0$, $\exists$ $\delta_N$ s.t. for any union of disjoint intervals $(a_1, b_1), \dots, (a_n, b_n)$ on $[a,b]$ with $n \le N$ we have

$$\Sigma_1^n b_i - a_i < \delta_N \Rightarrow \Sigma_1^n |(fg)(b_i) - (fg)(a_i)| < \varepsilon/2.$$

Define $\delta = \delta_N$. Then for any union of disjoint intervals $(a_1, b_1), \dots, (a_{N+1}, b_{N+1})$, with $\Sigma_1^{N+1} b_i - a_i < \delta$, we have $\Sigma_1^N b_i - a_i < \delta$ and $b_{N+1} - a_{N+1} < \delta$, giving

$$\begin{aligned}
\Sigma_1^{N+1} |(fg)(b_i) - (fg)(a_i)| &= \Sigma_1^N |(fg)(b_i) - (fg)(a_i)| + |(fg)(b_{N+1}) - (fg)(a_{N+1})| \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

□

09 Sp. 2

(b) $(fg)$ abs. cont. $\Rightarrow$ $\int_{[a,b]} (fg)' \, dx = (fg)(b) - (fg)(a)$

$$\Rightarrow \int_{[a,b]} fg' + f'g \, dx = f(b)g(b) - f(a)g(a)$$

$$\Rightarrow \int_{[a,b]} fg' \, dx = f(b)g(b) - f(a)g(a) - \int_{[a,b]} f'g \, dx. \qquad \square$$

(c) Let $f$ = devil's staircase let $g = 1$.

Then $f(0) = 0$, $f(1) = 1$, but $f' = 0$ a.e.

Thus,

$$\int_{[0,1]} fg' \, dx = \int_{[0,1]} f \cdot 0 \, dx = 0$$

But

$$f(1)g(1) - f(0)g(0) - \int_{[0,1]} f'g \, dx$$

$$= 1 \cdot 1 - 0 \cdot 1 - \int_{[0,1]} 0 \cdot 1 \, dx$$

$$= 1 \qquad \square$$

09 Sp. 3

$f: \mathbb{R} \to \mathbb{R}$ integrable, $f = 0$ outside $[-1, 1]$.
Define $f_n(x) = f(x + 1/n)$. Must $f_n \to f$ in measure?

$f_n \to f$ in measure if $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \to 0$ as $n \to \infty$, for a given $\varepsilon > 0$.

First, consider when $f = \chi_{(a,b)}$ for $(a,b) \subset [0,1]$.
$f_n \to f$ in $L^1$ $\Rightarrow$ $f_n \to f$ in measure.
Indeed:

$$\begin{aligned}
\int |f_n - f|\,dx &= \int |f(x + 1/n) - f(x)|\,dx \\
&= \int |\chi_{(a-1/n,\, b-1/n)} - \chi_{(a,b)}|\,dx \\
&= \int \chi_{(a-\frac{1}{n},\, a)} + \chi_{(b-\frac{1}{n},\, b)}\,dx, \quad \text{for large } n \\
&= 2/n
\end{aligned}$$

which $\to 0$ as $n \to \infty$.

Now, for arbitrary $f$ integrable,
Since we are using Lebesgue measure, for any $\varepsilon > 0$, we can find a simple function $\phi(x) \leq f(x)$ s.t.

$$\int |f - \phi|\,dx < \varepsilon/3.$$

and s.t. $\phi$ is defined on finitely many intervals $\{(a_i, b_i)\}_1^M$.
Define $\phi_n(x) = \phi(x + 1/n)$. Then $\int |f_n - \phi_n|\,dx < \varepsilon/3$ $\forall\, n$.
And $\phi_n \to \phi$ in $L^1$ since $\phi$ is the sum of $M$ indicator functions on intervals, which we know already converge in $L^1$. $\Rightarrow \int |\phi_n - \phi|\,dx \to 0$ as $n \to \infty$.

Thus, $\exists\, N$ s.t. $\forall\, n \geq N$, $\int |\phi_n - \phi|\,dx < \varepsilon/3$.

Hence, $\forall\, n \geq N$,

$$\begin{aligned}
\int |f_n - f|\,dx &\leq \int |f_n - \phi_n| + |\phi_n - \phi| + |\phi - f|\,dx \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
&= \varepsilon.
\end{aligned}$$

$\Rightarrow f_n \to f$ in $L^1$ $\Rightarrow$ $f_n \to f$ in measure.

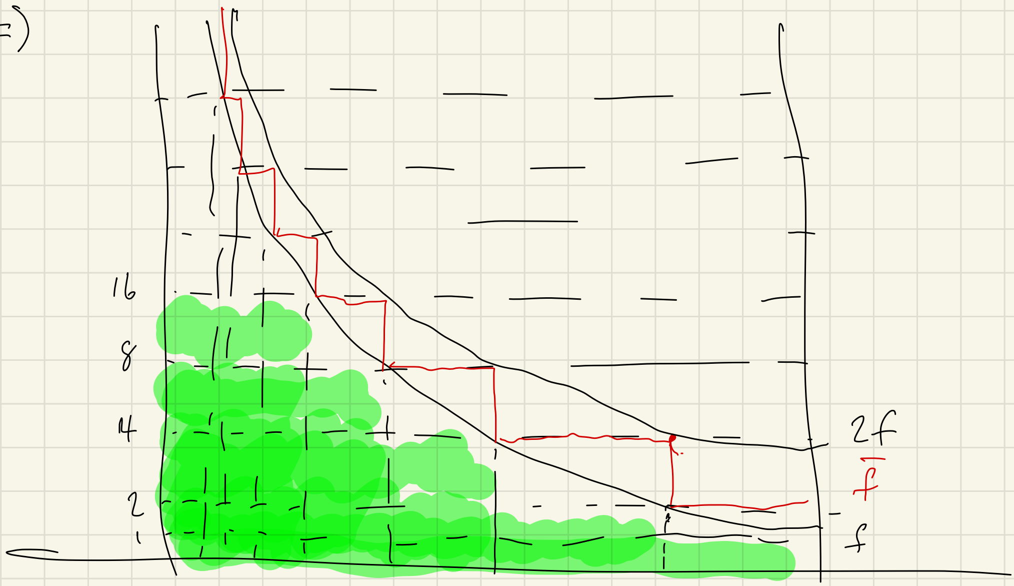
$\square$

# 09 Sp. 4

$\mu(X) < \infty$. $f \geq 0$ measurable on $X$.

Claim: $f$ $\mu$-integrable $\iff \sum_{n=0}^{\infty} 2^n \mu(\{x : f(x) \geq 2^n\}) < \infty$.

($\Leftarrow$)




Define $\bar{f} = 1 + \sum_{n=0}^{\infty} 2^n \chi_{\{x : f(x) \geq 2^n\}}$

We see that for any $x \in X$, we have $2^m \leq f(x) \leq 2^{m+1}$ for some $m = 0, 1, 2, \ldots$

Then $\bar{f}(x) = 1 + \sum_{n=0}^{m} 2^m = 1 + 2^{m+1} - 1 = 2^{m+1}$

$\Rightarrow$ $f(x) \leq \bar{f}(x)$ $\forall x \in X$.

Thus, $\int f(x)dx \leq \int \bar{f}(x)dx = \mu(X) + \sum_{n=0}^{\infty} 2^n \mu(\{x : f(x) \geq 2^n\}) < \infty$

which is to say $f$ is $\mu$-integrable.

($\Rightarrow$) for $x \in X$ with $2^m \leq f(x) \leq 2^{m+1}$, $\bar{f}(x) = 2^{m+1}$

$\Rightarrow$ $\bar{f}(x) \leq 2f(x)$ $\forall x \in X$

$\Rightarrow$ $\sum_{n=0}^{\infty} 2^n \mu(\{x : f(x) \geq 2^n\}) = \int \bar{f}(x)dx - \mu(X) \leq 2\int f(x)dx - \mu(X) < \infty$.

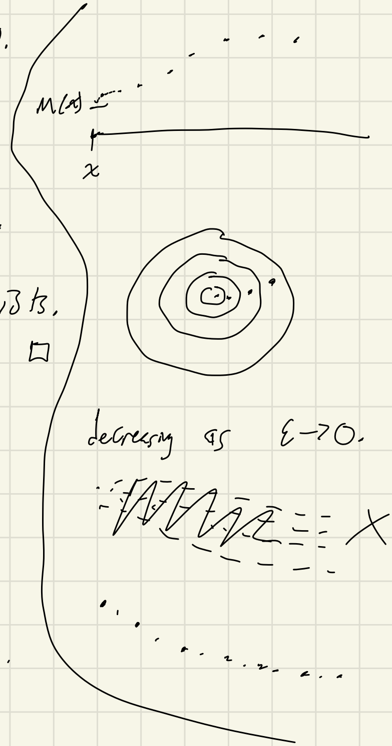
$\square$

# 09 Fa. 1

(1) Let $f$ be a bounded function on $\mathbb{R}^n$ and for $\epsilon > 0$ let $M_\epsilon(x) = \sup_{y:|y-x|<\epsilon} f(y)$.
(a) Show that $M(x) = \lim_{\epsilon \to 0} M_\epsilon(x)$ exists for all $x$.
(b) Show that $M$ is upper semicontinuous, that is, $\limsup_{y \to x} M(y) \le M(x)$.

a) $M_\varepsilon(x) = \sup_{y:|y-x|<\varepsilon} f(y) \geq f(x)$, $\forall \varepsilon > 0$.
and for $\varepsilon_1 > \varepsilon_2 > 0$, we have
$\{y : |y-x| < \varepsilon_1\} \supset \{y : |y-x| < \varepsilon_2\}$
$\Rightarrow M_{\varepsilon_1}(x) \geq M_{\varepsilon_2}(x)$.
That is, $M_\varepsilon(x)$ decreases as $\varepsilon \to 0$.
Thus, since it is also bounded below by $f(x)$ we have $\lim_{\varepsilon \to 0} M_\varepsilon(x)$ exists.

$\square$



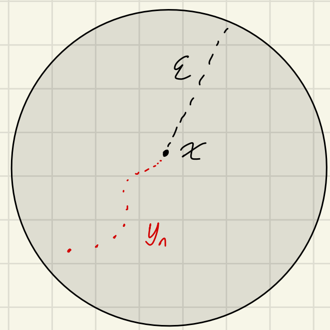

b) Let $y_n \to x$. $\forall \varepsilon > 0$, $\exists N_\varepsilon$ s.t.
$\forall n \geq N_\varepsilon$, $|y_n - x| < \varepsilon$. $\Rightarrow M(y_n) \leq M_\varepsilon(x)$
$\forall n \geq N_\varepsilon$, since $\exists$ ball of $y_n$ inside $\varepsilon$-ball of $x$.
$\limsup_{y_n \to x} M(y_n) = \lim_{N \to \infty} \sup_{y_n : n \geq N} M(y_n) =: L$, say.

For a given $N_\varepsilon$, $L \leq \sup_{y_n : n \geq N_\varepsilon} M(y_n) \leq M_\varepsilon(x)$.

This holds for any $\varepsilon$.

Thus, $L \leq \lim_{\varepsilon \to 0} M_\varepsilon(x) = M(x)$.

$\square$



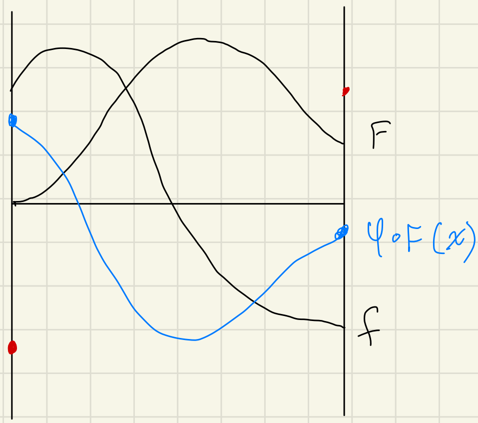
# 09 Fa. 2

(2) Let $m$ be Lebesgue measure on $[0,1]$, suppose $f \in L^1(m)$ and let $F(x) = \int_0^x f(t)\,dt$. Suppose $\varphi$ is a Lipshitz function, that is, for some $M$, $|\varphi(x) - \varphi(y)| \leq M|x-y|$ for all $x, y$. Show that there exists $g \in L^1(m)$ such that $\varphi(F(x)) = \int_0^x g(t)\,dt$.

Maybe $g = \varphi(f(t))$

Does $\varphi(F(x)) = \int_0^x \varphi(f(t))\,dt$ ?

$$\| \quad \varphi\left(\int_0^x f(t)\,dt\right)$$




at 0, $\varphi(F(0)) = \varphi(0)$

$\int_0^0 \varphi(f(t))\,dt = 0$ but $\varphi(0) \neq 0$ nec.

$$\frac{d}{dx}\varphi(F(x)) = F'(x)\,\varphi'(F(x))$$
$$= f(x)\underbrace{\varphi'(F(x))}_{\leq M}$$

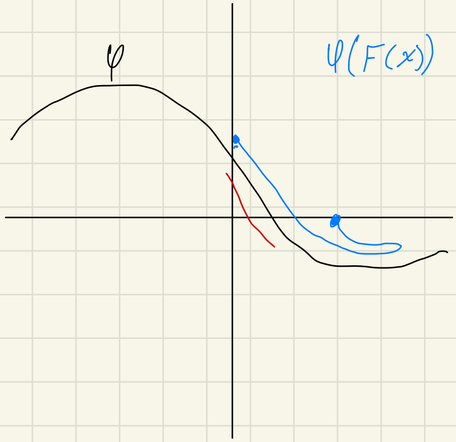
$$\frac{d}{dx}\int_0^x g(t)\,dt = g(x)$$

$$g(x) = f(x)\,\varphi'(F(x)).$$

Use Lipshitz twice.

FTC (b)

$\exists f$ s.t. $F(x) - F(a) = \int_a^x f(t)\,dt$ ☆



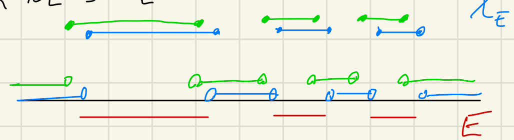
# 09Fa. 3

(3) Let $\chi_E$ denote the indicator function of a set $E$. Suppose $E \subset \mathbb{R}$ has finite Lebesgue measure and define

$$f(x) = \int_{\mathbb{R}} \chi_E(y)\chi_E(y-x)\,dy.$$

Show that $f$ is continuous.

Let $x_n \to x$.
Claim: $f(x_n) \to f(x)$ $\rightsquigarrow$ $\lim_{n\to\infty} \int_{\mathbb{R}} \chi_E(y)\chi_E(y-x_n) = \int_{\mathbb{R}} \chi_E(y)\chi_E(y-x)$ $\Rightarrow$ just need to show lim taken in.
or $|f(x_n) - f(x)| \to 0$.



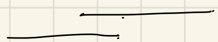

$$\chi_E(y-x) = \chi_{E+x}(y).$$

$$\chi_E(y)\,\chi_{E+x}(y) = \chi_{E\cap(E+x)}(y)$$

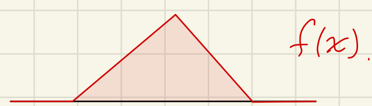
$$\Rightarrow \int_{\mathbb{R}} \chi_E(y)\chi_E(y-x)\,dy = \int_{\mathbb{R}} \chi_{E\cap(E+x)}(y)\,dy = \mu(E\cap(E+x)).$$

True for interval $(a,b)$:

$$f(x) = \mu((a,b)\cap(a+x,\, b+x))$$
$$= b-(a+x) = (b-a)-x \quad \text{OR } 0,$$
$$= \max\{(b-a)-x,\ 0\}.$$



which is continuous.




$E$ is just a countable union of open intervals. ?

09Fa.4

(4) Let $m$ be Lebesgue measure on $\mathbb{R}$ and let $f_n, f \in L^1(m)$. Suppose there is a constant $C$ such that $||f_n - f||_1 \leq \frac{C}{n^2}$ for all $n \geq 1$. Show that $f_n \to f$ a.e. HINT: Consider the sets

$$\{x : |f_n(x) - f(x)| > \epsilon \text{ for some } n \geq N\}.$$

Def $E(N,\varepsilon) = \{ x : |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq N \}$.

Def $F = \{ x : f_n(x) \not\to f(x) \}$
$= \{ x : \text{for some } \varepsilon, \forall N, \exists\ n \geq N \text{ s.t. } |f_n(x) - f(x)| \geq \varepsilon \}$
$= \bigcup_\varepsilon \{ x : \forall N, |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq N \}$

For a given $\varepsilon > 0$, $m(\underbrace{\{x : \forall N, |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq N\}}_{\ast}) = 0$.

For, consider when this set is not null. Say $m(\ast) > 0$.

Then $\forall N, \exists\ n \geq N$ s.t. $\int |f_n(x) - f(x)|\, dx \geq \int_\ast |f_n(x) - f(x)|\, dx$
$\geq \int_\ast \varepsilon\, dx$
$= \varepsilon m(\ast)$

But for $n$ large enough, $C/n^2 < \varepsilon m(\ast)$.

$\limsup_n$

$$\sum_{n=K}^{\infty} \underbrace{m(|f_n - f| > \varepsilon)}_{\leq \frac{C}{n^2}}$$

$\boxed{\text{Q: Why } n^2\text{?}}$

Borel-Catelli

10 Sp. 1

(1) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *upper semicontinuous* (or *u.s.c.*) if for all $x \in \mathbb{R}$ and all $\epsilon > 0$ there exists $\delta > 0$ such that $f(y) < f(x) + \epsilon$ whenever $|y - x| < \delta$.

(i) Show that every u.s.c. function is Borel measurable. HINT: Consider $\{x : f(x) < a\}$.

(ii) Suppose $\mu$ is a finite measure on $\mathbb{R}$ and $A$ is a closed subset of $\mathbb{R}$. Using (i) or otherwise, show that the function $x \mapsto \mu(x + A)$ is measurable. Here $x + A = \{x + y : y \in A\}$.

(i) Consider $E_a := \{x . f(x) < a\} = f^{-1}((-\infty, a))$

Let $x \in E_a$. Take $\varepsilon = (a - f(x))/2 > 0$.

$f$ u.s.c. $\Rightarrow$ $\exists$ $\delta > 0$ s.t. $f(y) < f(x) + \varepsilon < a$ whenever $|y - x| < \delta$.

which is to say $y \in E_a$ as well.

Thus $E_a$ is open and therefore Borel measurable

$\Rightarrow$ $f$ is Borel measurable. $\square$

(ii) Define $f(x) := \mu(x + A) = \mu(\mathbb{R}) - \mu((x+A)^c) = \mu(\mathbb{R}) - \mu(x + A^c)$.

Claim: $f$ is u.s.c., which implies $f$ is Borel measurable by (i).

Let $x \in \mathbb{R}$, $\varepsilon > 0$.

105p.2

(2) Suppose $\{f_n\}$ and $f$ are measurable functions on $(X, \mathcal{M}, \mu)$ and $f_n \to f$ in measure. Is it necessarily true that $f_n^2 \to f^2$ in measure if:

(a) $\mu(X) < \infty$

(b) $\mu(X) = \infty$.

In each case, prove or give a counterexample.

$f_n \to f$ in measure if $\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) \longrightarrow 0$ as $n \to \infty$, $\forall \varepsilon > 0$.

$$|f_n(x) - f(x)|^2 = f_n(x)^2 - 2f_n(x)f(x) + f(x)^2$$

NTS $\mu(\{x : |f_n(x)^2 - f(x)^2| > \varepsilon\}) \to 0$ as $n \to \infty$, $\forall \varepsilon > 0$.

(a) $\mu(X) < \infty \Rightarrow \mu(\{x : |f_n(x) - f(x)| \leq \varepsilon\}) \longrightarrow \mu(X)$ as $n \to \infty$, $\forall \varepsilon > 0$.

$\Rightarrow \exists N$ s.t. $\forall n \geq N$, $\mu(\{x : |f_n(x) - f(x)| \leq \varepsilon\}) > \mu(X) - \varepsilon'$. $\forall \varepsilon' > 0$.

$100 - 99.5 = .5$

$100^2 - 99.5^2 = 10000 - 9900.25 = 100.$

if $|f_n(x) - f(x)| < 1$ then $0 < f_n(x)^2 - 2f_n(x)f(x) + f(x)^2 < 1$

$f_n(x)^2 - f(x)^2 < 2f_n(x)f(x) - 2f(x)^2 + 1$

$= 2f(x)\underbrace{(f_n(x) - f(x))}_{< \varepsilon} + 1$

10Sp. 3

(3) Suppose $f : [0,1] \to \mathbb{R}$ is a strictly increasing absolutely continuous function. Let $m$ denote Lebesgue measure. If $m(E) = 0$ show that $m(f(E)) = 0$.

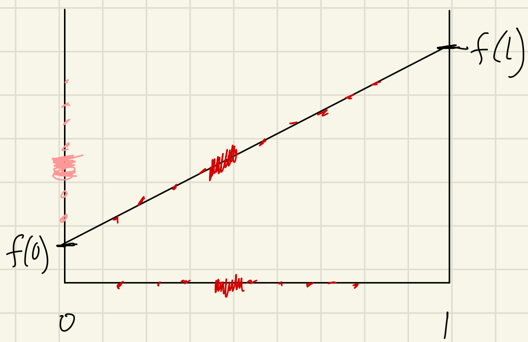
$f$ strictly increasing if $\forall\, x, y \in [0,1]$ with $x < y$, we have $f(x) < f(y)$.

$f$ abs. cont. if $\forall\, \varepsilon > 0$ $\exists\, \delta$ s.t. for any finite collection disjoint intervals $\{(x_i, y_i)\}_1^n$ the following holds;

$$\sum_1^n |y_i - x_i| < \delta \implies \sum_1^n |f(y_i) - f(x_i)| < \varepsilon.$$

$\iff$ $f(x) = f(0) + \int_0^x f'(t)\,dt$ by Fund. Thm.

$$\begin{aligned} f(E) &= \{f(x) : x \in E\} \\ &= \{f(0) + \textstyle\int_0^x f'(t)\,dt : x \in E\} \\ &= f(0) + \{\textstyle\int_0^x f'(t)\,dt : x \in E\} \\ m(f(E)) &= m(f(0) + \{\textstyle\int_0^x f'(t)\,dt : x \in E\}) \\ &= m(\{\textstyle\int_0^x f'(t)\,dt : x \in E\}). \end{aligned}$$



10Sp. 4

(4) For $n \geq 1$ define $h_n$ on $[0,1]$ by

$$h_n = \sum_{j=1}^{n} (-1)^j \chi_{(\frac{j-1}{n}, \frac{j}{n}]}.$$

Here $\chi_E$ denotes the characteristic function of $E$. If $f$ is Lebesgue integrable on $[0,1]$, show that

$$\lim_{n\to\infty} \int_{[0,1]} f h_n \, dm = 0.$$

HINT: First consider $f$ in a suitably smaller function space.

First consider simple functions defined on intervals, i.e. a function $\phi = \sum_1^K \alpha_i \chi_{A_i}$ where $\{A_i\}_1^K$ are disjoint intervals on $[0,1]$.

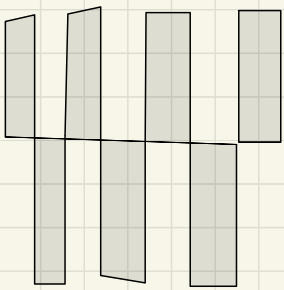
Then, $\int_0^1 \phi h_n \, dm = \sum_{i=1}^K \int_0^1 \alpha_i \chi_{A_i} h_n \, dm$

$= \sum_{i=1}^K \alpha_i \int_{A_i} h_n \, dm$

Consider $\left| \int_{A_i} h_n \, dm \right| < 2/n$

$\Rightarrow \int_{A_i} h_n dm \to 0$ as $n \to \infty$.

$\Rightarrow \int_0^1 \phi h_n \, dm \to 0$ as $n \to \infty$.



12Sp.1

1. Let $f$ and $g$ be real integrable functions on a $\sigma$-finite measure space $(X, \mathcal{M}, \mu)$, and for $t \in \mathbb{R}$ let

$$F_t = \{x \in \cancel{E}\ X : f(x) > t\} \quad \text{and} \quad G_t = \{x \in \cancel{E}\ X : g(x) > t\}.$$

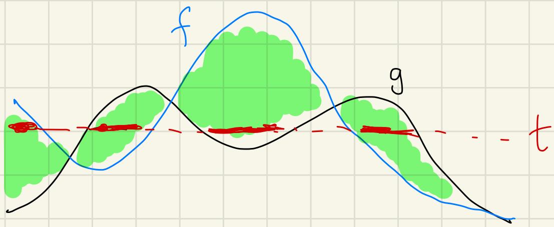
Show that

$$\int_X |f-g| d\mu = \int_{-\infty}^{\infty} \mu\left((F_t \backslash G_t) \cup (G_t \backslash F_t)\right) dt.$$

$$F_t \setminus G_t = \{x \in X : g(x) \le t < f(x)\}$$
$$G_t \setminus F_t = \{x \in X : f(x) \le t < g(x)\} \qquad \text{Disjoint.}$$

$$\Rightarrow \mu((F_t \setminus G_t) \cup (G_t \setminus F_t)) = \mu(F_t \setminus G_t) + \mu(G_t \setminus F_t).$$

$$LHS = \int_{f>g} f-g \, d\mu + \int_{g>f} g-f \, d\mu$$




$$NTS \quad \int_{f>g} f-g \, d\mu = \int_{-\infty}^{\infty} \mu(F_t \setminus G_t)\, dt$$

$$\mu(F_t \setminus G_t) = \int_X \chi_{F_t \setminus G_t} \, d\mu$$

$$\Rightarrow \int_{-\infty}^{\infty} \mu(F_t \setminus G_t)\, dt = \int_{-\infty}^{\infty} \int_X \chi_{F_t \setminus G_t} \, d\mu \, dt$$

$$= \int_X \int_{-\infty}^{\infty} \chi_{F_t \setminus G_t} \, dt \, d\mu \qquad \text{by Fubini}$$

$$= \int_{f>g} \int_{g(x)}^{f(x)} dt \, d\mu$$

$$= \int_{f>g} f-g \, d\mu.$$

125p.2

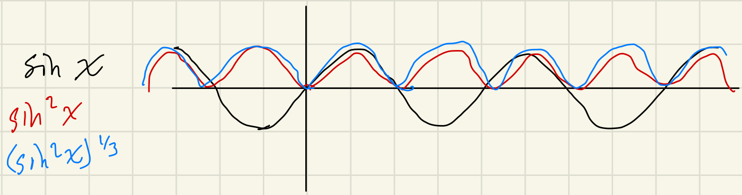
2. Show that

$$\int_{\pi}^{\infty} \frac{dx}{x^2(\sin^2 x)^{1/3}}$$

is finite.

?

$0 \leq \sin^2 x \leq 1 \quad \forall x. \qquad 0 \leq (\sin^2 x)^{1/3} \leq 1 \quad \forall x.$




$(\sin^2 x)^{1/3} \geq \sin^2 x \quad \forall x$ because $t^{1/3} \geq t$ for $t$ on $[0,1]$ and $\sin^2 x$ on $[0,1]$ $\forall x$.

$$\Rightarrow \frac{1}{x^2(\sin^2 x)^{1/3}} \leq \frac{1}{x^2 \sin^2 x} = \left(\frac{1}{x \sin x}\right)^2$$

$$\lim_{x \to \infty} \frac{1}{x^2(\sin^2 x)^{1/3}}$$

125 p.3

3. A collection of functions $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1(\mu)$ on the measure space $(X,\mathcal{M},\mu)$ is said to be *uniformly integrable* if

$$\lim_{M\to\infty} \sup_{\alpha\in\mathcal{A}} \int_{\{x:|f_\alpha(x)|>M\}} |f_\alpha| = 0.$$

**a**. Prove that if $f \in L^1$ then $\{f\}$ is uniformly integrable.

**b**. Prove that if $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{f_\beta\}_{\beta\in\mathcal{B}}$ are two collections of uniformly integrable functions then $\{f_\gamma\}_{\gamma\in\mathcal{A}\cup\mathcal{B}}$ is uniformly integrable.

**c**. Show that if $\mu(X) < \infty$ and $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1(\mu)$ is uniformly integrable then

$$\sup_{\alpha\in\mathcal{A}} \int |f|d\mu < \infty.$$

Give an example to show that the conclusion fails without the condition $\mu(X) < \infty$.

**d**. Again let $\mu(X) < \infty$ and suppose $\{f_n\}_{n=0}^\infty \subset L^1(\mu)$ such that $f_n \to f_0$ a.e. and $\int |f_n|d\mu \to \int |f_0|d\mu$. Prove that $\{f_n\}_{n=0}^\infty$ is uniformly integrable. Hint: Consider some $\phi_M$, a continuous, bounded function on $[0,\infty)$, equal to 0 on $[M,\infty)$, for which $|t|\mathbf{1}\{|t|>M\} \le |t| - \phi_M(|t|)$.

a) NTS $\lim_{M\to\infty} \int_{\{x: |f|>M\}} |f| = 0$ for $f \in L^1$.

We have for any $M \in \mathbb{N}$:

$$\int |f| = \sum_{k=1}^{\infty} \int_{\{x:\, k-1<|f|\le k\}} |f|$$
$$= \sum_{k=1}^{M} \int_{\{x:\, k-1<|f|\le k\}} |f| + \int_{\{x:|f|>M\}} |f|$$

$$\Rightarrow \lim_{M\to\infty} \int_{\{x:|f|>M\}} |f| = \lim_{M\to\infty} \left( \int |f| - \sum_{k=1}^{M} \int_{\{x:\, k-1<|f|\le k\}} |f| \right)$$
$$= \int |f| - \sum_{k=1}^{\infty} \int_{\{x:\, k-1<|f|\le k\}} |f|$$
$$= \int |f| - \int |f|$$
$$= 0.$$

$\square$

3. A collection of functions $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1(\mu)$ on the measure space $(X, \mathcal{M}, \mu)$ is said to be *uniformly integrable* if

$$\lim_{M\to\infty} \sup_{\alpha\in\mathcal{A}} \int_{\{x:|f_\alpha(x)|>M\}} |f_\alpha| = 0.$$

uniformly integrable

**b**. Prove that if $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{f_\beta\}_{\beta\in\mathcal{B}}$ are two collections of ~~uniformly integrable~~ functions then $\{f_\gamma\}_{\gamma\in\mathcal{A}\cup\mathcal{B}}$ is uniformly integrable.

b)
$$\lim_{M\to\infty} \sup_{\gamma\in A\cup B} \int_{\{x:|f_\gamma(x)|>M\}} |f_\gamma|$$
$$\leq \lim_{M\to\infty} \max\left\{ \sup_{\alpha\in A} \int_{\{x:|f_\alpha(x)|>M\}} |f_\alpha| , \sup_{\beta\in B} \int_{\{x:|f_\beta(x)|>M\}} |f_\beta| \right\}$$
$$\leq \lim_{M\to\infty} \sup_{\alpha\in A} \int_{\{x:|f_\alpha(x)|>M\}} |f_\alpha| + \sup_{\beta\in B} \int_{\{x:|f_\beta(x)|>M\}} |f_\beta|$$
$$= 0 + 0 = 0.$$

Thus $\displaystyle\lim_{M\to\infty} \sup_{\gamma\in A\cup B} \int_{\{x:|f_\gamma(x)|>M\}} |f_\gamma| = 0.$ $\square$

3. A collection of functions $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1(\mu)$ on the measure space $(X, \mathcal{M}, \mu)$ is said to be *uniformly integrable* if

$$\lim_{M\to\infty} \sup_{\alpha\in\mathcal{A}} \int_{\{x:|f_\alpha(x)|>M\}} |f_\alpha| = 0.$$

**c.** Show that if $\mu(X) < \infty$ and $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1(\mu)$ is uniformly integrable then

$$\sup_{\alpha\in\mathcal{A}} \int |\underset{f_\alpha}{f}| d\mu < \infty.$$

Give an example to show that the conclusion fails without the condition $\mu(X) < \infty$.

c) $\displaystyle\int |f_\alpha|\, d\mu = \int_{\{x:|f_\alpha|\le M\}} |f_\alpha|\, d\mu + \int_{\{x:|f_\alpha|>M\}} |f_\alpha|\, d\mu$

for any $M \in \mathbb{N}$. Thus, we have $\forall M$:

$$\begin{aligned}
\sup_\alpha \int |f_\alpha|\, d\mu &= \sup_\alpha \left( \int_{\{x:|f_\alpha|\le M\}} |f_\alpha|\, d\mu + \int_{\{x:|f_\alpha|>M\}} |f_\alpha|\, d\mu \right) \\
&\le \sup_\alpha \int_{\{x:|f_\alpha|\le M\}} |f_\alpha|\, d\mu + \sup_\alpha \int_{\{x:|f_\alpha|>M\}} |f_\alpha|\, d\mu \\
&\le \mu(X) M + \underbrace{\sup_\alpha \int_{\{x:|f_\alpha|>M\}} |f_\alpha|\, d\mu}_{\text{this is finite.}}
\end{aligned}$$

$\{f_\alpha\}$ uniformly integrable $\Rightarrow$ $\exists$ $M$ s.t.

Hence $\sup_\alpha \int |f_\alpha|\, d\mu$ is finite.

$\square$

Let $f_n = \chi_{[n,2n]}$

$$\lim_{M\to\infty} \sup_n \int_{\{x:|f_n(x)|>M\}} |f_n| = 0$$ since

$$\sup_n \int_{\{x:|f_n(x)|>M\}} |f_n| = 0 \quad \forall\ M \ge 1.$$

But $\displaystyle \sup_n \int |f_n|\, d\mu = \sup_n n = \infty.$

3. A collection of functions $\{f_\alpha\}_{\alpha \in \mathcal{A}} \subset L^1(\mu)$ on the measure space $(X, \mathcal{M}, \mu)$ is said to be *uniformly integrable* if

$$\lim_{M \to \infty} \sup_{\alpha \in \mathcal{A}} \int_{\{x : |f_\alpha(x)| > M\}} |f_\alpha| = 0.$$

**d**. Again let $\mu(X) < \infty$ and suppose $\{f_n\}_{n=0}^{\infty} \subset L^1(\mu)$ such that $f_n \to f_0$ a.e. and $\int |f_n| d\mu \to \int |f_0| d\mu$. Prove that $\{f_n\}_{n=0}^{\infty}$ is uniformly integrable. Hint: Consider some $\phi_M$, a continuous, bounded function on $[0, \infty)$, equal to 0 on $[M, \infty)$, for which $|t| \mathbf{1}\{|t| > M\} \le |t| - \phi_M(|t|)$.

d)

$$\int_0^1 x^{-1/2} = 2x^{1/2}\Big]_0^1 = 2$$

Must $\int |f_0| d\mu < \infty$ ?

$$\int_0^1 x^{-n/n+1} = (n+1) x^{1/n+1}\Big]_0^1 = n+1$$

$$f_n(x) = x^{-n/n+1}$$

$\phi_M$

$$f_n(x) \to \frac{1}{x} = f(x) \quad \text{a.e.} \quad \checkmark$$

$$\int_0^1 |f_n| \to \int_0^1 |f|$$
$$= n+1 \to = \infty \qquad \text{as } n \to \infty, \checkmark$$

$$x^{-n/n+1} = M$$
$$x = M^{-n+1/n}$$

$$\lim_{M \to \infty} \underbrace{\sup_n \int_{\{x : |f_n(x)| > M\}} |f_n(x)|}$$ =

$$= \sup_n \int_{\{x : |f_n(x)| > M\}} x^{-n/n+1} = \sup_n \int_0^{M^{-n+1/n}} x^{-n/n+1}$$
$$= \sup_n (n+1) x^{1/n+1}\Big]_0^{M^{-n+1/n}}$$
$$= \sup_n (n+1) M^{-1/n}$$
$$= \infty$$

$$\lim_{M \to \infty} \sup_{\alpha \in A} \int \phi_M(f_n) \to \int \phi_M(f_0)$$

DCT b/c bounded $\phi_M$ bounded

3. A collection of functions $\{f_\alpha\}_{\alpha \in \mathcal{A}} \subset L^1(\mu)$ on the measure space $(X, \mathcal{M}, \mu)$ is said to be *uniformly integrable* if

$$\lim_{M\to\infty} \sup_{\alpha\in\mathcal{A}} \int_{\{x:|f_\alpha(x)|>M\}} |f_\alpha| = 0.$$

**d**. Again let $\mu(X) < \infty$ and suppose $\{f_n\}_{n=0}^\infty \subset L^1(\mu)$ such that $f_n \to f_0$ a.e. and $\int |f_n| d\mu \to \int |f_0| d\mu$. Prove that $\{f_n\}_{n=0}^\infty$ is uniformly integrable. Hint: Consider some $\phi_M$, a continuous, bounded function on $[0,\infty)$, equal to 0 on $[M,\infty)$, for which $|t|\mathbf{1}\{|t| > M\} \leq |t| - \phi_M(|t|)$.

$$\phi_M(|t|) \leq |t|(1 - \mathbb{1}_{\{|t|>M\}}) = |t|\,\mathbb{1}_{\{|t|\leq M\}}$$

d) First, assume $f_0 \in L^1(\mu)$.

$\exists\ N$ s.t. $\forall\ n \geq N$, $\int |f_n|\, d\mu < \int |f_0|\, d\mu + \varepsilon$

By parts (a) and (b), using induction, we see that $\{f_n\}_{n=0}^N$ is uniformly integrable.

125p. 4

4. Let $\mathbb{M}$ be the collection of all finite measures on the measure space $(X, \mathcal{M})$.

**a**. Show that

$$d(\nu, \lambda) = 2 \sup_{E \in \mathcal{M}} |\nu(E) - \lambda(E)|$$

defines a metric on $\mathbb{M}$.

**b**. For any $\mu \in \mathbb{M}$ that dominates measures $\nu$ and $\lambda$ in $\mathbb{M}$ with $\nu(X) = \lambda(X) = 1$, let

$$p = \frac{d\nu}{d\mu} \quad \text{and} \quad q = \frac{d\lambda}{d\mu}.$$

Prove

$$d(\nu, \lambda) = \int |p(x) - q(x)| \, d\mu = 2\left(1 - \int (\min\{p(x), q(x)\}) \, d\mu\right).$$

Hint: notice that $\mu(E) - \lambda(E) = \lambda(E^c) - \nu(E^c)$.

a) • $d(\nu, \lambda) \geq 0$ ✓

• $\nu = \lambda \Rightarrow \quad d(\nu, \lambda) = 2 \sup |\nu(E) - \lambda(E)| = 2 \sup \{0\} = 0$. ✓

$d(\nu, \lambda) = 0 \Rightarrow \quad 2 \sup_E |\nu(E) - \lambda(E)| = 0$

$\Rightarrow \quad \nu(E) = \lambda(E) \quad \forall\, E \in \mathcal{M}$ ✓

• $d(\nu, \lambda) = d(\lambda, \nu)$ since $|\nu(E) - \lambda(E)| = |\lambda(E) - \nu(E)|$. ✓

• NTS $d(\nu, \mu) \leq d(\nu, \lambda) + d(\lambda, \mu)$

$2 \sup |\nu(E) - \mu(E)| \leq 2 \sup |\nu(E) - \lambda(E)| + 2 \sup |\lambda(E) - \mu(E)|$ ?

Yes, b/c $|\nu(E) - \mu(E)| \leq |\nu(E) - \lambda(E)| + |\lambda(E) - \mu(E)|$ ✓

□

**b**. For any $\mu \in \mathbb{M}$ that dominates measures $\nu$ and $\lambda$ in $\mathbb{M}$ with $\nu(X) = \lambda(X) = 1$, let

$$p = \frac{d\nu}{d\mu} \quad \text{and} \quad q = \frac{d\lambda}{d\mu}.$$

Prove

$$d(\nu, \lambda) = \int |p(x) - q(x)| \, d\mu = 2\left(1 - \int (\min\{p(x), q(x)\}) \, d\mu\right).$$

Hint: notice that $\mu(E) - \lambda(E) = \lambda(E^c) - \nu(E^c)$.

b)

13 Sp. 1

1. Suppose that $\{f_n\}$ is a sequence of of real valued continuously differentiable functions on $[0,1]$ such that

$$\lim_{n\to\infty}\int_0^1 |f_n(x)|\,dx = 0 \text{ and } \lim_{n\to\infty}\int_0^1 |f_n'(x)|\,dx = 0.$$

Show that $\{f_n\}$ converges to 0 uniformly on $[0,1]$.

Let $\varepsilon > 0$.

Let $N$ be large enough so that $\forall\, n \geq N$,

$$\int_0^1 |f_n(x)|\,dx < \varepsilon/2 \text{ AND } \int_0^1 |f_n'(x)|\,dx < \varepsilon/2.$$

Assume $\exists\; n \geq N,\; x \in [0,1]$ s.t. $|f_n(x)| \geq \varepsilon$.

$\forall\; y \leq x \in [0,1]$, we have

$$|f_n(x) - f_n(y)| = \left|\int_y^x f_n(t)\,dt\right| \leq \int_y^x |f_n(t)|\,dt < \varepsilon/2.$$

and $\forall\; y \geq x \in [0,1]$, we have

$$|f_n(y) - f_n(x)| = \left|\int_x^y f_n(t)\,dt\right| \leq \int_x^y |f_n(t)|\,dt < \varepsilon/2.$$

$\Rightarrow$ $|f_n(y)| > \varepsilon/2 \quad \forall\; y \in [0,1]$.

$\Rightarrow$ $\int_0^1 |f_n(x)|\,dx > \varepsilon/2$, a contradiction.

Thus, assumption is wrong.

$\Rightarrow$ $\forall\; n \geq N,\; x \in [0,1],\quad |f_n(x)| < \varepsilon$.

Hence, we have shown $f_n \to 0$ uniformly on $[0,1]$.

$\square$

13 Sp. 2

2. Investigate the convergence of $\sum_{n=0}^{\infty} a_n$, where

$$a_n = \int_0^1 \frac{x^n}{1-x} \sin(\pi x)\, dx.$$

$$\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} a_n = \lim_{N\to\infty} \sum_{n=0}^{N} \int_0^1 \frac{x^n}{1-x} \sin(\pi x)\, dx$$

$$= \lim_{N\to\infty} \int_0^1 \sum_{n=0}^{N} \frac{x^n}{1-x} \sin(\pi x)\, dx$$

* DCT $\Rightarrow$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{x^n}{1-x} \sin(\pi x)\, dx$$

$$= \int_0^1 \frac{1}{(1-x)^2} \sin(\pi x)\, dx$$

$$\sin(\pi x) = (\pi x) - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \cdots$$

## Sp 13.3

3. Let $(X, \mathcal{M}, \mu)$ be a measure space, $f_n, f \in L^1(\mu)$. Show that $\int_X |f_n - f|\, d\mu \to 0$ as $n \to \infty$ if and only if

$$\sup_{A \in \mathcal{M}} \left| \int_A f_n d\mu - \int_A f d\mu \right| \to 0$$

as $n \to \infty$.

$(\Longrightarrow)$:

$\forall A \in \mathcal{M}$,
$$\begin{aligned} \left| \int_A f_n \, d\mu - \int_A f \, d\mu \right| &= \left| \int_A f_n - f \, d\mu \right| \\ &\leq \int_A |f_n - f| \, d\mu \\ &\leq \int_X |f_n - f| \, d\mu \end{aligned}$$

$$\Rightarrow \sup_{A \in \mathcal{M}} \left| \int_A f_n d\mu - \int_A f d\mu \right| \leq \int_X |f_n - f| d\mu \longrightarrow 0 \text{ as } n \to \infty. \quad \square$$

$(\Longleftarrow)$: $\sup_{A \in \mathcal{M}} \left| \int_A f_n d\mu - \int_A f d\mu \right| \longrightarrow 0$ as $n \to \infty$

$\Rightarrow \exists\ N$ s.t. $\forall\ n \geq N$, $\sup_{A \in \mathcal{M}} \left| \int_A f_n d\mu - \int_A f d\mu \right| < \varepsilon$.

$\Rightarrow \left| \int_{\{x: f_n > f\}} f_n \, d\mu - \int_{\{x: f_n > f\}} d\mu \right|$

$= \left| \int_{\{x: f_n > f\}} f_n - f \, d\mu \right| = \int_{\{x: f_n > f\}} f_n - f \, d\mu < \varepsilon$.

And $\left| \int_{\{x: f_n \leq f\}} f_n - f \, d\mu \right| = \int_{\{x: f_n \leq f\}} f - f_n \, d\mu < \varepsilon$.

$$\Rightarrow \int_X |f_n - f| \, d\mu = \int_{\{x: f_n > f\}} f_n - f \, d\mu + \int_{\{x: f_n \leq f\}} f - f_n \, d\mu$$
$$< 2\varepsilon.$$

Thus $\int_X |f_n - f| \, d\mu \longrightarrow 0$ as $n \to \infty$. $\square$

13 Sp. 4

4. Let $\mu$ and $\nu$ be $\sigma$-finite positive measures, $\mu \geq \nu$ and assume that $\nu << \mu - \nu$ ($\nu$ is absolutely continuous with respect to $\mu - \nu$).

Prove that

$$\mu\left(\{\frac{d\nu}{d\mu} = 1\}\right) = 0.$$

$$\mu(\star) > 0$$

$$\nu(\star) = \int_\star d\nu = \int_\star \frac{d\nu}{d\mu}\, d\mu = \mu(\star)$$

13 Fa, 1

1. Let $\mu$ be a finite Borel measure on $\mathbb{R}$, which is absolutely continuous with respect to the Lebesgue measure $m$. Prove that $x \mapsto \mu(A+x)$ is continuous for every Borel set $A \subseteq \mathbb{R}$.

Let $A$ be a Borel set and let $c \in \mathbb{R}$.
We need to show $\lim_{x\to c} \mu(A+x) = \mu(A+c)$

$$\mu(A+x) = \int_{A+x} d\mu = \int_{A+x} \frac{d\mu}{dm}\, dm$$

If we show $\lim_{x\to c} \mu((A+x)\,\Delta\,(A+c)) = 0$,
then

$$\begin{aligned}\lim_{x\to c} |\mu(A+x) - \mu(A+c)| &= \lim_{x\to c} |\mu((A+x)\setminus(A+c)) - \mu((A+c)\setminus(A+x))| \\ &\le \lim_{x\to c} \mu((A+x)\setminus(A+c)) + \mu((A+c)\setminus(A+x)) \\ &= \lim_{x\to c} \mu((A+x)\,\Delta\,(A+c)) \\ &= 0\end{aligned}$$

$\Rightarrow \lim_{x\to c} \mu(A+x) = \mu(A+c)$, finishing the proof.

$$\int \chi_A(y-x)\, d\mu$$

$$f(x) = \int \frac{d\mu}{dm} \chi_A(y-x)\, dm(y)$$

Thm 3.5 says, since $\mu$ finite and $\mu \ll m$, $\forall \varepsilon > 0$, $\exists\, \delta > 0$ s.t. if $m(E) < \delta$ then $\mu(E) < \varepsilon$.
We can find an open $U \supset A$ s.t. $m(U \setminus A) < \delta^*$ $\Rightarrow \mu(U\setminus A) < \varepsilon$.
$U$ open $\Rightarrow U = \bigcup_1^\infty (a_i, b_i)$, disjoint open intervals.
$\mu$ finite $\Rightarrow \exists\, N$ s.t. $\mu(\bigcup_1^N (a_i,b_i)) > \mu(U) - \varepsilon$.
Then $|\mu(A) - \mu(\bigcup_1^N(a_i,b_i))| \le |\mu(U) - \mu(A)| + |\mu(U) - \mu(\bigcup_1^N(a_i,b_i))| < 2\varepsilon$.

approx. by step function.

Now, $m(\bigcup_1^N (a_i+x, b_i+x)\,\Delta\,\bigcup_1^N (a_i+c, b_i+c)) \le \sum_1^N m((a_i+x,b_i+x)\,\Delta\,(a_i+c,b_i+c))$
$= \sum_1^N 2\min\{|x-c|, (b_i - a_i)\}$
which $\to 0$ as $x \to c$. Thus, we can find a $\delta > 0$ s.t. $\forall x \in (c-\delta, c+\delta)$
$\mu(\bigcup_1^N (a_i+x, b_i+x)\,\Delta\,\bigcup_1^N(a_i+c, b_i+c)) < \varepsilon$.

$(A+x\,\Delta\,\bigcup_1^N(a_i+x, b_i+x)) = (A\,\Delta\,\bigcup_1^N(a_i,b_i))$
$\le$


$$f(x) = \mu(A+x) = \int \chi_{A+x}(y)\, d\mu(y) = \int \frac{d\mu}{dm}(y)\, \chi_{A+x}(y)\, dm(y)$$
$$= \int \frac{d\mu}{dm}(y)\, \chi_A(y-x)\, dm(y)$$

* True even if $m(A) = \infty$. Limit of finite case on $[-M, M]$.

$g(x) = \mu(A+x)$

$f$ = dens. of $\mu$ $\qquad f = \frac{d\mu}{dm}$

$$g(x) = \int \chi_A(y) f(y-x)\, dm(y)$$

OR

$$= \int \underbrace{\chi_A(y-x) f(y)}\, dm(y).$$

want to approx by a step func. of $y-x$.

$\Psi(x) = \int \chi_A(y-x)\, dm(y)$ $\qquad$ Can approx by step function

13 Fa. 2

2. Let $f$ be a Lebesgue integrable function on $\mathbb{R}$, and assume that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty.$$

Prove that $g(x) = \sum_{n=1}^{\infty} f(a_n x)$ converges almost everywhere and is integrable on $\mathbb{R}$. Also, find an example of a Lebesgue integrable function $f$ on $\mathbb{R}$ such that $g(x) = \sum_{n=1}^{\infty} f(nx)$ converges almost everywhere but is not integrable.

First, we find the example.

Let $f = \chi_{[0,1]}$. $\Rightarrow$ $g(x) = \sum_{n=1}^{\infty} \chi_{[0,1]}(nx)$.

$x \in (1/2, 1] \Rightarrow 2x \notin [0,1] \Rightarrow g(x) = 1$.

$x \in (1/3, 1/2] \Rightarrow 3x \notin [0,1] \Rightarrow g(x) = 2$.

This continues and we see that $g(x) = \lfloor \frac{1}{x} \rfloor$ on $(0,1]$.

And $g(x) = 0$ $\forall x \notin [0,1]$. Thus $g(x)$ converges almost everywhere.

But $\int g(x) = \int_0^1 \lfloor \frac{1}{x} \rfloor = \int_{1/2}^1 1 + \int_{1/3}^{1/2} 2 + \int_{1/4}^{1/3} 3 + \cdots$

$= 1/2 + 1/3 + 1/4 + \cdots$

$= \infty$. $\square$

Now, we prove the result.

Assume $g$ diverges on a non-null set, $E$

Then consider $\{a_n x : n \in \mathbb{N}, x \in E\}$, call this $A$.

Claim: $\int_A f = \int_E g = \infty$.

$$\int_E g = \int_E \sum_1^\infty f(a_n x)$$

$$\int g(x) = \int \sum f(a_n x)\, dx$$

$$\int f(a_n x)\, dm = \frac{1}{a_n} \int f(x)\, dm$$

$u = a_n x$ $\qquad \int \frac{1}{a_n} f(u)\, du$

$du = a_n\, dx$

13 Fa. 3

3. Assume $b > 0$. Show that the Lebesgue integral

$$\int_1^\infty x^{-b} e^{\sin x} \sin(2x)\, dx$$

exists if and only if $b > 1$.

?

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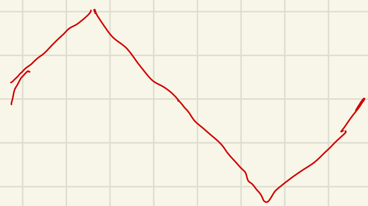
$$\left| \int_1^\infty x^{-b} e^{\sin x} \sin(2x)\, dx \right| \le \int_1^\infty |x^{-b} e|\, dx \le M$$

Find $\pm\infty$ for $b \le 1$.

Sin(2x)

Break down into periods of sin x.

$e^{\sin x}$



Section by Section.

$$\sum M n^{-b} = \infty$$

13 Fa. 4

4. Suppose that $F$ is the distribution function of a Borel measure $\mu$ on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$. Prove that

$$\int_{-\infty}^{\infty} (F(x+a) - F(x))\, dx = a$$

for all $a > 0$.

$$\begin{aligned}
\int_{-\infty}^{\infty} (F(x+a) - F(x))\, dx &= \int_{-\infty}^{\infty} \mu((x, x+a])\, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{(x,x+a]}(y)\, d\mu(y)\, dm(x) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{(x,x+a]}(y)\, dm(x)\, d\mu(y) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{[y-a,y)}(x)\, dm(x)\, d\mu(y) \\
&= \int_{-\infty}^{\infty} (y - (y-a))\, d\mu(y) \\
&= \int_{-\infty}^{\infty} a\, d\mu(y) = a\,\mu(\mathbb{R}) = a.
\end{aligned}$$

Because $y \in (x, x+a]$

$\iff x < y$ AND $y \le x+a$

$\iff x < y$ AND $y-a \le x$

$\iff x \in [y-a, y)$.

Because $\chi_{(x,x+a]}(y) \in L^+(\mathbb{R}\times\mathbb{R})$ and $\mu, m$ are $\sigma$-finite, so we can use Tonelli to swap iterated integrals.

14 Sp. 1

1. Suppose that $(X, \mathcal{B}, \mu)$ is a measure space with $\mu(X) < \infty$, and that $\{f_n\}_{n\geq 1}$ and $f$ are measurable functions on $X$ such that $f_n \to f$ almost everywhere.

(i) Suppose that $\int f^2\, d\mu < \infty$. Show that $f$ is integrable.

(ii) Suppose that there exists $C < \infty$ such that $\int f_n^2\, d\mu \leq C$ for all $n \geq 1$. Show that $f_n \to f$ in $L^1$.

(iii) Give an example where $\int |f_n|\, d\mu \leq 1$ for all $n \geq 1$ but $f_n \not\to f$ in $L^1$.

(i) $\int f^2 d\mu = \int_{\{x : |f(x)| \leq 1\}} f^2 d\mu + \int_{\{x : |f(x)| > 1\}} f^2 d\mu$

$\Rightarrow \int_{\{x : |f(x)| > 1\}} f^2 d\mu < \infty$

$\Rightarrow \int |f| d\mu = \int_{\{x : |f(x)| \leq 1\}} |f| d\mu + \int_{\{x : |f(x)| > 1\}} |f| d\mu$

$\leq \mu(X) + \int_{\{x : |f(x)| > 1\}} f^2 d\mu < \infty.$ $\square$

(ii) NTS $\int |f_n - f| d\mu \to 0$ as $n \to \infty$.

?

why is $|f_n(x) - f(x)| \leq 2\,|f(x)|$ a.e.

$M$ $\int_{\{x : f > M\}} f_n$

$M f_n \chi_{\{x : f_n \geq M\}} \leq f_n^2$ ✱

(iii) Let $f_n = n\chi_{[0, 1/n]}$, $f_n \to 0$ a.e. But $\int |f_n - 0| d\mu = 1$ $\forall n$.
So $f_n \not\to 0$ in $L^1$. $\square$

14Sp.2

2. For what non-negative integer $n$ and positive real $c$ does the integral

$$\int_1^\infty \ln\left(1+\frac{(\sin x)^n}{x^c}\right)dx$$

(a) exist as a (finite) Lebesgue integral? only depends on $C$.

(b) converge as an improper Riemann integral?

(a)

Is Kayla

$\ln(1+y) \approx y$ $\quad \longrightarrow \int \frac{(\sin x)^n}{x^c}\,dx$

$\ln(1+y) \le 2y$

$\boxed{\begin{array}{ll} c>1 & \text{integrable} \\ c\le 1 & \text{not.} \end{array}}$

$\{x : \sin x \ge \frac{1}{2}\}$

bound $\frac{(\sin x)^n}{x^c}$ below by $\frac{1}{x^c}$

pieces of intervals of length $2\pi$.

14 Sp. 3

3. Suppose $f$ is Lebesgue integrable on $\mathbb{R}$. Show that

$$\lim_{t\to 0}\int_{-\infty}^{\infty}|f(x+t)-f(x)|dx = 0.$$

First consider a continuous function $g$ that vanishes outside $[-M, M]$.

Do.

14 Sp. 4

4. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be measure spaces such that $\mu(X) > 0$ and $\nu(Y) > 0$. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be measurable functions (with respect to $\mathcal{A}$ and $\mathcal{B}$ respectively) such that

$$f(x) = g(y) \quad \mu \times \nu \text{ -almost everywhere on } X \times Y$$

Show that there exists a constant $\lambda$ such that $f(x) = \lambda$ for $\mu$-a.e. $x$ and $g(y) = \lambda$ for $\nu$-a.e. $y$.

Assume $\exists \lambda \in \mathbb{R}$ s.t. $\mu(f^{-1}(\lambda)) > 0$.

$f(x) = g(y)$ $\mu \times \nu$ - a.e.

$\Rightarrow \mu \times \nu \left( f^{-1}(\lambda) \times g^{-1}(\mathbb{R} \setminus \{\lambda\}) \right) = \mu(f^{-1}(\lambda)) \cdot \nu(g^{-1}(\mathbb{R} \setminus \{\lambda\})) = 0$

$\Rightarrow \nu(g^{-1}(\mathbb{R} \setminus \{\lambda\})) = 0 \Rightarrow g(y) = \lambda$ $\nu$-a.e.

$\Rightarrow \nu(g^{-1}(\lambda)) = \nu(Y) > 0$

$\Rightarrow$ by the same logic, $f(x) = \lambda$ $\mu$-a.e.

We get the same result if our starting assumption is that $\exists \lambda \in \mathbb{R}$ s.t. $\nu(g^{-1}(\lambda)) > 0$.

So, what happens if $\nexists \lambda$ s.t. $\mu(f^{-1}(\lambda)) > 0$ or $\nu(g^{-1}(\lambda)) > 0$? That is, $\forall \lambda \in \mathbb{R}$, $\mu(f^{-1}(\lambda)) = 0$ and $\nu(g^{-1}(\lambda)) = 0$.

Let $E \subset \mathbb{R}$ be a measurable subset.

$$\mu \times \nu (f^{-1}(E) \times g^{-1}(E^c)) = 0. \qquad \forall E \subset \mathbb{R}.$$

$\mu(f^{-1}(E)) > 0$ for some $E \subset \mathbb{R}$ $\Rightarrow \nu(g^{-1}(E^c)) = 0$

$\Rightarrow \nu(g^{-1}(E)) = \nu(Y) \Rightarrow \mu(f^{-1}(E^c)) = 0$

$\Rightarrow \mu(f^{-1}(E)) = \mu(X)$.

$\mu(f^{-1}(\mathbb{R})) > 0 \Rightarrow \mu(f^{-1}(\mathbb{R})) = \mu(X)$.

?

$\mu(f^{-1}(\mathbb{R})) = \mu(f^{-1}((-\infty, 0))) + \mu(f^{-1}([0, \infty))) = \mu(X)$

$\Rightarrow$ one of $=\mu(X)$.

14 Fa. 1

1. Assume that $f$ is integrable on $(0,1)$. Prove that

$$\lim_{a\to\infty}\int_0^1 f(x)\,x\,\sin(ax^2)\,dx = 0.$$

First let $f = \chi_{(b,c)}$ for $(b,c) \subset (0,1)$.

Then
$$\lim_{a\to\infty} \int_0^1 \chi_{(b,c)}(x)\, x \sin(ax^2)\, dx$$
$$= \lim_{a\to\infty} \int_b^c x \sin(ax^2)\, dx$$
$$= \lim_{a\to\infty} -\frac{1}{2a}\cos(ax^2)\Big]_b^c$$
$$= \lim_{a\to\infty} \frac{1}{2a}\left[\cos(ab^2) - \cos(ac^2)\right]$$

And $\left|\frac{1}{2a}\left[\cos(ab^2) - \cos(ac^2)\right]\right| \le \frac{1}{a} \to 0$ as $a \to \infty$.

$\Rightarrow \lim_{a\to\infty} \int_0^1 \chi_{(b,c)}(x)\, x \sin(ax^2)\, dx = 0$.

By linearity of the integral, this is true when $f$ is a simple function on sets which are finite unions of open intervals.

Since we have the Lebesgue measure, and since $f \in L^1$, for $\varepsilon > 0$, $\exists$ such a simple function $\phi$ with $\int_0^1 |f - \phi| < \varepsilon$.

Then,
$$\lim_{a\to\infty} \left| \int_0^1 f(x)\, x\sin(ax^2)\, dx - \int_0^1 \phi(x)\, x \sin(ax^2)\, dx \right|$$
$$\le \lim_{a\to\infty} \int_0^1 |(f(x) - \phi(x))\, x \sin(ax^2)|$$
$$\le \lim_{a\to\infty} \int_0^1 |f(x) - \phi(x)|\, dx \quad \text{since } |x \sin(ax^2)| \le 1 \text{ on } (0,1).$$
$$< \varepsilon$$

$\Rightarrow \lim_{a\to\infty} \left| \int_0^1 f(x)\, x \sin(ax^2)\, dx \right| < \varepsilon$, and since $\varepsilon$ was abitrary, we conclude

$$\lim_{a\to\infty} \int_0^1 f(x)\, x \sin(ax^2) dx = 0.$$

$\square$

14Fa.2

2. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f$ and $f_1, f_2, f_3 \ldots$ be real valued measurable functions on $X$. If $f_n \to f$ in measure and if $F\colon \mathbb{R} \to \mathbb{R}$ is uniformly continuous, prove that $F \circ f_n \to F \circ f$ in measure.

$f_n \to f$ in measure $\iff$ $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \to 0$ as $n \to \infty$.

NTS $\mu(\{x : |F\circ f_n(x) - F \circ f(x)| \geq \varepsilon\}) \to 0$ as $n \to \infty$.

$F$ uniformly cont. $\iff$ $\forall \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\forall\, x \in X$, if $y \in (x - \delta, x + \delta)$, then $|F(y) - F(x)| < \varepsilon$.

$\exists\, N$ s.t. $\forall\, n \geq N$ $\mu(\{x : |f_n(x) - f(x)| \geq \delta\}) < \varepsilon$.

$|f_n(x) - f(x)| < \delta \Rightarrow |F\circ f_n(x) - F\circ f(x)| < \varepsilon$

Thus, $\{x : |f_n(x) - f(x)| < \delta\} \subset \{x : |F \circ f_n(x) - F\circ f(x)| < \varepsilon\}$

$\Rightarrow \{x : |f_n(x) - f(x)| \geq \delta\} \supset \{x : |F\circ f_n(x) - F\circ f(x)| \geq \varepsilon\}$

$\Rightarrow \mu(\{x : |F\circ f_n(x) - F\circ f(x)| \geq \varepsilon\}) \leq \mu(\{x : |f_n(x) - f(x)| \geq \delta\}$

$\to 0$ as $n \to \infty$.

Hence, $F\circ f_n \to F \circ f$ in measure.

$\square$

14Fa.3

3. Let $f_n$ be **nonnegative** measurable functions on a measure space $(X, \mathcal{M}, \mu)$ which satisfy $\int f_n \, d\mu = 1$ for all $n = 1, 2, \ldots$. Prove that

$$\limsup_n (f_n(x))^{1/n} \leq 1$$

for $\mu$-a.e. $x$.

$$\lim_{n \to \infty} n^{1/n} = 1$$

$f_n(x) > n$ on a shrinking measure.

14 Fa. 4

4. Let $-\infty < a < b < \infty$. Suppose $F\colon [a,b] \to \mathbb{C}$.

(a) Define what it means for $F$ to be absolutely continuous on $[a,b]$.

(b) Give an example of a function which is uniformly continous but not absolutely continuous. (Remember to justify your answer.)

(c) Prove that if there exists $M$ such that $|F(x)-F(y)| \leq M|x-y|$ for all $x, y \in [a,b]$, then $F$ is absolutely continous. Is the converse true? (Again, remember to justify your answer.)

a)

15 sp. 1

1. Consider the sequence

$$f_n(x) = \left(1 + \frac{x}{n}\right)^{-n} \cos\left(\frac{x}{n}\right), \qquad n = 1, 2, \dots .$$

Evaluate

$$\lim_n \int_0^\infty f_n(x)\,dx,$$

being careful to justify your answer.

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{-n} = y$$

$$\lim_{n\to\infty} \ln\left[\left(1+\frac{x}{n}\right)^{-n}\right] = \ln y$$

$$\lim_{n\to\infty} -n \ln\left(1+\frac{x}{n}\right) = \ln y$$

$$\lim_{n\to\infty} -\frac{\ln\left(1+\frac{x}{n}\right)}{1/n} = \ln y$$

L'Hopital: $\lim_{n\to\infty} -\frac{\frac{n}{n+x}\left(-x/n^2\right)}{-1/n^2} = \ln y$

$$\lim_{n\to\infty} -\frac{xn}{n+x} = \ln y$$

L'Hopital: $\lim_{n\to\infty} -\frac{x}{1} = \ln y \Rightarrow y = e^{-x}$

$\frac{x}{n} = x$

$1 = n$

$\frac{x}{n} < x$ for $n > 1$.

And $\lim_{n\to\infty} \cos\left(\frac{x}{n}\right) = \cos(0) = 1.$

Thus if DCT applies then $\lim_n \int_0^\infty f_n(x)\,dx = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big]_0^\infty = 1.$

So let's examine $\left(1+\frac{x}{n}\right)^{-n}$. We can expand $\left(1+\frac{x}{n}\right)^n$:

$$\left(1+\frac{x}{n}\right)^n = 1 + n\left(\frac{x}{n}\right) + \binom{n}{2}\left(\frac{x}{n}\right)^2 + \cdots = 1 + x + \frac{n(n-1)}{2}\frac{x^2}{n^2} + \cdots = 1 + x + \frac{x^2(n-1)}{2n} + \cdots$$

$$\geq 1 + x + \frac{x^2(n-1)}{2n} \Rightarrow \left(1+\frac{x}{n}\right)^{-n} \leq \left(1 + x + \frac{x^2(n-1)}{2n}\right)^{-1} \text{ for } n > 2.$$

$$= \left(1 + x + \frac{x^2}{2} - \frac{x^2}{2n}\right)^{-1}$$

$$\leq \left(1 + x + \frac{x^2}{2} - \frac{x^2}{4}\right)^{-1}$$

$$= \left(1 + x + \frac{x^2}{4}\right)^{-1}$$

$$\leq \left(\frac{x^2}{4}\right)^{-1} = \frac{4}{x^2} \in L^1. \quad \square$$

15sp.2

2. Suppose that $f\colon [0,\infty) \to \mathbb{R}$ is Lebesgue integrable.
(i) Show that there exists a sequence $x_n \to \infty$ such that $f(x_n) \to 0$.
(ii) Is it true that $f(x)$ must converge to 0 as $x \to \infty$? Give a proof or a counterexample.
(iii) Suppose additionally that $f$ is differentiable and $f'(x) \to 0$ as $x \to \infty$. Is it true that $f(x)$ must converge to 0 as $x \to \infty$? Give a proof or a counterexample.

(i) We can construct such a sequence. Assume $\nexists\, x$ s.t. $|f(x)| < 1$. then $\int_0^\infty |f| dm \geq \int_0^1 dm = \infty \Rightarrow\Leftarrow$, So $\exists$ some $x_1$ with $f(x_1) < 1$. Now consider $(x_1, \infty)$ and apply the same logic to find some $x_2 \in (x_1, \infty)$ with $|f(x_2)| < \frac{1}{2}$. Then we find some $x_3 \in (x_2, \infty)$ with $|f(x_3)| < \frac{1}{3}$ and so on.

(ii) No. Consider $f(x) = \sum_1^\infty \chi_{[n, n+1/2^n]}$
$\int_0^\infty f\, dm = \sum_1^\infty m([n, n+1/2^n]) = \sum_1^\infty 1/2^n = 1 < \infty$.
But $f \not\to 0$ as $x \to \infty$.

(iii) Assume not. That is, $f \not\to 0$ as $x \to \infty$.
Then for some $\varepsilon > 0$, $\forall N \in \mathbb{N}$, $\exists\, x \geq N$ with $|f(x)| \geq \varepsilon$.
$f'(x) \to 0 \Rightarrow$ for some $M \in \mathbb{N}$, $\forall x \geq M$ $|f'(x)| < \varepsilon$.
There exists some $x_1 \geq M$ with $|f(x_1)| \geq \varepsilon$. Then $\int_{x_1}^{x_1+1} |f| dm \geq \varepsilon/2$ because $|f|$ cannot be less than the straight line function between the points of the graph $(x_1, \varepsilon)$ and $(x_1+1, 0)$. for if so then MVT would be violated.

We can continue to find points where this is true ad infinitum. Thus $\int_0^\infty |f| \geq \sum_1^\infty \varepsilon/2 = \infty$.

Hence, $f \to 0$ as $x \to \infty$.

15 sp. 3

3. Define $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$.

(i) Show that

$$\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)\, dx = 0$$

and

$$\int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\, dx = \log(b/a).$$

(ii) What can you deduce about the value of

$$\int_0^{\infty} \sum_{n=1}^{\infty} |f_n(x)|\, dx?$$

(i) $\sum_{n=1}^{\infty} \int_0^{\infty} ae^{-nax} - be^{-nbx}\, dx = \sum_{n=1}^{\infty} -\frac{1}{n} e^{-nax} + \frac{1}{n} e^{-nbx} \Big]_0^{\infty}$

$= \sum_{n=1}^{\infty} 0 + 0 + \frac{1}{n} e^{0} - \frac{1}{n} e^{0}$

$= \sum_{n=1}^{\infty} 0 = 0.$

$\sum_{n=1}^{\infty} a e^{-nax} = a \sum_{n=1}^{\infty} \left(\frac{1}{e^{ax}}\right)^n = a\left(\frac{1}{e^{ax}}\right) \Big/ \left(1 - \frac{1}{e^{ax}}\right) = \frac{ae^{-ax}}{1-e^{-ax}}$

$\Rightarrow \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\, dx = \int_0^{\infty} \frac{ae^{-ax}}{1-e^{-ax}} - \frac{be^{-bx}}{1-e^{-bx}}\, dx$ $\qquad u = 1 - e^{-ax}$, $du = ae^{-ax}\, dx$

$= \ln(1 - e^{-ax}) - \ln(1 - e^{-bx}) \Big]_0^{\infty}$

$= \lim_{K \to \infty} \ln(1 - e^{-ax}) - \ln(1 - e^{-bx}) \Big]_{1/K}^{\infty}$

$= \lim_{K \to \infty} \ln(1) - \ln(1) + \ln(1 - e^{-b/K}) - \ln(1 - e^{-a/K})$

$= \lim_{K \to \infty} \ln\left(\frac{1 - e^{-b/K}}{1 - e^{-a/K}}\right) = y$, say

$\lim_{K \to \infty} \frac{1 - e^{-b/K}}{1 - e^{-a/K}} = e^{y}$

L'Hopital $\Rightarrow$ $\lim_{K \to \infty} \frac{-b/K^2\, e^{-b/K}}{-a/K^2\, e^{-a/K}} = e^{y}$

$\Rightarrow \lim_{K \to \infty} \frac{b}{a} e^{(a-b)/K} = e^{y}$

$\Rightarrow b/a = e^{y} \Rightarrow y = \ln\left(\frac{b}{a}\right).$

(ii) $\int_0^{\infty} \sum_{n=1}^{\infty} |f_n(x)|\, dx = \infty$, for if not,
then $f_n(x) \in L^1(\nu \times m)$ where $\nu$ is the counting measure
$\Rightarrow$ by Fubini $\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)\, dx = \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\, dx$, a contradiction.

$\square$

15 sp. 4

4. Assume that $f$ is integrable on $[0,1]$ with respect to the Lebesgue measure $m$, and let $F(x) = \int_0^x f(t)\,dt$. Assume that $\phi\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz, i.e., there exists a constant $C \geq 0$ such that

$$|\phi(x_1) - \phi(x_2)| \leq C|x_1 - x_2|, \qquad x_1, x_2 \in \mathbb{R}.$$

Prove that there exists a function $g$ which is integrable on $[0,1]$ such that $\phi(F(x)) = \int_0^x g(t)\,dt$ for $x \in [0,1]$.

Assuming $\phi(0) = 0$

The Fundamental Thm of Calc. for Lebesgue Integrals tells us that $\phi(F(x))$ is absolutely continuous on $[0,1] \iff \exists\, g \in L^1([0,1])$ s.t. $\int_0^x g(t)\,dt = \phi(F(x)) - \phi(F(0)) = \phi(F(x)) - \phi(0) = \phi(F(x))$.

So we need to show $\phi(F(x))$ absolutely continuous.
F is absolutely continuous by this theorem.
Let $\varepsilon > 0$. $\exists\, \delta$ s.t. for a collection of disjoint intervals on $[0,1]$, $\{(a_i, b_i)\}_1^N$ we have

$$\sum_1^N (b_i - a_i) < \delta \Rightarrow \sum_1^N |F(b_i) - F(a_i)| < \varepsilon / C^*$$

$\phi$ Lipschitz $\Rightarrow$
$$\begin{aligned} \sum_1^N |\phi(F(b_i)) - \phi(F(a_i))| &\leq \sum_1^N C\,|F(b_i) - F(a_i)| \\ &< C\,\varepsilon/C \\ &= \varepsilon. \end{aligned}$$

Thus $\phi(F(x))$ is absolutely continuous as well.

$\square$

* Note if $C = 0$ then $\phi(F(x)) = F(x)$ and we are done.

15 Fa.1

1. Prove that for almost all $x \in [0,1]$, there are at most finitely many rational numbers with reduced form $p/q$ such that $q \geq 2$ and $|x - p/q| < 1/(q\log q)^2$. (Hint: Consider intervals of length $2/(q\log q)^2$ centered at rational points $p/q$.)

Consider the interval $\left(p/q - 1/(q\ln q)^2,\ p/q + 1/(q\ln q)^2\right)$ for $p/q$ reduced and $q \geq 2$. Call this set $E_{p/q}$.

Consider $\sum_{p/q} \chi_{E_{p/q}}$.

$$\int \sum_{p/q} \chi_{E_{p/q}} = \sum_{p/q} \int \chi_{E_{p/q}} = \sum_{p/q} 2/(q\ln q)^2$$
$$\leq \sum_{q=2}^{\infty} 4q/(q\ln q)^2 \quad *$$
$$\leq 4\sum_{q=2}^{\infty} 1/q(\ln q)^2$$
$$< \infty.$$

Now, consider $\{x \in [0,1] : \text{infinitely many } p/q \text{ s.t. } |x - p/q| < 1/(q\ln q)^2\}$
For $x$ in this set $\sum_{p/q} \chi_{E_{p/q}}(x) = \infty$.
Thus the measure of this set must be null since $\sum_{p/q} \chi_{E_{p/q}}$ is integrable.

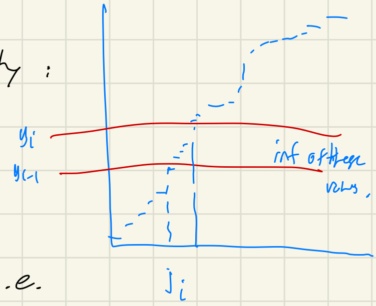
*

# 15 Fa. 2

2. Suppose that the real-valued function $f(x)$ is nondecreasing on the interval $[0,1]$. Prove that there exists a sequence of continuous functions $f_n(x)$ such that $f_n \to f$ pointwise on this interval.

Since $f$ nondecreasing, there are at most countably many discontinuities, $x_1, x_2, \dots$
There are 3 cases of discontinuity:

(i) $f(x-) < f(x) = f(x+)$
(ii) $f(x-) = f(x) < f(x+)$
(iii) $f(x-) < f(x) < f(x+)$

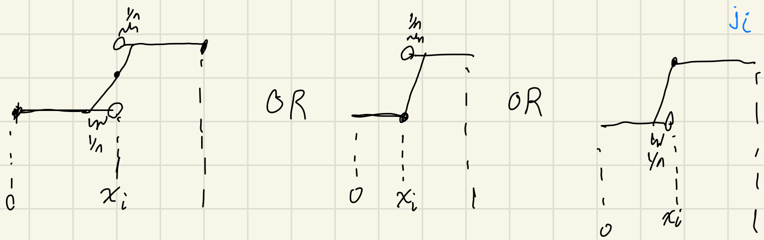



$f_n(x) = y_{i-1}$ on $j_i$ $\forall$ [illegible] $j_i$

$f$ nondecreasing $\Rightarrow$ $f$ differentiable a.e.

Define $g(x) = f(0) + \int_0^x f'(t)\,dt$.
This is basically $f$ without the discontinuities.

Define $h_{i,n} =$




corresponding to the discontinuity. Define $H_n = \sum_i h_{i,n}$

Define $f_n = g + H_n$

cont. because $g$, $h_{i,n}$ continuous. & $\sum$ cont. is cont.

$f_n(x) \to f(x)$ ?

Simple functions converging from below, similar

15 Fa. 3

3. Let $(X,\mu)$ be a finite measure space. Assume that a sequence of integrable functions $f_n$ satisfies $f_n \to f$ in measure, where $f$ is measurable. Assume that $f_n$ satisfies the following property: For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mu(E) \le \delta \implies \int_E |f_n|\,d\mu \le \epsilon. \qquad \forall n\,?$$

Prove that $f$ is integrable and that

$$\lim_n \int_X |f_n - f|\,d\mu = 0.$$

Let $\varepsilon > 0$. Then $\exists\ \delta > 0$ s.t. $\mu(E) \le \delta \Rightarrow \int_E |f_n|\,d\mu \le \varepsilon \quad \forall n$.
$f_n \to f$ in measure $\Rightarrow \exists$ subsequence $f_{n_k} \to f$ a.e.
And since $\mu(X) < \infty$, Egoroff $\Rightarrow \exists$ set $E$ s.t.
$f_{n_k} \to f$ uniformly on $E$ and $\mu(E^c) < \delta$. Check?
$\Rightarrow \int_{E^c} |f_n|\,d\mu \le \varepsilon \quad \forall n$.
Then $\int_{E^c} |f|\,d\mu = \int_{E^c} \liminf |f_{n_k}|\,d\mu \le \liminf \int_{E^c} |f_{n_k}|\,d\mu \le \varepsilon$.
$f_{n_k} \to f$ uniformly on $E \Rightarrow \exists\ N_K \in \{n_k\}$ s.t. $|f_{N_K}(x) - f(x)| < \varepsilon$
$\forall x \in E$. Then $\int_E |f|\,d\mu < \int_E |f_{N_K}| + \varepsilon\,d\mu < \infty$
Thus, $\int_X |f|\,d\mu = \int_E |f|\,d\mu + \int_{E^c} |f|\,d\mu < \infty$.

$\square$

Now, let $C := \int_X |f|\,d\mu$.
$f_n \to f$ in measure $\Rightarrow$ for $\varepsilon > 0$ $\mu(\{x : |f_n(x) - f(x)| \ge \varepsilon\}) \to 0$ as $n \to \infty$.
Call this set $E_n$. Then $\exists\ N \in \mathbb{N}$ s.t. $\forall n \ge N$, we have
$\mu(E_n) \le \delta \Rightarrow \forall n,\ \int_{E_n} |f_n|\,d\mu \le \varepsilon$.
Thus, $\forall n \ge N$,

$$\begin{aligned}\int_X |f_n - f|\,d\mu &= \int_{E_n} |f_n - f|\,d\mu + \int_{E_n^c} |f_n - f|\,d\mu \\ &\le \int_{E_n} |f_n|\,d\mu + \int_{E_n} |f|\,d\mu + \varepsilon\mu(X) \\ &\le \varepsilon + \int_{E_n} \liminf |f_{n_k}|\,d\mu + \varepsilon\mu(X) \\ &\le \varepsilon + \varepsilon + \varepsilon\mu(X)\end{aligned}$$

Since $\varepsilon$ arbitrary, $\lim_n \int_X |f_n - f|\,d\mu = 0$.

$\square$

# 15 Fa. 4

4. Consider the following two statements about a function $f\colon [0,1] \to \mathbb{R}$:
(i) $f$ is continuous almost everywhere
(ii) $f$ is equal to a continuous function $g$ almost everywhere.
Does (i) imply (ii)? Prove or give a counterexample. Does (ii) imply (i)? Prove or give a counterexample.

(ii) $\not\Rightarrow$ (i) Consider $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ and $g = 0$.

Then $f \neq g$ only on $\mathbb{Q}$, which has measure 0. So $f = g$ a.e.
But $f$ is continuous nowhere, since for any $x \in [0,1]$, there are sequences $x_n \to x$ s.t. $\{x_n\} \subset \mathbb{Q}$ or $\{x_n\} \subset [0,1] \setminus \mathbb{Q}$.

(i) $\not\Rightarrow$ (ii) Consider $f(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2] \\ 1 & \text{if } x \in (1/2, 1] \end{cases}$.

Assume there was some continuous $g$ s.t. $f = g$ a.e.

Consider $x \in [0, 1/2]$ If $g(x) \neq 0$, then $|g(x)| > 0$.
$g$ continuous $\Rightarrow$ $\exists\ \delta$ s.t. $\forall y \in (x-\delta, x+\delta) \cap [0, 1/2]$, $|g(y) - g(x)| < |g(x)|/2$.
$\Rightarrow$ $|g(y)| \neq 0$ $\forall\ y \in (x-\delta, x+\delta) \cap [0,1/2]$ contradicting $g = f$ a.e.
Since $(x-\delta, x+\delta) \cap [0, 1/2]$ is not a null set.
Thus $g(x) = 0$ $\forall\ x \in [0, 1/2]$

Similarly $g(x) = 1$ $\forall\ x \in (1/2, 1]$ $\Rightarrow$ $g = f$ not continuous $\Rightarrow \Leftarrow$.

Thus there cannot be a continuous $g$ s.t. $f = g$ a.e.

16 sp. 1

1. Let

$$f(y) = \sum_n \frac{x}{x^2 + yn^2}.$$

Show that $g(y) = \lim_{x\to\infty} f(x,y)$ exists for all $y > 0$. Find $g(y)$.

Kardaś

Doesn't work

Let $\varepsilon = \frac{1}{x}$. Then $f(y) = \sum_n \frac{1/\varepsilon}{1/\varepsilon^2 + yn^2} = \sum_n \frac{1}{\varepsilon(1+yn^2\varepsilon^2)/\varepsilon^2}$

$= \sum_n \frac{\varepsilon}{1+yn^2\varepsilon^2}$

$g(y) = \lim_{x\to\infty} f(x,y) = \lim_{\varepsilon\to 0} \sum_n \frac{\varepsilon}{1+yn^2\varepsilon^2}$

$\frac{1}{1+yu^2} = g \qquad g(n\varepsilon)$

$\int_0^\infty \frac{1}{1+yu^2}\, dy$

16sp. 2

2. Let $A \subseteq \mathbb{R}$ be Lebesgue measurable. Show that $n(\chi_A * \chi_{[0,\frac{1}{n}]}) \to \chi_A$ pointwise a.e. as $n \to \infty$. (Recall that $(f * g)(x) = \int f(x-y)g(y)\,dy$ for $x \in \mathbb{R}$.)

$$n(\chi_A * \chi_{[0,1/n]}) \longrightarrow \chi_A \qquad \text{as } n \to \infty$$

$$n \int \chi_A(x-y)\,\chi_{[0,1/n]}(y)\,dy \longrightarrow \chi_A(x)$$

$$= n \int_0^{1/n} \chi_A(x-y)\,dy$$

$$= n \int_0^{1/n} \chi_{x-A}(y)\,dy$$

$$= n\, m(x - A \cap [0, 1/n]) \qquad \leq 1 \quad \forall n.$$

Assume not. Assume $\exists$ set $E$ with $m(E) > 0$ and
$n(\chi_A * \chi_{[0,1/n]}) \not\to \chi_A \quad \forall\ x \in E.$

16 sp. 3

3. a) Prove that if a sequence of integrable functions $f_n$ on $[0,1]$ satisfies $\int_0^1 |f_n(x)|\,dx \le 1/n^2$ for $n \in \mathbb{N}$, then $f_n \to 0$ a.e. on $[0,1]$ as $n \to \infty$.
b) Show that the above fact is not true if $1/n^2$ is replaced by $1/\sqrt{n}$.

a) Assume instead that $f_n \not\to 0$ a.e. on $[0,1]$ as $n \to \infty$.
Then $\exists$ set $E \subset [0,1]$ with $m(E) > 0$ s.t.
$f_n(x) \not\to 0$ $\forall$ $x \in E$. So for each $x \in E$, $\forall N \in \mathbb{N}$,
$\exists$ $n \ge N$ s.t. $f_n(x) \ge \varepsilon$, for a given $\varepsilon > 0$.

$$\int_0^1 |f_n(x)|\,dx \le 1/n^2 \Rightarrow \sum_{n=1}^\infty \int_0^1 |f_n(x)|\,dx \le \sum_{n=1}^\infty 1/n^2 < \infty$$
$$\Rightarrow \int_0^1 \sum_{n=1}^\infty |f_n(x)|\,dx < \infty$$

But for each $x \in E$, $\sum_{n=1}^\infty |f_n(x)| \ge \sum_1^\infty \varepsilon = \infty$
$\Rightarrow \int_0^1 \sum_{n=1}^\infty |f_n(x)|\,dx = \infty$ $\Rightarrow \Leftarrow$. $\square$

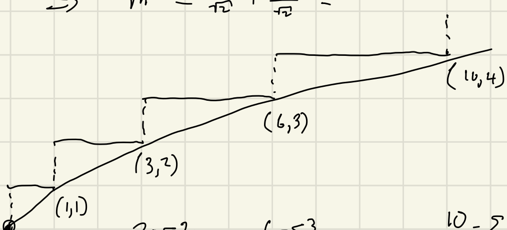
b) Consider the scanning function:
Define $g(k,l) := \chi_{[k/l,\, k+1/l]}$ for $l \in \mathbb{N}$, $k \in \{0, 1, \dots, l-1\}$.
Define $f_1 = g(0,1)$. For $f_n = g(k,l)$, define

$$f_{n+1} = \begin{cases} g(0, l+1) & \text{if } k = l-1 \\ g(k+1, l) & \text{otherwise.} \end{cases}$$

Thus, $\int_0^1 |f_n|\,dx = 1/l$ when $f_n = g(k,l)$.
We claim $\sqrt{n} \le l$ $\forall n$.
Indeed, $n \le \frac{l(l+1)}{2} = \frac{l^2}{2} + \frac{l}{2}$
$\Rightarrow \sqrt{n} \le \frac{l}{\sqrt{2}} + \frac{\sqrt{l}}{\sqrt{2}} =$

| $\int\lvert f_n\rvert$ | $1/\sqrt{n}$ |
|---|---|
| 1 | 1 |
| 1/2 | |
| 1/2 | |
| 1/3 | 1/2 |
| 1/3 | |
| 1/3 | |
| 1/4 | |
| 1/4 | |
| 1/4 | 1/3 |
| 1/4 | |

✓




$3 = \sum_1^2$ $6 = \sum_1^3$ $10 = \sum_1^4 = \frac{l(l+1)}{2}$

$y =$ $x = \sum_1^y n = \frac{y(y+1)}{2} = \frac{y^2+y}{2}$

16 sp.4

4. Let

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Also, let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rationals. Define

$$g_n(x) = \frac{1}{2^n} f(x - r_n), \qquad x \in \mathbb{R}$$

and let

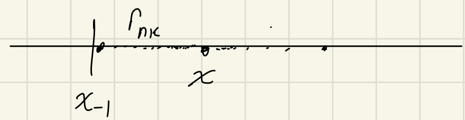
$$g(x) = \sum_{n=1}^{\infty} g_n(x), \qquad x \in \mathbb{R}$$

a) Prove that $g$ is integrable on $\mathbb{R}$.
b) Prove that $g$ is discontinuous at every point in $\mathbb{R}$.

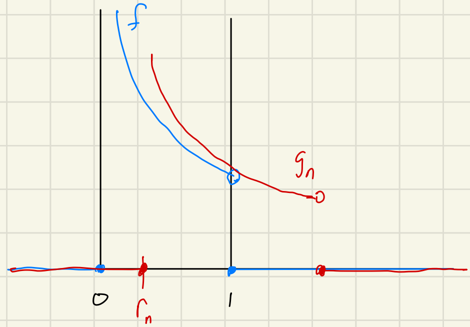
a)
$$\begin{aligned} \int |g(x)|\,dx &= \int \sum_{n=1}^{\infty} g_n(x)\,dx \\ &= \int \sum_{n=1}^{\infty} \frac{1}{2^n} f(x-r_n)\,dx \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \int f(x-r_n)\,dx \qquad \text{since } g_n \in L^+ \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \int f(x)\,dx \\ &= 2\sum_{n=1}^{\infty} \frac{1}{2^n} = 2 < \infty. \end{aligned}$$

b) Let $x \in \mathbb{R}$.

$$\begin{aligned} \lim_{y\to x} g(y) &= \lim_{y \to x} \sum_{n=1}^{\infty} \frac{1}{2^n} f(y-r_n) \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} \lim_{y\to x} f(y-r_n) \qquad \text{by DCT} \end{aligned}$$




Let $x, r_n$ be fixed.
Assume $f(x-r_n) > 0$.
$y \to x$. $f(y-r_n) \to f(x-r_n)$. ✓
Assume $f(x-r_n) = f(0) = 0$
$y \nearrow x \Rightarrow f(y-r_n) \to 0$
$y \searrow x \Rightarrow f(y-r_n) \to \infty$.



# 16 Fa. 1

**Problem 1.** Let $(X, \mathcal{F}, \mu)$ be a *finite* measure space, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of *nonnegative* measurable functions. Prove that $f_n \to 0$ in measure if and only if

$$\lim_{n\to\infty} \int \frac{f_n}{f_n+1} d\mu = 0.$$

First, assume $f_n \to 0$ in measure.
Then for $\varepsilon > 0$, $\mu(\{x : |f_n(x)| \geq \varepsilon\}) \longrightarrow 0$ as $n \longrightarrow \infty$.
Call these sets $E_n$.
Then

$$\begin{aligned} \int \frac{f_n}{f_n+1} d\mu &= \int_{E_n} \frac{f_n}{f_n+1} d\mu + \int_{E_n^c} \frac{f_n}{f_n+1} d\mu \\ &\leq \int_{E_n} 1 \, d\mu + \int_{E_n^c} \varepsilon \, d\mu \\ &\leq \mu(E_n) + \mu(X)\varepsilon \end{aligned}$$

So $\lim\limits_{n\to\infty} \int \frac{f_n}{f_n+1} d\mu \leq \mu(X)\varepsilon$. And since $\varepsilon$ was arbitrary,

$$\lim_{n\to\infty} \int \frac{f_n}{f_n+1} d\mu = 0.$$

$\square$

Second, assume $\lim\limits_{n\to\infty} \int \frac{f_n}{f_n+1} d\mu = 0$.
Then $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\int \frac{f_n}{f_n+1} d\mu < \varepsilon$, for a given $\varepsilon > 0$.
Assume $f_n \not\to 0$ in measure. Then for some $\varepsilon > 0$,
$\mu(\{x : |f_n(x)| \geq \varepsilon\}) \not\to 0$ as $n \longrightarrow \infty$. Call these sets $E_n$.

We know that

$$\frac{d}{dx} \frac{x}{x+1} = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0 \quad \text{for } x \neq -1$$

$$\Rightarrow \frac{f_n}{f_n+1} \geq \frac{\varepsilon}{\varepsilon+1} \text{ whenever } f_n \geq \varepsilon.$$

Thus, $\int \frac{f_n}{f_n+1} d\mu \geq \int_{E_n} \frac{\varepsilon}{\varepsilon+1} d\mu = \mu(E_n) \frac{\varepsilon}{\varepsilon+1} \not\to 0$ as $n \to \infty$.

This is a contradiction, so $f_n \longrightarrow 0$ in measure.

$\square$

16Fa.2

**Problem 2.** Let $(X, \mathcal{F}, \mu)$ be a *finite* measure space, and let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ be a sequence of sets. Assume that $\mu(A_n) \geq \delta$ for all $n \in \mathbb{N}$, where $\delta > 0$. Prove that there exists a set $S \in \mathcal{F}$ of positive measure such that for every $x \in S$, is in $A_j$ for infinitely many $j$.

Define $S := \{ x \in X : x \in A_j$ for infinitely many $j \}$.

Define $S_K := \{ x \in X : x \in A_j$ for exactly $K$ different $j \}$, $K = 0, 1, 2, \ldots$

Assume $\mu(S) = 0$.

We know $\sum_{K=0}^{\infty} \mu(S_K) = \mu(X)$

$f(x) = \sum_j \chi_{A_j}$ NTS $f(x) = \infty$ on non null set.

$f = \sum_{n=1}^{\infty} \frac{1}{n} \chi_{A_n}$

$$\int \sum_{n=1}^{\infty} \frac{1}{n} \chi_{A_n} \, d\mu = \sum_{n=1}^{\infty} \frac{1}{n} \int \chi_{A_n} \, d\mu = \sum_{n=1}^{\infty} \frac{1}{n} \mu(A_n) \geq \sum_{n=1}^{\infty} \frac{1}{n} \delta = \infty.$$

$$\mu(X) \geq \int \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{A_n} \, d\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \mu(A_n) \geq \delta$$

1.8

16 Fa. 3

**Problem 3.** Let $f_n : [0,1] \to [0,\infty)$ be Lebesgue measurable and such that $f_n(x) \to 0$ for almost every $x$. Assume that

$$\sup_n \int_0^1 \varphi(f_n(x))dx \le 1$$

for some continuous $\varphi : [0,\infty) \to [0,\infty)$ which satisfies $\varphi(t)/t \to \infty$ as $t \to \infty$. Prove that $\int_0^1 f_n(x)dx \to 0$ as $n \to \infty$. (Provide a detailed proof).

$\varphi(t)/t \longrightarrow \infty$ as $t \to \infty$ $\Rightarrow$ for $M \in \mathbb{N}$, $\exists\, T \in \mathbb{N}$ s.t. $\forall\, t \ge T$, $\varphi(t)/t \ge M$.

Define $E_n^T := \{x : f_n(x) \ge T\}$

16 Fa. 4

**Problem 4.** Let $h : [0, \infty) \to \mathbb{R}$ be continuous with compact support. Prove that

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \frac{h(\alpha x) - h(\beta x)}{x} dx = h(0) \log \frac{\alpha}{\beta}$$

for every $\alpha, \beta > 0$.

? DCT

$\boxed{\text{Sign}}$

First, take $\beta \geq \alpha$.

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \frac{h(\alpha x) - h(\beta x)}{x} dx = \lim_{\varepsilon \to 0^+} \lim_{M \to \infty} \int_\varepsilon^M \frac{h(\alpha x)}{x} dx - \int_\varepsilon^M \frac{h(\beta x)}{x} dx$$

$$= \lim_{\varepsilon \to 0^+} \lim_{M \to \infty} \int_{\alpha\varepsilon}^{\alpha M} \frac{h(u)}{u} du - \int_{\beta\varepsilon}^{\beta M} \frac{h(u)}{u} du$$

$$= \lim_{\varepsilon \to 0^+} \lim_{M \to \infty} \int_{\alpha\varepsilon}^{\beta\varepsilon} \frac{h(u)}{u} du - \int_{\alpha M}^{\beta M} \frac{h(u)}{u} du$$

$\cancel{u = x\varepsilon}$ $\quad du =$

$\cancel{x = u\varepsilon}$

$$= \lim_{\varepsilon \to 0^+} \lim_{M \to \infty} \int_\alpha^\beta \frac{h(\overset{x\varepsilon}{\cancel{x/\varepsilon}})}{x} dx - \int_\alpha^\beta \frac{h(\overset{xM}{\cancel{x/M}})}{x} dx$$

$x = u/\varepsilon \qquad u = x\varepsilon$

$du = \varepsilon\, dx$

$$= \lim_{\varepsilon \to 0^+} \lim_{M \to \infty} \int_\alpha^\beta \frac{h(\overset{x\varepsilon}{\cancel{x/\varepsilon}}) - h(\overset{xM}{\cancel{x/M}})}{x} dx$$

$$= \int_\alpha^\beta \lim_{\varepsilon \to 0^+} \lim_{M \to \infty} \frac{h(x/\varepsilon) - h(x/M)}{x} dx \qquad *$$

$$= \int_\alpha^\beta \frac{0^* - h(0)}{x} dx$$

$$= -h(0)\left(\ln(\beta) - \ln(\alpha)\right) = h(0) \ln\left(\frac{\alpha}{\beta}\right).$$

If $\alpha \geq \beta$, we have $\lim_{\varepsilon \to 0^+} \lim_{M \to \infty} \int_{\beta M}^{\alpha M} \frac{h(u)}{u} du - \int_{\beta\varepsilon}^{\alpha\varepsilon} \frac{h(u)}{u} du$

$$= \int_\beta^\alpha \frac{h(0)}{x} dx = h(0)\left(\ln(\alpha) - \ln(\beta)\right) = h(0) \ln\left(\frac{\alpha}{\beta}\right).$$

$\square$

* DCT

* $h(x/\varepsilon) \to 0$ as $\varepsilon \to 0$ b/c $h$ has compact $\Rightarrow$ bounded support.

17 sp. 1

**Problem 1.** Assume that $f$ is a positive absolutely continuous function on $[0, 1]$. Prove that $1/f$ is also absolutely continuous on $[0, 1]$.

$f$ continuous on $[0,1]$, compact $\Rightarrow$ $f$ has a minimum.
$f$ positive $\Rightarrow$ $\min\{f(x)\} > 0$. Call this $\ell$.

Let $\varepsilon > 0$. $f$ absolutely continuous $\Rightarrow$ $\exists\ \delta > 0$ s.t. for any collection of intervals $\{(a_i, b_i)\}$, we have

$$\sum (b_i - a_i) < \delta \Rightarrow \sum |f(b_i) - f(a_i)| < \varepsilon \ell^2$$

Thus for any collection of intervals with $\sum (b_i - a_i) < \delta$, we also have

$$\begin{aligned} \sum \left| \frac{1}{f(b_i)} - \frac{1}{f(a_i)} \right| &= \sum \frac{|f(a_i) - f(b_i)|}{f(a_i) f(b_i)} \\ &\leq \sum \frac{|f(a_i) - f(b_i)|}{\ell^2} \\ &= 1/\ell^2 \sum |f(a_i) - f(b_i)| \\ &< \varepsilon. \end{aligned}$$

Hence, $1/f$ is absolutely continuous.

$\square$

17 sp. 2

**Problem 2.** Assume that $E$ is Lebesgue measurable.

(a) Suppose $m(E) < \infty$, where $m$ is the Lebesgue measure. Show that

$$f(x) = \int \chi_E(y)\chi_E(y-x)dm(y)$$

is continuous. (Here, $\chi_A$ denotes the characteristic function of a set $A \subset \mathbb{R}$).

(b) Suppose $0 < m(E) \le \infty$. Show that $S = E - E = \{x - y : x, y \in E\}$ contains an open interval $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

?

Looks like Kayla's works

a) Note $f(x) = \int \chi_E(y)\chi_E(y-x)\,dm(y) = \int \chi_E(y)\chi_{E+x}(y)\,dm(y) = m(E \cap E+x)$.

First, assume $E = (a,b)$ is an open interval.
Then $f(x) = m((a,b) \cap (a+x, b+x)) = \chi_{0<x\le b-a}(x)(b-a-x) + \chi_{a-b\le x\le 0}(x)(b-a+x)$
which is continuous.

Try again.

By linearity, $f(x)$ is continuous when $E$ is a ~~countable~~ finite union of disjoint open intervals, with $m(E)<\infty$.

Now, consider arbitrary measurable $E$ with $m(E)<\infty$.
We can find a countable union of disjoint open intervals s.t.
$U = \bigcup (a_i, b_i) \supset E$ and $m(U \setminus E) < \varepsilon$, for any $\varepsilon > 0$.
For clarity denote $f_E(x) := m(E \cap E+x)$ and $f_U(x) = m(U \cap U+x)$.

$$\begin{aligned} f_U(x) - f_E(x) = m(U \cap U+x) - m(E \cap E+x) &= m((U \cap U+x) \setminus (E \cap E+x)) \\ &\le m(U \setminus E) + m(U+x \setminus E+x) \\ &< 2\varepsilon. \qquad \forall x. \end{aligned}$$

Thus, for $z \to x$ we can make $f_E(z) \to f_E(x)$, since $\varepsilon$ is arbitrary, and $f_U$ is continuous $\forall$ $U$ constructed as above.

b) $S = E - E = \bigcup_{y\in E} E - y$.

Try Kayla's.

17 sp. 3

**Problem 3.** Assume that $f$ is a continuous function on $[0,1]$. Prove that

$$\lim_{n\to\infty}\int_0^1 nx^{n-1}f(x)dx = f(1).$$

Let $\varepsilon > 0$. $f$ continuous $\Rightarrow$ $\exists\ \delta > 0$ s.t. $\forall\ x \in (1-\delta, 1]$, we have $|f(x) - f(1)| < \varepsilon$.

Then $\lim\limits_{n\to\infty} \int_0^1 nx^{n-1} f(x)dx = \lim\limits_{n\to\infty} \int_0^{1-\delta} nx^{n-1} f(x)dx + \lim\limits_{n\to\infty} \int_{1-\delta}^1 nx^{n-1} f(x)dx$

First, assume $f(1) = 0$.

Then
$$\begin{aligned}\lim_{n\to\infty} \left| \int_{1-\delta}^1 nx^{n-1} f(x)dx \right| &\le \lim_{n\to\infty} \int_{1-\delta}^1 nx^{n-1} |f(x)|\, dx \\ &\le \varepsilon \lim_{n\to\infty} \int_{1-\delta}^1 nx^{n-1} dx \\ &= \varepsilon \lim_{n\to\infty} x^n\Big]_{1-\delta}^1 \\ &= \varepsilon.\end{aligned}$$

Thus. $\int_{1-\delta}^1 nx^{n-1} f(x)dx \to 0 = f(1)$.

Now assume $f(1) > 0$. Assume we've taken $\varepsilon$ smaller than $f(1)$.

Then
$$\begin{aligned}\lim_{n\to\infty} \int_{1-\delta}^1 nx^{n-1} f(x)dx &\ge (f(1)-\varepsilon)\lim_{n\to\infty} \int_{1-\delta}^1 nx^{n-1} dx \\ &= f(1) - \varepsilon.\end{aligned}$$

Similarly, $\lim\limits_{n\to\infty} \int_{1-\delta}^1 nx^{n-1} f(x)dx \le f(1) + \varepsilon$.

Thus, since $\varepsilon$ arbitrary, $\lim\limits_{n\to\infty} \int_{1-\delta}^1 nx^{n-1} f(x)dx = f(1)$.

This reasoning works when $f(1) < 0$ as well.

And $\lim\limits_{n\to\infty} \int_0^{1-\delta} nx^{n-1} f(x)dx = \int_0^{1-\delta} f(x) \lim\limits_{n\to\infty} nx^{n-1} dx = 0$ by DCT.

$f$ continuous on $[0, 1-\delta]$ $\Rightarrow$ $f$ has a maximum, say at $x$. Then,
$\frac{d}{dn} nx^{n-1} = x^{n-1} + nx^{n-1}\ln(x) = x^{n-1}(1 + n\ln(x)) = 0$ when $n = \frac{-1}{\ln x}$.
So for $n > \frac{-1}{\ln(x)}$, $nx^{n-1}$ decreases.
Thus $\forall\ n$, $ny^{n-1} f(y) \le \left(\frac{-1}{\ln(x)}\right) x^{\left(-\frac{1}{\ln(x)} - 1\right)} f(x)$ which is integrable.

Hence, $\lim\limits_{n\to\infty} \int_0^1 nx^{n-1} f(x)dx = 0 + f(1) = f(1)$.

$\square$

17 sp.4

**Problem 4.** Let $(\Omega, \mathcal{F}, \mu)$ be a $\sigma$-finite measure space. Let $f, g$ be measurable real valued functions. Show that

$$\int |f-g|d\mu = \int_{-\infty}^{\infty} \int \left|\chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x))\right| d\mu(x)dt.$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int \left| \chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x)) \right| d\mu(x)\, dt \\
= &\int \int_{-\infty}^{\infty} \left| \chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x)) \right| dt\, d\mu(x) \\
= &\int \int_{-\infty}^{\infty} \left| \chi_{(-\infty, f(x))}(t) - \chi_{(-\infty, g(x))}(t) \right| dt\, d\mu(x) \\
= &\int \int_{-\infty}^{\infty} \chi_{[\min\{f(x), g(x)\}, \max\{f(x), g(x)\})}(t)\, dt\, d\mu(x) \\
= &\int |f(x) - g(x)|\, d\mu
\end{aligned}$$

We can swap integrals because of Tonelli, since $\left|\chi_{(t,\infty)}(f(x)) - \chi_{(t,\infty)}(g(x))\right| \in L^+(\mathbb{R} \times \Omega)$. Indeed, this function is measurable because the composition of measurable functions is measurable.

$\square$.

17 Fa. 1

**Problem 1.** Let $(X, \mathcal{A}, \mu)$ be a measure space and $f, g, f_n, g_n$ measurable so that $f_n \to f$ and $g_n \to g$ in measure. Is it true that $f_n^3 + g_n \to f^3 + g$ in measure if

(a) $\mu(X) = 1$

(b) $\mu(X) = \infty$

In both cases prove the statement or provide a counter example.

a) Since $\mu(X) < \infty$, $\exists\ M > 1$ s.t. $\mu(\{x : |f| \geq M\}) < \varepsilon$. ? True ?

$\exists\ N$ s.t. $\forall\ n \geq N$, $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon/12M^2\}) < \varepsilon$
and $\mu(\{x : |g_n(x) - g(x)| \geq \varepsilon/4\}) < \varepsilon$

If $|f(x)| < M$, $|f_n(x) - f(x)| < \varepsilon/12M^2$, and $|g_n(x) - g(x)| < \varepsilon/4$

$$\begin{aligned}
|f_n^3(x) - f^3(x) + g_n(x) - g(x)| &\leq |f_n^3(x) - f^3(x)| + |g_n(x) - g(x)| \\
&= |(f_n - f)^3 + 3f_nf(f_n - f)| + |g_n - g| \\
&= |(f_n - f)^3 + 3(f + (f_n - f))f(f_n - f)| + |g_n - g| \\
&= |(f_n - f)^3 + 3f^2(f_n - f) + 3f(f_n - f)^2| + |g_n - g| \\
&\leq |f_n - f|^3 + 3M^2|f_n - f| + 3M|f_n - f|^2 + |g_n - g| \\
&< \varepsilon^3/12^3M^6 + 3M^2\,\varepsilon/12M^2 + 3M\,\varepsilon^2/12^2M^4 + \varepsilon/4 \\
&\leq \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 \\
&= \varepsilon.
\end{aligned}$$

Thus, $\{x : |f_n^3 + g_n - f^3 - g| \geq \varepsilon\} \subset \{x : |f_n - f| \geq \varepsilon/12M^2\} \cup \{x : |g_n - g| \geq \varepsilon/4\} \cup \{x : |f| \geq M\}$

$$\begin{aligned}
\Rightarrow \quad \mu(\{x : |f_n^3 + g_n - f^3 - g| \geq \varepsilon\}) &\leq \mu(\{x : |f_n - f| \geq \varepsilon/12M^2\}) + \mu(\{x : |g_n - g| \geq \varepsilon/4\}) \\
&\quad + \mu(\{x : |f| \geq M\}) \\
&< 3\varepsilon
\end{aligned}$$

Since $\varepsilon$ arbitrary, $f_n^3 + g_n \longrightarrow f^3 + g$ in measure.

$\square$

b) Consider $f_n(x) = x + \frac{1}{n}$, $f(x) = x$ on $(\mathbb{R}, \mathcal{M}, m)$.
Let $\varepsilon > 0$. $m(\{x : |f_n(x) - f(x)| \geq \varepsilon\})$
$= m(\{x : \frac{1}{n} \geq \varepsilon\}) \longrightarrow 0$ as $n \to \infty$.
Thus, $f_n \longrightarrow f$ in measure.

But, $f_n^3(x) = x^3 + 3x^2\frac{1}{n} + 3x\frac{1}{n^2} + \frac{1}{n^3}$.
Let $\varepsilon > 0$. $m(\{x : |f_n^3(x) - f^3(x)| \geq \varepsilon\})$
$= m(\{x : 3x^2\frac{1}{n} + 3x\frac{1}{n^2} + \frac{1}{n^3} \geq \varepsilon\}) = \infty$
for each $n$, since $\forall\ x \geq n\sqrt{\varepsilon}$,
we have $3x^2\frac{1}{n} + 3x\frac{1}{n^2} + \frac{1}{n^3} \geq 3n\varepsilon + 3\sqrt{\varepsilon}\frac{1}{n} + \frac{1}{n^3} \geq \varepsilon$.
Take $g_n = g$ $\forall\ n$, and the counterexample is shown.

<u>17Fa.2</u>

**Problem 2.** Let $f \in L^1(\mathbb{R})$. Show that the series

$$\sum_{n=1}^{\infty} f(x+n)$$

converges absolutely for Lebesgue almost every $x \in \mathbb{R}$.

For $M \in \mathbb{Z}$,

$$\begin{aligned}\int_M^{M+1} \sum_{n=1}^{\infty} |f(x+n)|\,dx &= \sum_{n=1}^{\infty} \int_M^{M+1} |f(x+n)|\,dx \quad * \\ &= \sum_{n=1}^{\infty} \int_{M+n}^{M+n+1} |f(x)|\,dx \\ &= \int_{M+1}^{\infty} |f(x)|\,dx \\ &< \infty\end{aligned}$$

Thus, for almost every $x \in [M, M+1]$, $\sum_{n=1}^{\infty} |f(x+n)| < \infty$. And since $M$ was arbitrary, this is true for almost every $x \in \mathbb{R}$.

$\square$

$*$ We can pull the sum out of the integral because $|f(x+n)| \in L^+$.

17Fa.3

**Problem 3.** Assume that $E \subset \mathbb{R}$ is such that $m(E \cap (E+t)) = 0$ for all $t \neq 0$, where $m$ is the Lebesgue measure on $\mathbb{R}$. Prove that $m(E) = 0$.

$$\begin{aligned} m(E \cap (E+t)) &= \int \chi_E \chi_{E+t} \, dm \\ &= \int \chi_E(x) \chi_E(x-t) \, dm(x) \\ &= 0 \qquad\qquad \forall\, t \neq 0. \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & \int \int \chi_E \chi_{E+t} \, dm(x) \, dm(t) = 0 \\ &= \int\int \chi_E \chi_{E+t} \, dm(t) \, dm(x) \\ &= \int \chi_E \int \chi_{E+t} \, dm(t) \, dm(x) \\ &= \int \chi_E(x) \int \chi_{x-E}(t) \, dm(t) \, dm(x) \\ &= \int \chi_E(x) \, m(x-E) \, dm(x) \\ &= m(E) \int \chi_E(x) \, dm(x) \\ &= m(E)^2 \\ &= 0 \\ \Rightarrow \quad & m(E) = 0. \end{aligned}$$

# 17 Fa. 4

**Problem 4.** Let $(X, \mathcal{A}, \mu)$ be a measure space and $f_n$ a sequence of non-negative measurable functions. Prove that if $\sup_n f_n$ is integrable, then

$$\limsup_n \int_X f_n d\mu \le \int_X \limsup_n f_n d\mu.$$

Also show that

(a) the inequality may be strict and

(b) that the inequality may fail unless $\sup_n f_n \in L^1$.

For fixed $K \in \mathbb{N}$,

$$f_n \le \sup_{n \ge K} f_n \quad \forall \; n \ge K$$

$$\Rightarrow \quad \int_X f_n \, d\mu \le \int_X \sup_{n \ge K} f_n \, d\mu \quad \forall \; n \ge K$$

$$\Rightarrow \quad \sup_{n \ge K} \int_X f_n \, d\mu \le \int_X \sup_{n \ge K} f_n \, d\mu$$

Thus, $\lim_{K \to \infty} \sup_{n \ge K} \int_X f_n \, d\mu \le \lim_{K \to \infty} \int_X \sup_{n \ge K} f_n \, d\mu = \int_X \lim_{K \to \infty} \sup_{n \ge K} f_n \, d\mu$

by DCT, since $\sup_{n \ge K+1} f_n \le \sup_{n \ge K} f_n$ and $\sup_{n \ge 1} f_n$ integrable. $\square$

a) Consider $f_n = \begin{cases} \chi_{[0, 1/2]} & \text{if } n \text{ odd} \\ \chi_{[1/2, 1]} & \text{if } n \text{ even}. \end{cases}$

Then $\limsup_n \int_0^1 f_n \, d\mu = \limsup_n \frac{1}{2} = \frac{1}{2}$

And $\int_0^1 \limsup_n f_n \, d\mu = \int_0^1 1 \, d\mu = 1.$

? These examples good?

b) Consider $f_n = \chi_{[n, n+1]}$, $\sup_n f_n = \chi_{[0, \infty]} \notin L^1$.

$\limsup_n \int_0^\infty f_n \, d\mu = \limsup_n 1 = 1.$

$\int_0^\infty \limsup_n f_n \, d\mu = \int_0^\infty 0 \, d\mu = 0.$

Simpler then Kaplas.

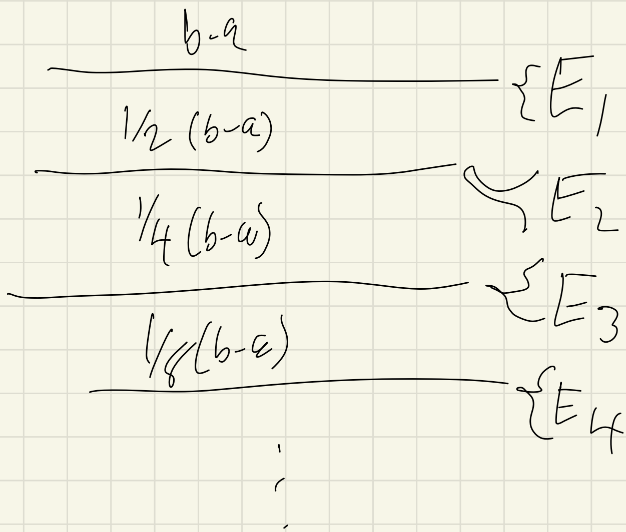
18 sp. 1

**Problem 1.** Let $-\infty < a < b < \infty$ and suppose $\mathcal{B}$ is a countable collection of closed subintervals of $(a, b)$. Give the proof that there is a countable pairwise-disjoint subcollection $\mathcal{B}' \subset \mathcal{B}$ such that

$$\bigcup_{I \in \mathcal{B}} I \subset \bigcup_{I \in \mathcal{B}'} \tilde{I},$$

where $\tilde{I}$ denotes the 5-times enlargement of $I$; thus if $I = [x - \rho, x + \rho]$ then $\tilde{I} = [x - 5\rho, x + 5\rho]$.

Let $E_n := \{ I$



18 sp. 2

**Problem 2.** Assume that $f$ is absolutely continuous on $[0,1]$, and assume that $f' = g$ a.e., where $g$ is a continuous function. Prove that $f$ is continuously differentiable on $[0,1]$.

Let $\varepsilon > 0$, $x \in [0,1]$. $g$ continuous $\Rightarrow \exists\, \delta > 0$ s.t.
$\forall\, y \in (x-\delta, x+\delta) \cap [0,1]$, $|g(y) - g(x)| < \varepsilon$.

$f$ absolutely continuous and $g = f'$ a.e. $\Rightarrow$ on any closed interval $[a,b] \subset [0,1]$, we have $\int_a^b g\, dm = f(b) - f(a)$.

We have $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} g\, dm$

Thus as the sequence $h_n \longrightarrow 0$, $\exists\, N \in \mathbb{N}$ s.t.
$\forall\, n \geq N$, $|h_n| < \delta \Rightarrow |g(x+h_n) - g(x)| < \varepsilon \quad \forall\, n \geq N$.
$\Rightarrow (g(x)-\varepsilon)h = \int_x^{x+h} g(x) - \varepsilon\, dm \leq \int_x^{x+h} g\, dm \leq \int_x^{x+h} g(x) + \varepsilon\, dm = (g(x)+\varepsilon)h$

$$\Rightarrow \quad g(x) - \varepsilon \leq f'(x) \leq g(x) + \varepsilon$$

$\Rightarrow \quad f'(x) = g(x)$

$\Rightarrow \quad f' = g \quad \forall\, x \quad \Rightarrow \quad f'$ continuous $\Rightarrow f$ cont. diff.

18sp.3

**Problem 3.** Let $(X, \mathcal{M}, \mu)$ be a measure space such that $\mu(X) = 1$. Let $A_1, A_2, ..., A_{50} \in \mathcal{M}$. Assume that almost every point in $X$ belongs to at least 10 of these sets. Prove that at least one of the sets has measure greater than or equal to 1/5.

Consider $\int_X \sum_1^{50} \chi_{A_i} \, d\mu = \sum_1^{50} \mu(A_i)$ since $\sum_1^{50} \chi_{A_i} \in L^+$

$\sum_1^{50} \chi_{A_i}(x) \geq 10$ a.e. $\Rightarrow \int_X \sum_1^{50} \chi_{A_i} \, d\mu \geq \int_X 10 \, d\mu = 10.$

Thus, $\sum_1^{50} \mu(A_i) \geq 10$

If $\mu(A_i) < \frac{1}{5}$ $\forall i$, then $\sum_1^{50} \mu(A_i) < \sum_1^{50} \frac{1}{5} = 10$ $\Rightarrow\Leftarrow$

Hence $\mu(A_i) \geq \frac{1}{5}$ for some $i$.

18 sp. 4

**Problem 4.** Let $f : [0, \infty) \to \mathbb{R}$ be absolutely continuous on every closed subinterval of $[0, \infty)$ and

$$f(x) = f(0) - \int_0^x g(t)dt, \qquad \text{for } x \geq 0,$$

where $g \in \mathcal{L}^1([0, \infty))$. Show that

$$\int_0^\infty \frac{f(2x) - f(x)}{x} dx = (\log 2) \int_0^\infty g(t)dt.$$

Sign?

$$\int_0^\infty \frac{f(2x)-f(x)}{x} dx = \int_0^\infty \frac{f(0) - \int_0^{2x} g(t)\,dt - f(0) + \int_0^x g(t)\,dt}{x} dx$$

$$= \int_0^\infty \frac{-\int_x^{2x} g(t)\,dt}{x} dx$$

$$= \int_0^\infty \int_0^\infty -\frac{1}{x} g(t) \chi_{[x,2x]}(t)\,dt\,dx$$

$$= \int_0^\infty \int_0^\infty -\frac{1}{x} g(t) \chi_{[t/2,t]}(x)\,dt\,dx$$

$$= \int_0^\infty -g(t) \int_0^\infty \frac{1}{x} \chi_{[t/2,t]}(x)\,dx\,dt \quad *$$

$$= \int_0^\infty -g(t) \left[\ln(t) - \ln(t/2)\right] dt$$

$$= \ln(2) \int_0^\infty g(t)\,dt.$$

* Tonell $\Rightarrow$ $\int \left| -\frac{1}{x} g(t) \chi_{[t/2,t]}(x) \right| (dm(t) \times dm(x))$

$$= \int_0^\infty \int_0^\infty \left| -\frac{1}{x} g(t) \chi_{[t/2,t]}(x) \right| dx\,dt$$

$$= \int_0^\infty |-g(t)| \int_0^\infty \left|\frac{1}{x} \chi_{[t/2,t]}(x)\right| dx\,dt$$

$$= \ln(2) \int_0^\infty |g(t)| < \infty$$

$\Rightarrow$ $-\frac{1}{x} g(t) \chi_{[t/2,t]}(x) \in L^1(m(t) \times m(x))$

$\Rightarrow$ Fubini may be used.

18 Fa. 1

**Problem 1.** Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function. Let

$$g(x) = \int_0^1 f(xt)dt, \qquad x \in [0,1].$$

Show that $g$ is an absolutely continuous function.

$f$ absolutely continuous $\Rightarrow$ for $0 \le a \le b \le 1$, $\exists$ $h \in L^1$ s.t.
$f(b) - f(a) = \int_a^b h(t)\,dt.$

NTS $\exists$ $\bar{g} \in L^1$ s.t. $g(x) - g(0) = \int_0^x \bar{g}\, d\cancel{t}$.

$$\begin{aligned} g(x) - g(0) &= \int_0^1 f(xt)\,dt - \int_0^1 f(0)\,dt \\ &= \int_0^1 [f(xt) - f(0)]\,dt \\ &= \int_0^1 \int_0^{xt} h(s)\,ds\,dt \\ &= \int_0^1 \int_0^1 \chi_{[0,xt]}(s)\,h(s)\,ds\,dt \\ &= \int_0^1 \int_0^1 \chi_{[s/x,1]}(t)\,h(s)\,dt\,ds \\ &= \int_0^1 h(s)\,(1 - s/x)\,ds \end{aligned}$$

$s \in [0, xt]$

$0 \le s \le xt$

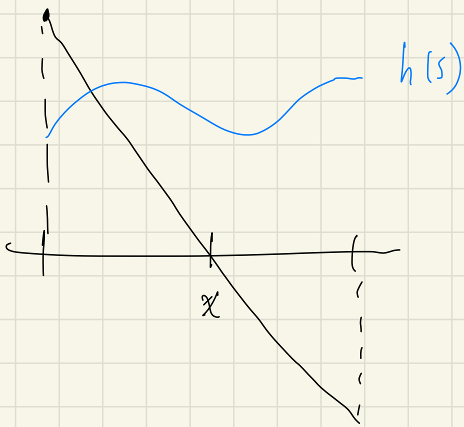
$1 \ge t \ge \frac{s}{x}$

$x = 0$?

$g(x) = g(0)$ ✓

$u = 1 - \frac{s}{x}$ $\qquad s = (1-u)x$

$du = -\frac{1}{x}$




$1 - \frac{1}{x} = \frac{x-1}{x}$

18Fa.2

**Problem 2.** Let $f \in L^1(\mathbb{R})$, and let

$$S_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} f\left(x + \frac{j}{n}\right), \qquad x \in \mathbb{R},$$

$$S(x) = \int_x^{x+1} f(y)dy, \qquad x \in \mathbb{R}.$$

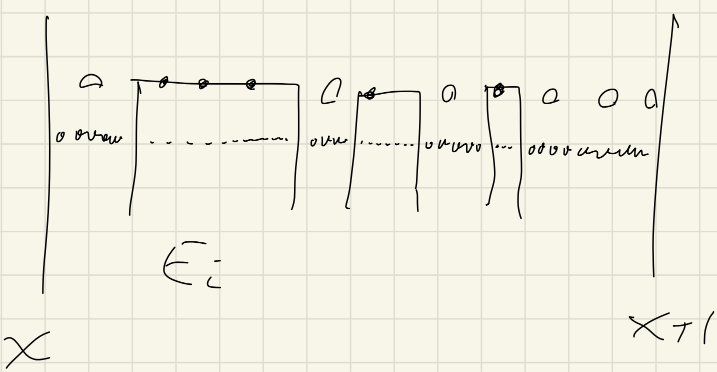
Show that $f_n \to f$ in $L^1(\mathbb{R})$.

> ***We assume that the question meant to say "Show that $S_n \to S$ in $L^1(\mathbb{R})$" since no $f_n$ is ever defined.

$$\begin{aligned}
\int |S_n(x) - S(x)|\,dx &= \int_{\mathbb{R}} \left| \frac{1}{n}\sum_{j=0}^{n-1} f(x+\tfrac{j}{n}) - \int_x^{x+1} f(y)\,dy \right| dx \\
&= \int_{\mathbb{R}} \left| \frac{1}{n}\sum_{j=0}^{n-1} f(x+\tfrac{j}{n}) - \sum_{j=0}^{n-1} \int_{x+j/n}^{x+(j+1)/n} f(y)\,dy \right| dx \\
&= \int_{\mathbb{R}} \left| \sum_{j=0}^{n-1} \frac{1}{n} f(x+\tfrac{j}{n}) - \int_{x+j/n}^{x+(j+1)/n} f(y)\,dy \right| dx \\
&= \int_{\mathbb{R}} \left| \sum_{j=0}^{n-1} \int_{x+j/n}^{x+(j+1)/n} f(x+j/n) - f(y)\,dy \right| dx
\end{aligned}$$

First, we prove the result for a simple function, $\phi = \sum_1^k \alpha_i \chi_{E_i}$

$$\begin{aligned}
\int_{\mathbb{R}} |S_n(x) - S(x)|\,dx &= \int_{\mathbb{R}} \left| \frac{1}{n}\sum_{j=0}^{n-1} \sum_{i=1}^{k} \alpha_i \chi_{E_i}(x+\tfrac{j}{n}) - \int_x^{x+1} \sum_{i=1}^{k} \alpha_i \chi_{E_i}(y)\,dy \right| dx \\
&= \int_{\mathbb{R}} \left| \frac{1}{n}\sum_{j=0}^{n-1} \sum_{i=1}^{k} \alpha_i \chi_{E_i}(x+\tfrac{j}{n}) - \sum_{i=1}^{k} \alpha_i\, m(E_i \cap [x,x+1]) \right| dx \\
&= \int_{\mathbb{R}} \left| \sum_{i=1}^{k} \alpha_i \left[ \frac{1}{n}\sum_{j=0}^{n-1} \chi_{E_i}(x+\tfrac{j}{n}) - m(E_i \cap [x,x+1]) \right] \right| dx
\end{aligned}$$



18 Fa.3

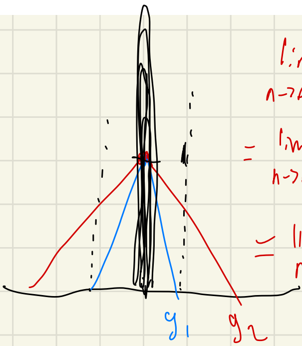
**Problem 3.** Assume that $f_n$ is a sequence of integrable functions on $\mathbb{R}$ such that

$$\lim_{n\to\infty} \int f_n(x)g(x)dx = g(0)$$

for all $g$ continuous with compact support.

Prove that $f_n$ is not a Cauchy sequence in $L^1(\mathbb{R})$.

Kaylen wrong?

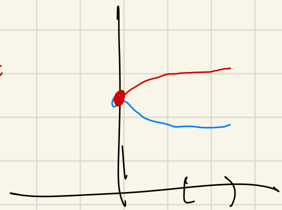



$$\lim_{n\to\infty} \int f_n(x)g(x)\,dx$$

$$= \lim_{n\to\infty} \int_{\{f_n(x)\to 0\}} f_n(x)g(x)\,dx + \int_{\{f_n \not\to 0\}} f_n g\,dx$$

$$= \lim_{n\to\infty} \int_{f_n(x)\to 0} f_n \tilde{g}\,dx + \int_{f_n \not\to 0} f_n \tilde{g}\,dx$$

$$\int_{[\varepsilon,\varepsilon]} f_n g \not\to 0.$$




$$\lim_{n\to\infty} \int f_n(x)g_1(x)\,dx \approx \lim_{n\to\infty}\int_0^1 f_n(x)\,dx$$

$$\lim_{n\to\infty} \int f_n(x)g_2(x)\,dx \approx \lim_{n\to\infty}\int_{-1}^0 f_n(x)\,dx$$

$= 1$

doesn't go to 0

$$\lim_{n\to\infty} \int f_n(x)g_1(x)\,dx = \lim_{n\to\infty}\int f_n(x)g_2(x)\,dx = 1$$

$$\lim_{n\to\infty} \int_{-1}^1 f_n(x)g_1(x)\,dx = \lim_{n\to\infty}\int_{-2}^2 f_n(x)g_2(x)\,dx$$

$f_n = \chi_{[-\frac{1}{n},\frac{1}{n}]}\, n/2$ Cauchy? $f_n \to 0$ a.e. but $\int f_n - 0 \not\to 0$.

$2/n$ $n/2$

$$\lim_{n\to\infty}\int f_n g = \lim_{n\to\infty}\int \frac{n}{2}\chi_{[-\frac{1}{n},\frac{1}{n}]}\, g$$

$=$

19 Sp. 1

**1.** Find the limit
$$\lim_{j\to\infty}\int_{-1}^{1}\frac{1-e^{-\frac{x^2}{j}}}{x^2}dx.$$

$$\lim_{j\to\infty}\int_{-1}^{1}\frac{1-e^{-x^2/j}}{x^2}dx = \int_{-1}^{1}\lim_{j\to\infty}\frac{1-e^{-x^2/j}}{x^2}dx \quad \text{by DCT}$$
$$= \int_{-1}^{1} 0\, dx = 0.$$

DCT:

$$\frac{d}{dj}\frac{1-e^{-x^2/j}}{x^2} = \frac{-e^{-x^2/j}\,(x^2/j^2)}{x^2} \le 0 \quad \forall x\neq 0,$$
$$\Rightarrow \frac{1-e^{-x^2/j}}{x^2} \le \frac{1-e^{-x^2}}{x^2} \quad \forall x\neq 0,\ \forall j\ge 1.$$

and $\frac{1-e^{-x^2}}{x^2}$ is continuous on $[-1,0)$ and $(0,1]$, with $\lim_{x\to 0}(1-e^{-x^2})/x^2 = 1$. Thus $\exists\ M$ s.t. $(1-e^{-x^2})/x^2 \le M \quad \forall x\in[-1,1]\setminus\{0\}$,

$$\Rightarrow \int_{-1}^{1}\frac{1-e^{-x^2/j}}{x^2}dx \le \int_{-1}^{1} M < \infty$$

Hence, DCT may be used.

---

Fix $x$. $\quad \lim_{j\to\infty}\frac{1-e^{-x^2/j}}{x^2} = 0$

Fix $j$. $\quad \lim_{x\to 0}\frac{1-e^{-x^2/j}}{x^2} = \frac{0}{0}$

$$LH \Rightarrow \lim_{x\to 0}\frac{(2x/j)e^{-x^2/j}}{2x} = \frac{0}{0}$$

$$LH \Rightarrow \lim_{x\to 0}\frac{\frac{2}{j}e^{-x^2/j} + \cancel{(2x/j)(-2x/j)}e^{-x^2/j}}{2} = \frac{\frac{2}{j}}{2} = \frac{1}{j}$$

19sp.2

**2.** Let $F$ be absolutely continuous and $F, F' \in L^1(\mathbb{R}, m)$, where $m(dx) = dx$ is the Lebesgue measure. Prove that
$$\int_{-\infty}^{\infty} F'(x)\,dx = 0.$$

For $M \in \mathbb{N}$, on $[-M, M]$, we have

$$\int_{-M}^{M} F'(x)\,dx = F(M) - F(-M).$$

Assume $\int_{-\infty}^{\infty} F'(x)\,dx = C > 0$. ($>0$ WLOG).

Since $F' \in L^1$, $\exists\ M \in \mathbb{N}$ s.t. $\forall\ m \geq M$,
$\int_{\mathbb{R}\setminus[-m,m]} F'(x)\,dx < C/2$.

Then $\forall\ m \geq M$,
$$\int_{-\infty}^{\infty} F'(x)\,dx = \int_{[-m,m]} F'(x)\,dx + \int_{\mathbb{R}\setminus[-m,m]} F'(x)\,dx = C$$

$$\Rightarrow \int_{[-m,m]} F'(x)\,dx = C - \int_{\mathbb{R}\setminus[-m,m]} F'(x)\,dx$$
$$> C/2$$

$$\Rightarrow \quad F(m) - F(-m) > C/2 \qquad \forall\ m \geq M.$$

Then $\int_{m \geq M} F(m) - F(-m)\,dm(m) = \infty$

But $\int_{m \geq M} F(m) - F(-m)\,dm(m) \leq \int_{m \geq M} |F(m)| + |F(-m)|$
$$\leq 2\int_{\mathbb{R}} |F| < \infty \qquad \Rightarrow\Leftarrow.$$

19 sp. 3

**3.** Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume that $h \circ f$ is integrable for every continuous function $h : \mathbb{R} \to \mathbb{R}$. Prove that there is $a > 0$ so that

$$\mu(\{x : |f(x)| > a\}) = 0.$$

Let $h(x) = h(-x) > 0 \ \forall x$. Let $h(n) \geq n / \mu(\{x : |f(x)| > n\}) \ \forall n \in \mathbb{N}$, with $\mu(\{x : |f(x)| > n\}) > 0$,
and make sure $h$ is continuous and increasing as $|x|$ increases.

Then
$$\begin{aligned} \int_X |h \circ f| \, d\mu &\geq \int_X |h \circ f| \chi_{\{x : |f(x)| > n\}} \, d\mu \\ &\geq \int_X h(n) \chi_{\{x : |f(x)| > n\}} \, d\mu \\ &= h(n) \mu(\{x : |f(x)| > n\}) \\ &\geq n \qquad \forall n \in \mathbb{N} \\ &\qquad \text{s.t. } \mu(\{x : |f(x)| > n\}) > 0. \end{aligned}$$

Thus, since $\int_X |h \circ f| \, d\mu < \infty$, there must be some $a > 0$ s.t. $\mu(\{x : |f(x)| > a\}) = 0$.

19 sp.4

**4.** (i) Show that for every $\varepsilon > 0$ there exists a non-negative $f \in L^1([0,1], m)$ such that $f(x) = 0$ on the set of measure $\geq 1 - \varepsilon$ and

$$\int_a^b f(x)\,dx > 0$$

for all $0 < a < b < 1$.

(ii) Show that for each $\varepsilon > 0$ there exists an absolutely continuous strictly increasing $h$ on $[0,1]$ so that $h'(x) = 0$ on a set of measure $\geq 1 - \varepsilon$.

(i) Enumerate the rationals: $r_1, r_2, \ldots$

Let $f(x) = \sum_{n=1}^{\infty} \chi_{[r_n, r_n + \varepsilon/2^{n+1}] \cap [0,1]}(x)$

Then $f(x) = 1$ on a set of measure $\leq \sum_{n=1}^{\infty} \varepsilon/2^{n+1} < \varepsilon$.
$\Rightarrow$ $f(x) = 0$ on a set of measure $\geq 1 - \varepsilon$.

Let $0 < a < b < 1$. $\exists$ rational $r_n$ s.t. $a < r_n < b$.

Then $\int_a^b f(x)\,dx \geq \int_a^b \chi_{[r_n, r_n + \varepsilon/2^{n+1}]}(x)\,dx$
$\geq \min\{b - r_n, \varepsilon/2^{n+1}\}$
$> 0$.

(ii) Let $f$ satisfy (i). Define $h(x) = \int_0^x f(y)\,dy$. for $x \in [0,1]$

Then $h$ is absolutely continuous on $[0,1]$, and $f(x) = h'(x)$ a.e. $\Rightarrow$ $h'(x) = 0$ on a set of measure $\geq 1 - \varepsilon$.
And $\forall$ $0 < a < b < 1$, we have $h(b) - h(a) = \int_a^b f(x)\,dx > 0$
$\Rightarrow$ $h$ is strictly increasing.

# 19 Fa.1

1. Let $f$ be a real valued function on a set $X$. Let $\mathcal{M}$ be the smallest $\sigma$-algebra on $X$ for which $f$ is measurable. Prove that $\{x\} \in \mathcal{M}$ for all $x \in X$ if and only if $f$ is one-to-one.

First, assume $f$ is one-to-one.
$f$ measurable $\Rightarrow$ $f^{-1}(U) \in \mathcal{M}$ for $U$ measurable in $\mathbb{R}$.

Let $x \in X$. $f$ one-to-one $\Rightarrow$ $f^{-1}(f(x)) = \{x\}$.
$\{f(x)\}$ is measurable in $\mathbb{R}$, and $f$ measurable $\Rightarrow \{x\} \in \mathcal{M}$.

Now assume $\{x\} \in \mathcal{M}$ $\forall x \in X$. Let $x, y$ be two points in $X$ s.t. $f(x) = f(y)$.

$\mathcal{M}' = \{E \in \mathcal{M} : x, y \in E \text{ OR } x, y \notin E\}$. Note $x, y \in E$ means Both.

$E \in \mathcal{M}' \Rightarrow E^c \in \mathcal{M}'$ since $x, y \in E \Leftrightarrow x, y \notin E^c$.

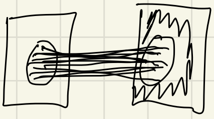
$\{E_i\}_1^\infty \subset \mathcal{M}'$.

If $x, y \notin E_i$ $\forall i$, then $x, y \notin \bigcup E_i$.

Otherwise $\exists\, i$ s.t. $x, y \in E_i \Rightarrow x, y \in \bigcup E_i$.

$X, \emptyset \in \mathcal{M}'$ as well.

$\{f^{-1}(A) : A \in \mathcal{B}_{\mathbb{R}}\}$
$\sigma$-algebra?
$f^{-1}(A^c) = f^{-1}(A)^c$?



$\Rightarrow \mathcal{M}'$ is a $\sigma$-algebra $\Rightarrow \mathcal{M} \subset \mathcal{M}'$ by minimality.

$\Rightarrow \{x\}, \{y\} \in \mathcal{M}'$ a contradiction.

Thus $f$ must be one-to-one.

Is $f$ measurable for $\mathcal{M}'$? $f^{-1}((b, \infty)) \ni x, y$ if $a \in (b, \infty)$
$\not\ni x, y$ if $a \notin (b, \infty)$

$\Rightarrow f^{-1}((b, \infty)) \in \mathcal{M}'$ $\forall b \in \mathbb{R}$ $\Rightarrow f$ measurable.

19Fa.2

2. Let $m$ be the Lebesgue measure on $[0,1]$. Let $f_n\colon [0,1] \to \mathbb{R}$ be measurable functions. Assume that $\sum_{n=1}^{\infty} \int_0^1 |f_n(x)|\, dx \le 1$. Prove that $f_n \to 0$ as $n \to \infty$ almost everywhere.

? Seems too short.

$$\sum_{n=1}^{\infty} \int_0^1 |f_n(x)|\, dx \overset{*}{=} \int_0^1 \sum_{n=1}^{\infty} |f_n(x)|\, dx \le 1$$

$\Rightarrow \sum_{n=1}^{\infty} |f_n(x)|$ converges almost everywhere

$\Rightarrow |f_n(x)| \longrightarrow 0 \Rightarrow f_n(x) \longrightarrow 0$ almost everywhere.

$*$ We can exchange $\sum_{n=1}^{\infty}$ and $\int_0^1$ since $|f_n(x)| \in L^+$.

$$\begin{aligned} \sum_{n=1}^{\infty} \int_0^1 |f_n(x)|\, dx &= \lim_{k\to\infty} \sum_{n=1}^{k} \int_0^1 |f_n(x)|\, dx \\ &= \lim_{k\to\infty} \int_0^1 \sum_{n=1}^{k} |f_n(x)|\, dx \\ &= \int_0^1 \lim_{k\to\infty} \sum_{n=1}^{k} |f_n(x)|\, dx \quad \text{by MCT.} \end{aligned}$$

19Fa.3

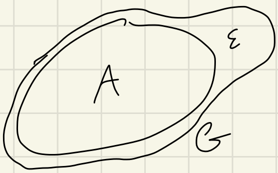
3. Let $m$ be the Lebesgue measure on $[0,1]$. Prove that there does not exists a measurable set $A \subseteq [0,1]$ such that $m(A \cap [a,b]) = (b-a)/2$ for all $0 \leq a < b \leq 1$.

? Zejing's disjoint union of closed intervals.

Let $G$ open $\supset A$ s.t. $m(G \setminus A) < \varepsilon$.

$G = \bigsqcup_1^\infty (a_i, b_i)$

$G \setminus A = \left(\bigsqcup_1^\infty (a_i, b_i)\right) \setminus A = \bigsqcup_1^\infty \left((a_i, b_i) \setminus A\right)$




$$
\begin{aligned}
m((a_i, b_i) \setminus A) &= m\left((a_i, b_i) \setminus (A \cap (a_i, b_i))\right) \\
&= m((a_i, b_i)) - m(A \cap (a_i, b_i)) \\
&= (b_i - a_i) - \frac{b_i - a_i}{2} \\
&= \frac{b_i - a_i}{2}.
\end{aligned}
$$

Then, $m(G \setminus A) = \sum_1^\infty m((a_i, b_i) \setminus A) = \sum_1^\infty \frac{b_i - a_i}{2} = \frac{1}{2} m(G)$

$\Rightarrow \quad \frac{1}{2} m(G) < \varepsilon$

$\Rightarrow \quad m(A) < m(G) < 2\varepsilon$, which is a contradiction since

$m(A) = m(A \cap [0,1]) = \frac{1}{2}$ and $\frac{1}{2} \geq 2\varepsilon$ for small $\varepsilon$.

19Fa.4

4. Let $V_f(0,x)$ be the total variation of $f$ on $[0,x]$. Prove that if $f(x)$ is absolutely continuous on $[0,1]$, then so is $V_f(0,x)$.

$$V_f(0,x) = \sup_{0=a_1<a_2<\cdots<a_n=x} \sum |f(a_{i+1}) - f(a_i)|$$

$f$ abs. cont. $\Rightarrow$ $f$ differentiable a.e. on $[0,1]$ and $f' \in L^1([0,1])$ with $\int_a^b f'(x)\,dx = f(b) - f(a)$ for $0 \le a \le b \le 1$.

$V_f(0,x)$ increasing $\Rightarrow$ discontinuities are at most countable
$\Rightarrow$ $V_f'(0,x)$ defined a.e. on $[0,1]$.

$\exists$ sequence $0 = a_1 < \cdots < a_n = x$ s.t. $\sum |f(a_{i+1}) - f(a_i)| > V_f(0,x) - \varepsilon$.
Then, for this sequence,
$\int_0^x |f'(t)|\,dt = \sum \int_{a_i}^{a_{i+1}} |f'(t)|\,dt \le \sum \left| \int_{a_i}^{a_{i+1}} f'(t)\,dt \right| = \sum |f(a_{i+1}) - f(a_i)| > V_f(0,x) - \varepsilon$.
Since $\varepsilon$ arbitrary, $\int_0^x |f'(t)|\,dt \ge V_f(0,x)$.

For any $x < y \in [0,1]$, we have $|f(y) - f(x)| \le V_f(y) - V_f(x)$
Then for $x \in [0,1]$ where $f'(x)$ and $V_f'(0,x)$ exist,
we have $|f'(x)| = \left| \lim_{y \to x} \frac{f(y)-f(x)}{y-x} \right| = \lim_{y \to x} \frac{|f(y)-f(x)|}{|y-x|} \le \lim_{y \to x} \frac{V_f(y) - V_f(x)}{|y-x|} = V_f'(0,x)$.

$$\Rightarrow \int_0^x |f'(t)|\,dt \le \int_0^x V_f'(0,t)\,dt \le V_f(0,x)$$

* since $V_f$ increasing.

? This right.

Hence $\int_0^x |f'(t)|\,dt = V_f(0,x)$ and $|f'(t)| \in L^1([0,1])$
$\Rightarrow$ $V_f(0,x)$ absolutely continuous.

20sp. 1

$f: \mathbb{R} \to \mathbb{R}$ strictly incr., cont.

T/F,

If $A \subset \mathbb{R}$ Lebesgue measurable then $f^{-1}(A)$ Lebesgue measurable.

False. Let $C \subset [0,1]$ be the cantor set. Let $g(x)$ be the cantor function. Define $h(x) = g(x) + x$, $h: [0,1] \to [0,2]$.
$g, x$ continuous $\Rightarrow$ $h$ continuous. $h'(x) = 1$ a.e. since $g'(x) = 0$ a.e.
$\Rightarrow$ $h$ strictly increasing. Thus $h^{-1}: [0,2] \to [0,1]$ well defined, continuous, and strictly increasing.

Consider
$$
\begin{aligned}
m(h([0,1] \setminus C)) &= m(h(\cup_1^\infty (a_i, b_i))), \\
&= m(\cup_1^\infty h((a_i, b_i))) \\
&= \textstyle\sum_1^\infty m(h((a_i, b_i))), \quad \text{since } h \text{ strictly increasing} \\
&= \textstyle\sum_1^\infty m((h(a_i), h(b_i))) \\
&= \textstyle\sum_1^\infty m((g(a_i) + a_i, g(b_i) + b_i)) \\
&= \textstyle\sum_1^\infty b_i - a_i, \quad \text{since } g(a_i) = g(b_i) \\
&= m([0,1] \setminus C) \\
&= 1
\end{aligned}
$$

[where $\{(a_i, b_i)\}$ are the intervals taken out to form $C$.]

Thus, $m(h(C)) = m([0,2]) - m(h([0,1] \setminus C)) = 2 - 1 = 1$.

Then, $\exists$ non-measurable set $E \subset h(C)$, since $m(h(C)) > 0$.

But $h^{-1}(E) \subset h^{-1}(h(C)) = C$, a null set in Lebesgue measure.
Since Lebesgue measure complete, $h^{-1}(E)$ Lebesgue measurable.

So take $f := h^{-1}$, with tails that go to $\pm\infty$ on $\mathbb{R}$, and take $A := h^{-1}(E)$, Lebesgue measurable.

Then $f^{-1}(A) = h(h^{-1}(E)) = E$, not Lebesgue measurable.

Note, $h^{-1}(E)$ special because it's Lebesgue measurable, but not Borel.
If $A$ is just Borel measurable, this is true.

20 sp. 2

? How

$$\lim_{n\to\infty} n\int_{1/n}^{1} \frac{\cos(x+\frac{1}{n})-\cos x}{x}\,dx$$

$$= \lim_{h\to 0} \frac{1}{h}\int_{h}^{1} \frac{\cos(x+h)-\cos x}{x}\,dx$$

$$= \int_0^1 \frac{1}{x} \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h} \chi_{[h,1]}\,dx \qquad * \text{ by DCT}$$

$$= \int_0^1 \frac{\frac{d}{dx}\cos(x)}{x}\,dx$$

$$= \int_0^1 \frac{-\sin(x)}{x}\,dx < \infty, \quad \text{since } \frac{-\sin(x)}{x} \text{ continuous on } (0,1]$$

and $\lim_{x\to 0} \frac{-\sin(x)}{x} = \lim_{x\to 0} \frac{-\cos(x)}{1} = -1$.

* Let $f_h = \frac{1}{x}\frac{\cos(x+h)-\cos x}{h}\chi_{[h,1]}$

By MVT, $\left|\frac{\cos(x+h)-\cos x}{h}\right| \le |\cos'(x+h)|$, for $x\in(0,1]$, $h$ small.

$= |-\sin(x+h)|$

Thus, $|f_h| \le \frac{\sin(x+h)}{x}\chi_{[h,1]}$ $\forall x\in(0,1]$, $\forall h$.

$\lim_{h\to 0} f_h(h) = \lim_{h\to 0} \frac{\sin(2h)}{h} = \lim_{h\to 0} 2\sin(h) = 2.$

$\frac{d}{dh} f_h(h) = \frac{d}{dh}\frac{\sin(2h)}{h} = \frac{2h\cos(2h)-\sin(2h)}{h^2} < 0$ when $h>0$, because

$$2h\cos(2h)-\sin(2h)<0 \iff 2h\cos(2h)<\sin(2h)$$
$$\iff \tan(2h)>2h$$
$$\Leftarrow 2h\in(0,\pi/2).$$

So, $f_h(h) \nearrow 2$ as $h \searrow 0$.

Similarly $\frac{d}{dx} \frac{\sin(x+h)}{x} = \frac{x\cos(x+h) - \sin(x+h)}{x^2} < 0$ for $x \in (0, \pi/2)$.

Hence $|f_h| \leq 2 \quad \forall \ x \in (0,1], \ \forall h.$

which is clearly integrable on $[0,1]$.

## 20 sp. 3

$f: \mathbb{R} \to \mathbb{R}$, $M > 0$. Prove (a) $\Longleftrightarrow$ (b) where
(a) $|f(x) - f(y)| \leq M|x-y| \quad \forall\, x, y \in \mathbb{R}$.
(b) $f$ abs. cont. with $|f'(x)| \leq M \quad \forall\, x \in \mathbb{R}$.

(a) $\Rightarrow$ (b): NTS $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. for any finite collection of disjoint intervals, $\{(a_i, b_i)\}$, we have

$$\sum (b_i - a_i) < \delta = \sum |f(b_i) - f(a_i)| < \varepsilon.$$

Take $\delta = \varepsilon / M$, then
$$\sum |f(b_i) - f(a_i)| \leq \sum M|b_i - a_i| = M \sum (b_i - a_i) < M\delta = \varepsilon$$

And $|f'(x)| = \left| \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \right| = \lim_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} \leq M$
since $\frac{|f(y) - f(x)|}{|y - x|} \leq M \quad \forall\, x, y \in \mathbb{R}$.

(b) $\Rightarrow$ (a): $f$ abs cont $\Rightarrow$ $f$ abs. cont. on $[-k, k]$ for any $k \in \mathbb{Z}$.

$\Rightarrow$ $f$ diff a.e. on $[-k, k]$ and
$$\int_a^b f'(x)\,dx = f(b) - f(a) \quad \forall\, -k \leq a \leq b \leq k.$$

$$|f'(x)| \leq M \;\forall\, x \in \mathbb{R} \Rightarrow \int_a^b f'(x)\,dx \leq M(b-a)$$
$$\Rightarrow f(b) - f(a) \leq M(b-a)$$

and
$$\int_a^b f'(x)\,dx \geq -M(b-a)$$
$$\Rightarrow f(b) - f(a) \geq -M(b-a)$$

Which is to say $|f(b) - f(a)| \leq M|b-a| \quad \forall\, a, b \in [-k, k]$.

Since $k$ arbitrary, for any $x, y \in \mathbb{R}$, the result holds, since $\exists\, k \in \mathbb{Z}$, s.t. $x, y \in [-k, k]$.

20 sp. 4

$m$ Lebesgue measure, $m^*$ Lebesgue outer measure.

$$m_*(A) = \sup\{m(K) : K \subset A,\ K \text{ compact}\}.$$

Prove $m^*(A) = m_*(A)$ and $m^*(A) < \infty \Rightarrow A$ Lebesgue measurable.

---

$$m^*(A) = \inf\{\textstyle\sum b_i - a_i : A \subset \bigcup_i^\infty (a_i, b_i)\}$$
$$= \sup\{m(K) : K \subset A,\ K \text{ compact}\} < \infty$$

Claim: $A$ is $m^*$-measurable $\Rightarrow$ $A$ is Lebesgue measurable.

Claim: $\forall\, E \subset \mathbb{R}$, $m^*(E) = m^*(E \cap A) + m^*(E \cap A^c)$.

Know $m^*(E) \leq m^*(E \cap A) + m^*(E \cap A^c)$ since $m^*$ outer measure.

$\exists$ $K$ compact, $\{(a_i, b_i)\}$ s.t. $K \subset A \subset \bigcup_i^\infty (a_i, b_i)$
and $\sum (b_i - a_i) - m(K) < \varepsilon$.

Let $E \subset \mathbb{R}$. $m^*(E) = \inf\{\sum b_i - a_i : E \subset \bigcup_i^\infty (a_i, b_i)\}$

$m^*(E \cap A) = \inf\{\sum b_i - a_i : E \cap A \subset \bigcup_i^\infty (a_i, b_i)\}$

$m^*(E \cap A^c) = \inf\{\sum b_i - a_i : E \cap A^c \subset \bigcup_i^\infty (a_i, b_i)\}$

$\exists$ $K$ compact s.t. $m(K) > m^*(A) - \varepsilon$.

$K$ compact $\Rightarrow K \in \mathcal{B}_{\mathbb{R}} \Rightarrow K$ $m^*$-measurable $\Rightarrow m^*(E) = m^*(K \cap E) + m^*(K^c \cap E)$.

$$E \cap A = (E \cap K) \cup (E \cap (A \setminus K))$$

$$\begin{aligned} m^*(E \cap A) &\leq m^*(E \cap K) + m^*(E \cap (A \setminus K)) \\ &= m^*(E) - m^*(K^c \cap E) + m^*(E \cap (A \setminus K)) \\ &\leq m^*(E) - m^*(K^c \cap E) + m^*(A \setminus K) \\ &< m^*(E) - m^*(K^c \cap E) + \varepsilon \\ &\leq m^*(E) - m^*(A^c \cap E) + \varepsilon \end{aligned}$$

$\Rightarrow m^*(E \cap A) + m^*(A^c \cap E) \leq m^*(E) + \varepsilon$

$\Rightarrow m^*(E \cap A) + m^*(A^c \cap E) \leq m^*(E)$


Done.

$F(x) = x$.

$\mathcal{A} = \{$ finite disjoint unions of h-intervals $\}$, an algebra.

$\mu_0 \left( \bigcup_1^n (a_j, b_j] \right) = \sum_1^n F(b_j) - F(a_j) = \sum_1^n b_j - a_j$, a premeasure on $\mathcal{A}$.

$\mathcal{B}_{\mathbb{R}}$ = Borel $\sigma$-algebra $= \sigma(\mathcal{A})$.

$\mu^*(E) = \inf \{ \sum_1^\infty \mu_0(A_j) : A_j \in \mathcal{A}, E \subset \bigcup_1^\infty A_j \}$, an outer measure.
$= \inf \{ \sum_1^\infty b_j - a_j : E \subset \bigcup_1^\infty (a_j, b_j) \}$.

$\mathcal{M} = \{ \mu^*$-measurable sets $\}$

Caratheodory $\Rightarrow$ $\mathcal{M}$ $\sigma$-algebra, $\mu^*|_{\mathcal{M}}$ complete measure.

1.13 $\Rightarrow$ $\mathcal{A} \subset \mathcal{M} \Rightarrow \mathcal{B}_{\mathbb{R}} \subset \mathcal{M}$

Define Lebesgue measure $m = \mu^*|_{\mathcal{M}}$

$\boxed{\text{Fa 20.1}}$

$f_n \geq 0$. $f_n$ measurable on $(X, \mu)$.

$\int f_n \, d\mu = 1 \quad \forall n \in \mathbb{N}$. Claim: $\limsup_n f_n(x)^{1/n} \leq 1$ $\mu$-a.e.

$$\limsup_n f_n(x)^{1/n} = \lim_{k\to\infty} \sup_{n\geq k} f_n(x)^{1/n}$$

$$\int \lim_{k\to\infty} \sup_{n\geq k} f_n(x)^{1/n} dx = \lim_{k\to\infty} \int \sup_{n\geq k} f_n(x)^{1/n} dx$$

Assume not: assume $\limsup_n f_n(x)^{1/n} > 1$ on $E \subset X$ w/ $\mu(E) > 0$.

Then for each $x \in E$

$\exists$ subsequence $n_k$ and $\varepsilon > 0$ s.t. $f_{n_k}(x)^{1/n_k} > 1+\varepsilon \quad \forall n_k$.

$f_{n_k}(x) > (1+\varepsilon)^{n_k}$

$\lim_n \int f_n \, d\mu = 1 = \int \lim f_n \, d\mu$ not true nec.

take $\chi_{[n,n+1]}$ on $\mathbb{R}$. not dominated

$$\int \lim f_n \, d\mu \leq 1$$

$$\int \liminf f_n \, d\mu \leq \liminf \int f_n \, d\mu = 1$$

---

Root test: $\limsup_n |x_n|^{1/n} > 1 \implies \sum |x_n| = \infty$

So $\limsup_n f_n(x)^{1/n} > 1$ on $E$, $\mu(E) > 0$.

Then $\sum f_n(x) = \infty \quad \forall x \in E$

Then $\int \sum f_n(x) = \infty$

$\lim_{n\to\infty} n^{2/n}$

$g_n = \frac{f_n}{n^2}$ $\quad \limsup_n g_n^{1/n} = \limsup_n \left(\frac{f_n}{n^2}\right)^{1/n}$

$= \limsup \frac{f_n^{1/n}}{n^{2/n}} = \limsup f_n^{1/n}$

$\implies \sum g_n(x) = \infty \implies \int \sum g_n(x) = \infty = \sum \int g_n(x) = \sum \frac{1}{n^2} < \infty$.

$\implies \Leftarrow$.

# 20Fa.1 Try 2

$f_n$ measurable, $\geq 0$. $\int f_n \, d\mu = 1$ $\forall n \in \mathbb{N}$.
Claim: $\limsup_n f_n(x)^{1/n} \leq 1$ $\mu$-a.e.

Assume instead, for some $\varepsilon > 0$, $\limsup_n f_n(x)^{1/n} > 1+\varepsilon$ $\forall x \in E$
where $\mu(E) > 0$.

$f_n > (1+\varepsilon)^n$

$\limsup_n f_n(x)^{1/n} = \lim_{k\to\infty} \sup_{n \geq k} f_n(x)^{1/n} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty}$

Let $E_n = \{x : f_n(x)^{1/n} > 1+\varepsilon\}$

$E = \limsup_n E_n = \{x : \limsup_n f_n(x)^{1/n} > 1+\varepsilon\}$ ?

$$\begin{aligned}
&\limsup_n E_n \\
&\quad \| \\
&\bigcap_k \bigcup_{n \geq k} E_n \\
&\quad \| \\
&\bigcap_k \{x : f_n(x)^{1/n} > 1+\varepsilon, \text{ for some } n \geq k\} \\
&\quad \| \\
&\{x : f_n(x)^{1/n} > 1+\varepsilon, \text{ for some } n \geq k, \forall k\} \\
&\quad \Updownarrow \\
&\{x : \limsup_n f_n(x)^{1/n} > 1+\varepsilon\}. \checkmark
\end{aligned}$$

$$\begin{aligned}
m(E) &= m\left(\bigcap_{k=1}^{\infty} \bigcup_{n \geq k} E_n\right) \\
&= \lim_{k\to\infty} m\left(\bigcup_{n \geq k} E_n\right) \quad * \\
&\leq \lim_{k\to\infty} \sum_{n \geq k} \frac{1}{(1+\varepsilon)^n} \\
&= 0.
\end{aligned}$$

$\varepsilon$ arbitrary. $\Rightarrow$ result.

$\mu(E_n) \leq \frac{1}{(1+\varepsilon)^n}$ $\forall n$.

$\mu(E_n) \to 0$.

$$\begin{aligned}
1 = \int f_n \, d\mu &= \int_{E_n} f_n \, d\mu + \int_{E_n^c} f_n \, d\mu \\
&\geq \int_{E_n} (1+\varepsilon)^n \, d\mu + \int_{E_n^c} f_n \, d\mu \\
&= (1+\varepsilon)^n \mu(E_n) + \int_{E_n^c} f_n \, d\mu \\
&\geq (1+\varepsilon)^n \mu(E_n)
\end{aligned}$$

So $(1+\varepsilon)^n \mu(E_n) \leq 1$ $\forall n$

$$\begin{aligned}
m\left(\bigcup_{n \geq k} E_n\right) &\leq \sum_{n \geq k} m(E_n) \\
&\leq \sum_{n \geq k} \frac{1}{(1+\varepsilon)^n} \\
&< \infty
\end{aligned}$$

since $1+\varepsilon > 1$.

$\Rightarrow$ cont. from above

20 Fa. 2

$$x = 0.n_1 n_2 n_3 \ldots \quad \in [0,1].$$
$$f(x) = \min_i (n_i). \quad \text{Claim: } f \text{ measurable, constant a.e.}$$

Define $f_K(x) = f_K(0.n_1 n_2 n_3 \ldots) = \min \{n_i\}_{i=1}^{K}$.

$\lim_{K \to \infty} f_K(x) = f(x) \quad \forall x.$

$f_K$ measurable, since $f_K^{-1}(\{d\}) = \bigcup_{\{n_1, \ldots, n_K : \min\{n_i\} = d\}} [0.n_1 n_2 \cdots n_K 000\ldots, 0.n_1 n_2 \cdots (n_K+1) 000 \ldots)$

$=$ finite union of intervals.

$f_K$ dominated by $g(x) = 9$, integrable.
DCT $\Rightarrow$ $f_K$ approaches a measurable function $\Rightarrow$ $f$ measurable.

$f_1(x) = n_1$
$f_2(x) = \min\{n_1, n_2\}$
$\vdots$

$f(x) \overset{?}{=} \lim_{n \to \infty} f_n(x)$ ✓

$\exists\, n_i$ s.t. $f(x) = n_i$

Claim: $f = 0$ a.e.

$f_K^{-1}(\{0\}) = \bigcup [0.n_1 n_2 \cdots n_K 000\ldots, 0.n_1 n_2 \cdots (n_K+1) 000 \ldots)$ measure $\frac{1}{10^K}$

in how many of $\{n_i\}_{i=1}^{K}$ is one of the $n_i$'s $= 0$. $1 - \frac{9^K}{10^K} = 1 - \left(\frac{9}{10}\right)^K$

$\Rightarrow m(f_K^{-1}(\{0\})) = 1 - \left(\frac{9}{10}\right)^K \longrightarrow 1$ as $K \to \infty$.

20 Fa. 3

$f$ cont. on $[0,1]$. $F(x) = \sup_{0 \le y \le x} f(y)$, $x \in [0,1]$.

(a) Is $F$ continuous?

$\lim_{z \to x} F(z) = \lim_{z \to x} \sup_{0 \le y \le z} f(y)$

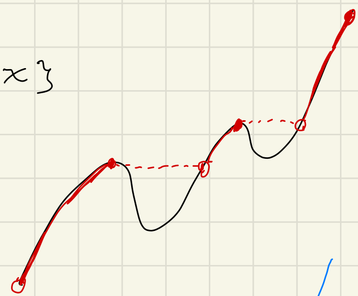
$F$ is non-decreasing, since sup is being taken over a larger set as $x$ increases.

$\lim_{z \nearrow x} F(z) = \lim_{z \nearrow x} \sup_{0 \le y \le z} f(y)$

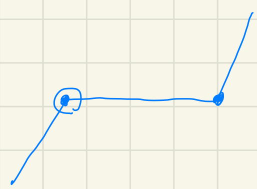
Since $f$ is continuous on a compact set, the supremum is acheived: $\exists\ c \in [0,x]$ s.t. $F(x) = f(c)$. If $c \in [0,x)$, then $\exists\ N \in \mathbb{N}$ s.t. $\forall n \ge N$, $\sup_{0 \le y \le z_n} f(y) = f(c) = F(x)$ for $z_n \nearrow x$.
If $c = x$, then since $f$ continuous, $\forall\ z_n \nearrow x$ $f(z_n) \nearrow f(x)$
$\Rightarrow \sup_{0 \le y \le z_n} f(y) \nearrow f(x) = F(x)$. Similarly $F$ is continuous as $z \searrow x$.

Thus $F$ continuous $\Rightarrow$ $F$ Borel function.

(b) Define $S = \{x \in (0,1] : f(x) > f(y)\ \forall\ 0 \le y < x\}$



Since $F$ is non-decreasing and continuous,
$F'$ defined a.e. and
$S =$



Consider $F^{-1}$ which is non decreasing and has countably many discontinuities,
$S = \bigcup F^{-1}$

continuous
$\{f - F = 0\}$

$S = \{f = F\} \setminus \bigcup \{F^{-1}(\{c\}) \mid m(F^{-1}(\{c\})) > 0\}$

$\bigcup_{c \in \mathbb{R}} \{F^{-1}(\{c\})\} = [0,1]$

20 Fa. 4

$$f(x) = \sum_{n=1}^{\infty} a_n e^{-nx} \quad x \geq 0 \quad a_n > 0 \quad \forall n,$$

$\sum_{n=1}^{\infty} a_n < \infty$. Claim: $\sum_{n=1}^{\infty} n a_n < \infty \iff f'(x+)$ exists at 0.

Note $f'(x+) = \lim_{y \searrow x} \frac{f(y) - f(x)}{y - x}$

$$\Rightarrow f'(0+) = \lim_{y \searrow 0} \frac{f(y) - f(0)}{y}$$

$$= \lim_{y \searrow 0} \frac{1}{y}\left(\sum_{n=1}^{\infty} a_n e^{-ny} - a_n\right)$$

$$= \lim_{y \searrow 0} \frac{1}{y} \sum_{n=1}^{\infty} a_n (1 - e^{-ny})$$

$$= \sum_{n=1}^{\infty} a_n \lim_{y \searrow 0} \frac{1}{y}(1 - e^{-ny}) \quad *$$

$$= \sum_{n=1}^{\infty} n a_n \quad *$$

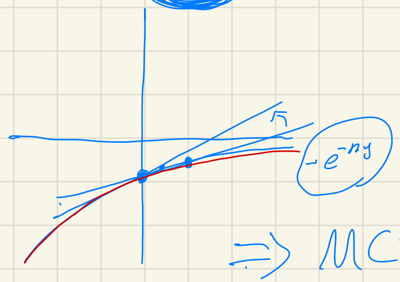
$\lim_{y \searrow 0} \frac{1}{y} \sum_{n=1}^{\infty} a_n (e^{-ny} - 1)$

$= \sum_1^{\infty} a_n \lim_{y \searrow 0} \frac{1}{y}(e^{-ny} - 1)$

$= \sum_1^{\infty} a_n n$ exists.

* DCT on $f_n(y) = \frac{1}{y} a_n (1 - e^{-ny})$

$\frac{d}{dy} -e^{-ny}\Big|_0 = \lim_{y \to 0} \frac{-e^{-ny} + 1}{y} = \lim_{y \to 0} \frac{1 - e^{-ny}}{y}$




$\Rightarrow$ MCT.

If $\sum_{n=1}^{\infty} n a_n < \infty$, then

$\frac{d}{dn}(1 - e^{-ny}) = y e^{-ny} > 0$

$\frac{1}{y} a_n (1 - e^{-ny}) \leq \frac{1}{y} a_n$

bounded for $y > \varepsilon$.

$$\frac{d}{dy} \frac{1}{y} a_n (1 - e^{-ny}) = \left(-\frac{1}{y^2}\right) a_n (1 - e^{-ny}) + \frac{1}{y} a_n n e^{-ny}$$

$$= \frac{1}{y} a_n e^{-ny}\left(\frac{1}{y} + n\right) - \frac{1}{y^2} a_n$$

$$= \frac{1}{y} a_n \left(e^{-ny}\left(\frac{1}{y} + n\right) - \frac{1}{y}\right) < 0.$$

$f_n \downarrow$ as $y \nearrow$.

$\frac{1}{y} \geq e^{-ny}\left(\frac{1}{y} + n\right)$

$1 \geq \underbrace{e^{-ny}}_{\leq 1} \underbrace{(1 + yn)}_{\geq 1} \iff e^{ny} \geq 1 + yn$

$e^x \geq 1 + x$? Yes.

$$* \quad \lim_{y \searrow 0} \frac{1}{y}(1 - e^{-ny}) = \lim_{y \searrow 0} n e^{-ny} = n.$$

$\frac{1}{x}(1 - e^{-x})$

$$\frac{d}{dx} = \frac{x e^{-x} - (1 - e^{-x})}{x^2}$$

$$= \frac{e^{-x}(1 + x) - 1}{x^2} = 0$$

$e^x - 1 = x \qquad 1 + x = e^x \qquad e^{-x} = \frac{1}{1+x}$

Note $f'(x+) = \lim_{y \searrow x} \frac{f(y) - f(x)}{y - x}$

$$\begin{aligned}
\Rightarrow \quad f'(0+) &= \lim_{y \searrow 0} \frac{f(y) - f(0)}{y} \\
&= \lim_{y \searrow 0} \frac{1}{y}\left(\sum_{n=1}^{\infty} a_n e^{-ny} - a_n\right) \\
&= \lim_{y \searrow 0} \frac{1}{y} \sum_{n=1}^{\infty} a_n \left(1 - e^{-ny}\right) \\
&= \sum_{n=1}^{\infty} a_n \lim_{y \searrow 0} \frac{1}{y}\left(1 - e^{-ny}\right) \\
&= \sum_{n=1}^{\infty} n\, a_n
\end{aligned}$$

$e^{-nh}$ close to 1.