Real analysis! **Measures**

$\limsup x_n = \inf_{k \geq 1} \left(\sup_{n \geq k} x_n \right)$
$\liminf x_n = \sup_{k \geq 1} \left(\inf_{n \geq k} x_n \right)$

$\mathcal{B}_X$ - generated by open sets
$\mathcal{B}_{\mathbb{R}}$ - generated by: open int, closed int, h-open int, open rays, closed rays

Continuity from below: $E_1 \subseteq E_2 \subseteq \cdots \Rightarrow \mu(\bigcup_j E_j) = \lim_{j \to \infty} \mu(E_j)$
" " above: $E_1 \supseteq E_2 \supseteq \cdots$ and $\mu(E_1) < \infty \Rightarrow \mu(\bigcap_j E_j) = \lim_{j \to \infty} \mu(E_j)$

* Complete measure = all subsets of null sets are measurable (m is complete)

MEASURABILITY!!!

$f: X \to Y$ measurable if $f^{-1}(E)$ meas. for all meas. $E \subseteq Y$.
(f_j) seq. of $\overline{\mathbb{R}}$-valued measurable functions $\Rightarrow \sup f_j, \inf f_j, \limsup f_j, \liminf f_j$ all measurable

If μ complete: (a) f meas, $f = g$ a.e. $\Rightarrow$ g meas.
(b) f_n meas, $f_n \to f$ μ-a.e. $\Rightarrow$ f meas.

$\mu(E) = \inf\{\mu(U) : U \supseteq E, U \text{ open}\}$ ∋ $\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ cpt}\}$

$E \Delta F = (E \setminus F) \cup (F \setminus E)$:
$E \subseteq \mathbb{R}$ meas and $\mu(E) < \infty \Rightarrow \exists A$ finite union of open int. s.t. $\mu(E \Delta A) < \varepsilon$ ($E \in \mathcal{M}_{\mathbb{R}}$

Integration:

Approx. by simple: ① f meas, $f \geq 0 \Rightarrow \exists (\phi_n)$ simple s.t. $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f$, and $\phi_n \to f$ pointwise and $\phi_n \to f$ unif on set where f bdd

simple = linear combo of characteristic fn

② f meas, real. $\Rightarrow \exists (\phi_n)$ simple s.t. $0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \leq |f|$, $\phi_n \to f$ pt wise, $\phi_n \to f$ unif. on set f bdd

$\boxed{\text{MCT}}$ $(f_n) \subseteq L^+$ s.t. $f_j \leq f_{j+1}$ $\forall j$ $\Longrightarrow \int \lim_{n \to \infty} f_n = \lim_{n \to \infty} \int f_n$

$\boxed{\text{FATOU}}$ $(f_n) \subseteq L^+ \Longrightarrow \int \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int f_n$

$\boxed{\text{DCT}}$ $(f_n) \subseteq L^1$ s.t. $|f_n| \leq g$ a.e. $\forall n \Rightarrow \int \lim_{n \to \infty} f_n = \lim_{n \to \infty} \int f_n$ ($g \in L^1$)

Corollaries: ① $(f_n) \subseteq L^+ \Rightarrow \int \Sigma f_n = \Sigma \int f_n$

* Integrable simple functions are dense in L^1

Derivative/integral exchange: $f: X \times [a,b] \to \mathbb{C}$ and $f(\cdot, t): X \to \mathbb{C}$ integrable $\forall t \in [a,b]$

(a) If $|f(x,t)| \leq g(x) \in L^1(\mu)$ and $\lim_{t \to t_0} f(x,t) = f(x,t_0)$ $\forall x$, then
$\lim_{t \to t_0} \int_X f(x,t) \, d\mu(x) = \int_X f(x,t_0) \, d\mu(x)$ (i.e. if $f(x,\cdot)$ cont. for all x, then $F(t) = \int_X f(x,t) \, d\mu(x)$ cont.)

(b) If $\frac{\partial f}{\partial t}$ exists ∋ $\left|\frac{\partial f}{\partial t}\right| \leq g(x) \in L^1(\mu)$ then

$$\boxed{\frac{d}{dt} \int_X f(x,t) \, d\mu(x) = \int \frac{\partial f}{\partial t}(x,t) \, d\mu(x)}$$

Modes of convergence

Convergent in measure: for each $\varepsilon > 0$, $\lim_{n\to\infty} \mu(\{|f_n(x) - f(x)| \geq \varepsilon\}) = 0$

Convergence in L^1: for each $\varepsilon > 0$, $\exists N$ s.t. $n \geq N \Rightarrow \int |f_n - f| \, d\mu < \varepsilon$

Unif. convergent: for each $\varepsilon > 0$, $\exists N$ s.t. $n \geq N \Rightarrow |f - f_n| < \varepsilon$ (unif $\Rightarrow$ ptwise)
$\hookrightarrow$ for all x

- $f_n \to f$ in L^1 $\Rightarrow$ $f_n \to f$ in measure
- $f_n \to f$ in measure $\Rightarrow$ $\exists$ subseq f_{n_j} converging to f a.e.

FUBINI / TONELLI $(X, \mu), (Y, \nu)$ σ-finite

① Tonelli: $f \in L^+(X \times Y)$; then $\int f \, d(\mu \times \nu) = \int \left[\int f \, d\nu(y) \right] d\mu(x) = \int \left[\int f \, d\mu(x) \right] d\nu$

② Fubini: $f \in L^1(\mu \times \nu)$; same

Differentiation

$\mu \perp \nu$: If $\exists$ E, F s.t. $E \cup F = X$, $E \cap F = \emptyset$, $\mu(E) = 0$, $\nu(F) = 0$

Jordan decomp: $\nu = \nu^+ - \nu^-$ s.t. $\nu^+ \perp \nu^-$, both positive ($|\nu| = \nu^+ + \nu^-$)

$\nu \ll \mu$: $\mu(E) = 0 \Rightarrow \nu(E) = 0$ ($\{\text{null sets } \mu\} \subseteq \{\text{null sets } \nu\}$)

Theorem: $\nu \ll \mu$ $\Leftrightarrow$ for each $\varepsilon > 0$, $\exists \delta > 0$ s.t. $\mu(E) < \delta \Rightarrow |\nu(E)| < \varepsilon$
(ν finite, μ positive)

$\nu(E) = \int_E f \, d\mu$ (notation: $d\nu = f \, d\mu$), then clearly $\nu \ll \mu$

Radon-Nikodym ν signed measure, μ σ-finite positive

$\exists!$ measure λ and $f \in L^1(\mu)$ s.t. $\boxed{d\nu = d\lambda + f \, d\mu}$ — Lebesgue decomp

and $\lambda \perp \mu$

and $\exists$ such that $d\nu_a = f \, d\mu$

If $\nu \ll \mu$, then $d\nu = f \, d\mu$ (i.e. $f = \frac{d\nu}{d\mu}$ exists) — R-N derivative

(i.e. part of ν that is abs. cont w.r.t. μ can be differentiated)

Properties: ν signed, μ, λ pos., $\nu \ll \mu \ll \lambda$.

① $g \in L^1(\nu) \Rightarrow g \frac{d\nu}{d\mu} \in L^1(\nu)$ and $\int g \, d\nu = \int g \frac{d\nu}{d\mu} \, d\mu$

② $\nu \ll \lambda$ and $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\lambda}$

Lebesgue Differentiation $f \in L^1_{loc}$

$$\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y)-f(x)|\,dy = 0 \ \ni\ \lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} f(y)\,dy = f(x)$$

for every family (E_r) shrinking nicely to x.

- R-N for complex: ν complex, μ positive. $\exists \lambda \ni f \in L^1(\mu)$ s.t. $\lambda \perp \mu$ $\ni$ $d\nu = d\lambda + f\,d\mu$
- ν reg. signed or complex Borel meas., $d\nu = d\lambda + f\,dm$ L-R-N decomp.

  Then: $\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = f(x)$ for E_r shrinking nicely to x (a.e.)

FTOC Lebesgue: $F: [a,b] \to \mathbb{C}$ TFAE:

(a) F abs. cont. on $[a,b]$

(b) $F(x) - F(a) = \int_a^x f(t)\,dt$ for some $f \in L^1([a,b])$

(c) F diff a.e. on $[a,b]$ and $F(x) - F(a) = \int_a^x F'(t)\,dt$

Continuity

- f unif cont if for each $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$ for all x,y
- f abs. cont if for each $\varepsilon > 0$, $\exists \delta > 0$ s.t. $\sum(b_j - a_j) < \delta \Rightarrow \sum |f(b_j) - f(a_j)| < \varepsilon$
- f Lipschitz cont: if $\exists M$ s.t. $|f(x)-f(y)| \le M|x-y|$

  f Lipschitz $\iff$ f abs. cont $\ni$ derivative bdd a.e.

Chebyshev: $f \in L^p$, then for $a > 0$, $\mu\{|f| > a\} \le \left(\frac{\|f\|_p}{a}\right)^p$

Minkowski: $f, g \in L^p$: $\|f+g\|_p \le \|f\|_p + \|g\|_p$

Hölder: $\frac{1}{p} + \frac{1}{q} = 1$, f, g meas $\Rightarrow$ $\|fg\|_1 \le \|f\|_p \cdot \|g\|_q$

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1-x}$$

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}, \quad |x| < 1$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n = \{x : x \in E_n \text{ for infinitely many } n\}$$

$$\liminf E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n = \{x : x \in E_n \text{ for all but finitely many } n\}$$

de Morgan: $\left(\bigcup_{\alpha \in A} E_\alpha\right)^c = \bigcap_{\alpha \in A} E_\alpha^c$

$\left(\bigcap_{\alpha \in A} E_\alpha\right)^c = \bigcup_{\alpha \in A} E_\alpha^c$

semicontinuity: $x_n \to x$

u.s.c. $\limsup f(x_n) \le f(x)$

at x • $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow f(y) \le f(x) + \varepsilon$

• $\{f(x) < a\}$ open for all $a \in \mathbb{R}$

l.s.c: • $f(x) \le \liminf f(x_n)$

at x • $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow f(x) - \varepsilon \le f(y)$

• $\{f(x) < a\}$ open for all $a \in \mathbb{R}$

$$\limsup_{x \to a} f(x) = \inf_{\delta > 0} \left(\sup_{|x-a|<\delta} f(x) \right)$$

$$\liminf_{x \to a} f(x) = \sup_{\delta > 0} \left(\inf_{|x-a|<\delta} f(x) \right)$$

Finite intersection prop: $\{F_\alpha\}_{\alpha \in A}$; $\bigcap_{\alpha \in B} F_\alpha \neq \emptyset$ for every finite $B \subseteq A$

Prop: X cpt $\Longleftrightarrow$ every family of closed subsets $\{F_\alpha\}$ with fin-intersxn prop has $\bigcap F_\alpha \neq \emptyset$

$f \in L^1(\mu)$: ① $\varepsilon > 0$, $\exists$ int. simple fn. $\phi = \sum a_j \chi_{E_j}$ s.t. $\|f - \phi\|_{L^1} < \varepsilon$

(I.E. Int simple functions are dense in L^1)

② μ = Leb-Stieltjes on $\mathbb{R}$: • E_j = finite unions of open int

• $\exists$ cont g w/ bdd support s.t. $\|f - g\|_{L^1} < \varepsilon$ (i.e. cont. fns. dense in Lebesgue L^1)

$$m\left(\limsup_{n\to\infty} A_n\right) \ge \limsup_{n\to\infty} m(A_n) \quad \text{(cont from below)}$$

$$m\left(\liminf_{n\to\infty} A_n\right) \le \liminf_{n\to\infty} m(A_n) \quad \text{(cont from above)}$$

$A = \{x : f_n(x) \not\to f(x)\}$; $A_{\varepsilon,N} = \{|f_n(x) - f(x)| > \varepsilon$ for some $n \ge N\}$

$$\Rightarrow A = \bigcup_{\varepsilon > 0} \bigcap_{N=1}^{\infty} A_{\varepsilon,N} = \limsup_{n\to\infty}(\ldots)$$

Egoroff: $\mu(X) < \infty$, $f_n \to f$ a.e. Then for $\varepsilon > 0$, $\exists E \subseteq X$ s.t. $\mu(E) < \varepsilon$ and $f_n \to f$ unif on $X \setminus E$.

Leb diff: $f \in L^1_{loc}$; [illegible] $= 0$ [illegible] $= f(x)$

FALL 2003 REAL ANALYSIS (MATH 525A) QUALIFYING EXAM
WEDNESDAY, SEPTEMBER 24, 2003

DIRECTIONS. Do exactly four of the following five problems. Start each problem on a *fresh* sheet of paper, and write on only one side. When you have completed the exam, be sure your name is printed on each page. You may keep this printed page.

Problem 1. Let E be a Lebesgue-measurable subset of $\mathbb{R}$ which has the property that $x \in E, y \in E, x \neq y$ implies that $(x+y)/2$ is *not* in E. Prove: E has Lebesgue measure zero.
Hint: Show that for an interval (a,b), for a fixed $x_0 \in (a,b) \cap E$,

$$\frac{1}{2}x_0 + \frac{1}{2}((a,b) \cap E)$$

is a subset of (a,b) which has half the measure of $(a,b) \cap E$ and is disjoint from $(a,b) \cap E$. Conclude that the measure of $(a,b) \cap E$ therefore does not exceed $\frac{2}{3}(b-a)$.

Problem 2.

(a) Show that the class of all step functions, i.e. those of the form $\sum_{j=1}^{n} c_j \chi_{(a_j,b_j)}$ with a_j, b_j finite, is dense in $L^1(\mu)$, where μ is Lebesgue on $\mathbb{R}$. [**Hint**: why is the corresponding statement true for simple functions?] Folland thm. 2.26

(b) Suppose $f \in L^1(\mu)$. Use the result in (a) to show that

$$\lim_{h \to 0} \int |f(x+h) - f(x)|\, dx = 0.$$

Problem 3. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be "lower semi-continuous" if

$$\liminf_{n \to \infty} f(x_n) \geq f(x) \quad \text{whenever } x_n \to x.$$

Prove that a lower semi-continuous function is Borel measurable.

Problem 4. Show that for $a > -1$

$$\int_0^1 x^a (1-x)^{-1} \ln x\, dx = -\sum_{k=1}^{\infty} (a+k)^{-2},$$

being careful to justify your calculations.

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \int_{-\infty}^{+\infty} \frac{\sin(tx)}{1+t^2}\, dt.$$

Prove: f is continuous on $\mathbb{R}$.

Fall 2003

(1.) $E \subseteq \mathbb{R}$ Lebesgue measurable such that $x, y \in E, x \neq y \Rightarrow (x+y)/2 \notin E$. Prove $m(E)=0$

Consider an interval (a,b) and point $x_0 \in E \cap (a,b)$

Now consider the set $\frac{1}{2}x_0 + \frac{1}{2}((a,b)\cap E) = \frac{1}{2}(x_0 + (a,b) \cap E)$

$= \frac{1}{2}((a+x_0, b+x_0)\cap E) \subseteq (a,b)$ (since $x_0 \in (a,b)$).

Thus $m(\frac{1}{2}x_0 + \frac{1}{2}(a,b)\cap E) = \frac{1}{2} m((a+x_0, b+x_0)\cap E)$

$= \frac{1}{2} m((a,b)\cap E)$ since Leb meas translation invariant

Now, by hypothesis,
$[\frac{1}{2}x_0 + \frac{1}{2}((a,b)\cap E)] \cap [(a,b)\cap E] = \emptyset$, hence:

$$m((a,b)) \geq m((a,b)\cap E \cup \tfrac{1}{2}x_0 + \tfrac{1}{2}((a,b)\cap E))$$
$$= m((a,b)\cap E) + \tfrac{1}{2} m((a,b)\cap E)$$
$$= \tfrac{3}{2} m((a,b)\cap E).$$

$$\Rightarrow m((a,b)\cap E) \leq \tfrac{2}{3} m((a,b)) = \tfrac{2}{3}(b-a).$$

__Lemma__: For E with $m(E) > 0$, for $\alpha < 1$, $\exists I$ interval s.t. $m(E\cap I) > \alpha m(I)$.

Since $\mu(E) = \inf\{m(U) : E \subseteq U, U \text{ open}\}$, we can choose $U \supseteq E$ s.t. $\underline{m(E)+\varepsilon \geq m(U)}$ $(*)$ Then, for $x \in E$, choose interval $x \in I \subseteq U$ s.t. $m(I) < \frac{\varepsilon}{1-\alpha}$. Then by $(*)$:

$$m(E\cap I) + \varepsilon \geq m(U \cap I) = m(I)$$ since $I \subseteq U$

Then:
$$\underline{m(E\cap I)} \geq m(I) - \varepsilon > m(I) - (1-\alpha)m(I) = \alpha\, m(I)$$

$\Rightarrow$ Now let $\alpha = \frac{2}{3}$; then $(E\cap I) \leq \frac{2}{3} m(I)$, a contradiction.

__Hence $m(E)=0$__

(2) (a) Show that step functions are dense in $L^1(\mathbb{R})$

→ Let (ϕ_n) be a sequence of integrable simple functions such that $0 \le |\phi_1| \le \cdots \le |f|$, $\phi_n \to f$ ptwise.

Now, $|\phi_n - f| \le |\phi_n| + |f| \le 2|f| \in L^1$, hence we may apply DCT:

$0 = \int \lim_{n\to\infty} |\phi_n - f| = \lim_{n\to\infty} \int |\phi_n - f|$, hence for $\varepsilon > 0$, $\exists N$ s.t. $n \ge N \Rightarrow \int |\phi_n - f| < \varepsilon$.

So simple functions are dense in L^1.

→ Now, let $\phi = \sum a_j \chi_{E_j}$ be simple; then, since $\int_{E_j} |\phi| = \int_{E_j} |a_j|$, we have $|a_j|^{-1} \int_{E_j} |\phi| = \int_{E_j} d\mu = \mu(E_j)$, hence $\mu(E_j) = |a_j|^{-1} \int_{E_j} |\phi| \le |a_j|^{-1} \int_{E_j} |f| < \infty$ since f is L^1.

Since $\mu(E_j) < \infty$, we can choose A = finite union of intervals with $\mu(E \Delta A) < \varepsilon$, i.e. $\int |\chi_E - \chi_A| < \varepsilon$.

Since A is finite union of intervals, χ_A is a step function.

Then: $\int |\chi_A - f| = \int |\chi_A - \phi + \phi - f| \le \int |\chi_A - \phi| + \int |f - \phi| < 2\varepsilon$,

hence step functions are dense.

2 (b) $f \in L^1(\mu)$; Show $\lim_{h\to 0} \int |f(x+h) - f(x)| dx = 0$:

Since step functions are dense in L^1, for $\varepsilon > 0$ $\exists \phi$ step fn. st. $\|f - \phi\|_{L^1} < \varepsilon$. Now see the following:

$$\lim_{h\to 0} \int |\phi(x+h) - \phi(x)| dx = \lim_{h\to 0} \int \Big| \sum_{j=0}^{n} c_j \chi_{(a_j,b_j)}(x+h) - \sum_{j=0}^{n} c_j \chi_{(a_j,b_j)}(x) \Big| dx$$

$$= \lim_{h\to 0} \int \Big| \sum_{j=0}^{n} c_j \left(\chi_{(a_j-h,b_j-h)}(x) - \chi_{(a_j,b_j)}(x) \right) \Big| dx$$

$$\leq \lim_{h\to 0} \int \sum_{j=0}^{n} |c_j| \, | \chi_{(a_j-h,b_j-h)}(x) - \chi_{(a_j,b_j)}(x) | \, dx$$

$$= \int \lim_{h\to 0} \sum_{j=0}^{n} |c_j| \, | \chi_{(a_j-h,b_j-h)}(x) - \chi_{(a_j,b_j)}(x) | \, dx$$

MCT (all summands are positive)

$$= \int \sum_{j=0}^{n} |c_j| (0) \, dx = 0.$$

Now:

$$\lim_{h\to 0} \int |f(x+h) - f(x)| dx = \lim_{h\to 0} \int |f(x+h) - \phi(x+h) + \phi(x+h) - \phi(x) + \phi(x) - f(x)| dx$$

$$\leq \lim_{h\to 0} \int |f(x+h) - \phi(x+h)| + |f(x) - \phi(x)| + |\phi(x) - \phi(x+h)|$$

$$< \varepsilon + \varepsilon + \lim_{h\to 0} \int |\phi(x) - \phi(x+h)| = 2\varepsilon.$$

hence $= 0$ since ε arbitrary.

(3) $f:\mathbb{R}\to\mathbb{R}$ l.s.c. if $\liminf_{n\to\infty} f(x_n) \geq f(x)$ for $x_n \to x$.

Show f is measurable:

Recall that f is L.S.C. at x if for $\varepsilon>0$, $\exists\, \delta>0$ s.t. $|y-x|<\delta \Rightarrow f(x)-\varepsilon \leq f(y)$.

Choose $x \in f^{-1}([a,\infty))$.

For $\varepsilon>0$, $\exists\, \delta>0$ s.t. $|y-x|<\delta \Rightarrow f(x)-\varepsilon \leq f(y)$,

i.e. $y\in(x-\delta, x+\delta) \Rightarrow a-\varepsilon \leq f(x)-\varepsilon \leq f(y)$

now, letting $\varepsilon\to 0$, $\exists\, \delta'$ s.t. $y\in(x-\delta', x+\delta') \Rightarrow a \leq f(y)$,

here there is an open interval $(x-\delta', x+\delta') \subseteq f^{-1}([a,\infty))$ containing x, hence since x was arbitrary, the set $f^{-1}([a,\infty))$ is open, hence Borel, hence f is measurable.

4) Show for $a > -1$:

$$\int_0^1 x^a (1-x)^{-1} \ln(x)\, dx = -\sum_{k=1}^{\infty} (a+k)^{-2}$$

$$\int_0^1 x^a \ln(x) \frac{1}{1-x}\, dx = \int_0^1 x^a \ln(x) \sum_{k=0}^{\infty} x^k\, dx$$

By MCT since all summands are positive.

$$= \sum_{k=0}^{\infty} \int_0^1 x^{k+a} \ln(x)\, dx$$

Now see the following:

$$\int_0^1 x^{k+a} \ln(x)\, dx = \left[\frac{\ln(x)\, x^{k+a+1}}{k+a+1} - \int \frac{x^{k+a+1}\, dx}{(k+a+1)\, x} \right]_0^1$$

int. by parts with $u = \ln(x)$, $dv = x^{k+a} dx$

$$= \left[\frac{\ln(x)\, x^{k+a+1}}{k+a+1} - \frac{x^{k+a+1}}{(k+a+1)^2} \right]_0^1 \quad (*)$$

Now, see that $\lim_{x \to 0} \ln(x)\, x^{k+a+1} = \lim_{x \to 0} \frac{\ln(x)}{\frac{1}{x^{k+a+1}}}$

L'Hopital

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{-(k+a+1)\frac{1}{x^{k+a+2}}} = \lim_{x \to 0} \frac{1}{-(k+a+1)\frac{1}{x^{k+a+1}}} = \lim_{x \to 0} \frac{-x^{k+a+1}}{(k+a+1)} = 0$$

hence $(*) = \left[0 - \frac{1}{(k+a+1)^2} - 0 + 0 \right] = \frac{1}{(k+a+1)^2}$.

So $$\int_0^1 x^a \ln(x) \frac{1}{1-x}\, dx = \sum_{k=0}^{\infty} \frac{1}{(k+a+1)^2} = \underline{\underline{\sum_{k=1}^{\infty} \frac{1}{(k+a)^2}}}$$

(5.) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \int_{-\infty}^{+\infty} \frac{\sin(tx)}{1+t^2} dt$

Show f is continuous :

Consider a sequence such that $x_n \to x$.

Now :

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \int_{-\infty}^{+\infty} \frac{\sin(tx_n)}{1+t^2} dt \quad (*)$$

Now note that $\left| \frac{\sin(tx_n)}{1+t^2} \right| \leq \frac{1}{1+t^2} \in L^1$, hence we may apply DCT,

$$(*) = \int_{-\infty}^{+\infty} \lim_{n\to\infty} \frac{\sin(tx_n)}{1+t^2} dt = \int_{-\infty}^{+\infty} \frac{\sin(tx)}{1+t^2} dt$$

$$= f(x)$$

So $\lim_{n\to\infty} f(x_n) = f(x)$

hence f continuous.

Incomplete
#4

REAL ANALYSIS QUALIFYING EXAM
SPRING 2004

Answer all four questions. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

check

1. Show that

$$\lim_{n\to\infty} n \int_{1/n}^{1} \frac{\cos(x+\frac{1}{n}) - \cos(x)}{x^{3/2}}\, dx$$

exists.

2. Suppose f_n, $n = 1, 2, \ldots$, and f are non-negative measurable functions on a measure space $(X, \mathcal{M}, \mu)$ with $f_n \to f$ a.e. and

$$\int_X f_n(x)\, d\mu(x) \to \int_X f(x)\, d\mu(x).$$

Show that

$$\int_X f_n(x) g(x)\, d\mu(x) \to \int_X f(x) g(x)\, d\mu(x)$$

for every bounded measurable function g. [Hint: use Fatou's Lemma].

3. Let $f \in L^1(\mathbf{R}^d)$. Evaluate

$$\lim_{y\to\infty} \int_{\mathbf{R}^d} |f(x+y) - f(x)|\, dx.$$

[Note that the limit is NOT as $y \to 0$.]

4. Let E_n be the set of all $f \in C([0,1])$ for which there exists $x_0 \in [0,1]$ (depending on f) such that

$$|f(x) - f(x_0)| \le n|x - x_0| \quad \text{for all } x \in [0,1].$$

(i) Show that E_n is nowhere dense in $C([0,1])$.
(ii) Show that the set of nowhere differentiable functions $f \in C([0,1])$ is non-empty.

→ Baire category theorem

2- show $f_n \to f$ in L^1

then $f_n \to f$ a.e. + fatou

$\int f_n \to \int f$

$$\int |g f_n| - |g f| < \int |g f_n - g f| = \int |g| |f_n - f| \to 0$$

as $n \to \infty$

3- use step fns dense in L^1

$$\left| \frac{\cos(x+\frac{1}{n}) - \cos(x)}{\frac{1}{n}} \right| \le |\sin(\theta_n)| \le \sin(x) + \frac{1}{n} \le 3 \sin x$$

3

$\theta_n \in [x, x+\frac{1}{n}]$

$x \in [\frac{1}{n}, 1]$

$\sin(x) \ge \frac{1}{2n}$

$\sin(x) \ge \frac{1}{n}$

$\cos(x)$

$x \ge 1/n$

$\Rightarrow \sin(x) \ge 1/2n$

$\sin(x) = \cos(x) x$

Spring 04:

① $\lim_{n\to\infty} n\int_{1/n}^{1} \frac{\cos(x+\frac{1}{n})-\cos(x)}{x^{3/2}}\,dx$ exists

$= \lim_{n\to\infty} \int_0^1 n\chi_{[\frac{1}{n},1]}(x)\left(\frac{\cos(x+\frac{1}{n})-\cos(x)}{x^{3/2}}\right)dx$

(*) Hence if $\left|\frac{\cos(x+\frac{1}{n})-\cos(x)}{\frac{1}{n}}\right| \le |g(x)|$ for $g\in L^1$, $x\in[\frac{1}{n},1]$, for all n, then we may apply DCT:

$= \int_0^1 \lim_{n\to\infty} \chi_{[\frac{1}{n},1]}(x)\left(\frac{\cos(x+\frac{1}{n})-\cos(x)}{\frac{1}{n}x^{3/2}}\right)dx$

$= \int_0^1 \frac{-\sin(x)}{x^{3/2}}\,dx$; now, $\frac{\sin(x)}{x}$ is bdd on $[0,1]$,

hence $\le \int_0^1 -\frac{1}{x^{1/2}}\,dx$, which is integrable on $[0,1]$.

So, need to show (*):

See that, by the MVT,

$\left|\chi_{[\frac{1}{n},1)}\frac{\cos(x+\frac{1}{n})-\cos(x)}{\frac{1}{n}}\right| \le |\sin(\theta_n)|$ for some $\theta_n\in[x,x+\frac{1}{n}]$

Note that $\sin(x)$ is increasing on $[\frac{1}{n},1]$, (approximately linearly)

hence $|\sin\theta_n| \le |\sin(x+\frac{1}{n})| \le \sin(x)+\frac{1}{n}$; and since increasing on $[\frac{1}{n},1]$, $\sin(x)$ is minimal at $\frac{1}{n}$. Since $\frac{1}{n}$ is close to 0, $\sin(x)$ is approximately linear, hence $\sin(x_1) \ge \frac{1}{2}(\frac{1}{n})$, i.e. $2\sin(x) \ge \frac{1}{n}$, hence $|\sin\theta_n| \le |3\sin(x)|$,

which is integrable, and the statement follows from above.

(2.) $(f_n) \subseteq L^+(\mu)$, $f_n \to f$ a.e., $\int_X f_n \, d\mu \to \int_X f \, d\mu$ as $n \to \infty$.

Show: $\int_X f_n g \, d\mu \to \int_X fg \, d\mu$ for g bdd, measurable.

Since $|f_n - f| \leq |f_n| + |f|$, consider the sequence of positive measurable functions $g_n = |f_n| + |f| - |f_n - f|$.

Clearly $g_n \to 2|f|$ a.e.; now apply Fatou:

$$\int \liminf_{n\to\infty} g_n \leq \liminf_{n\to\infty} \int g_n$$

$$\Rightarrow \int 2|f| \leq \liminf_{n\to\infty} \int |f_n| + |f| - |f_n - f|$$

$$= \liminf_{n\to\infty} \left(\int |f_n| + \int |f| - \int |f_n - f| \right)$$

$$= 2\int |f| - \liminf_{n\to\infty} \int |f_n - f|$$

$\Rightarrow \liminf_{n\to\infty} \int |f_n - f| \leq 0$, hence $= 0$, hence $f_n \to f$ in L^1.

Now; for g meas and $|g(x)| \leq M$ bdd:

$$\left| \int g f_n - \int g f \right| \leq \int |g f_n - g f| \leq \int |g| \, |f_n - f| \leq M \int |f_n - f|$$

and since $f_n \to f$, for $\frac{\varepsilon}{M} > 0$, $\exists N$ s.t. $n \geq N \Rightarrow \int |f_n - f| < \frac{\varepsilon}{M}$

hence for $n \geq N$, $\left| \int g f_n - \int g f \right| < \varepsilon$, hence $\lim_{n\to\infty} \int g f_n = \int g f$.

③ $f \in L^1(\mathbb{R}^d)$. Evaluate $\lim_{y \to \infty} \int_{\mathbb{R}^d} |f(x+y) - f(x)|\, dx$

Suppose ϕ is an integrable simple function, i.e. $\phi(x) = \sum a_i \chi_{E_i}(x)$.
Now see the following:

$$\int |\sum a_i \chi_{E_i}(x+y) - \sum a_i \chi_{E_i}(x)| = \int |\sum a_i \chi_{E_i - y}(x) - \sum a_i \chi_{E_i}(x)|$$

For y large enough, we have $E_i - y \cap E_i = \phi$

$$= \int |\sum a_i \chi_{E_i - y}(x)| + |\sum a_i \chi_{E_i}(x)|$$

$$= \int |\sum a_i \chi_{E_i - y}(x)| + \int |\sum a_i \chi_{E_i}(x)|$$

$$= \int |\phi(x+y)| + \int |\phi(x)|$$

Since Lebesgue int. is translation invariant

$$= 2\int |\phi(x)|$$

Now, since f is measurable, $\exists$ sequence (ϕ_n) of int. simple functions s.t. $0 \le |\phi_1| \le |\phi_2| \le \cdots \le |f|$, $\phi_n \to f$ pointwise.

Then $|\phi_n(x+y) - \phi_n(x)| \longrightarrow |f(x+y) - f(x)|$ pointwise, and:

$|\phi_n(x+y) - \phi_n(x)| \le |\phi_n(x+y)| + |\phi_n(x)| \le |f(x+y)| + |f(x)|$, which is L^1,

hence for $y \to \infty$ and apply DCT:

$$2\int |f| = \lim_{n \to \infty} 2\int |\phi_n| = \lim_{n \to \infty} \int |\phi_n(x+y) - \phi_n(x)|$$

$$= \int \lim_{n \to \infty} |\phi_n(x+y) - \phi_n(x)|$$

$$= \int |f(x+y) - f(x)|$$

REAL ANALYSIS GRADUATE EXAM

FALL 2004

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n\to\infty} \left(\int_a^b |f(x)|^n \, dx \right)^{1/n} = \sup_{x\in[a,b]} |f(x)|.$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Prove that

$$g(x) = \sum_{n=1}^{\infty} f\left(2^n x + \frac{1}{n}\right)$$

do for f_+, f_- & apply Tonelli

just need weak

is integrable and

$$\int_{-\infty}^{\infty} g(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx.$$

3. A Lebesgue integrable function f defined on the interval $[0, 4]$ has the property that $\int_E f(x)\, dx = 0$ for all measurable E with $m(E) = \pi$. Must $f = 0$ a.e.?

4. Let $f(x) = x^2 \sin(1/x^2)$ and $g(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = g(0) = 0$.
   (i) Show that f and g are each differentiable everywhere (including $x = 0$).
   (ii) Show that $f \notin BV([-1, 1])$.
   (iii) Show that $g \in BV([-1, 1])$.

$$\lim_{p\to\infty} a^{1/p} = \lim_{p\to\infty} e^{\log(a)\frac{1}{p}} = e^{\lim_{p\to\infty}\log(a)\frac{1}{p}} = e^{0} = 1$$

$$\lim_{p\to\infty} a^{\frac{p-q}{p}} = \lim_{p\to\infty} e^{\log(a)\frac{p-q}{p}} = e^{\lim_{p\to\infty}\log(a)\frac{p-q}{p}}$$

$$= e^{\log(q)} = q$$

Fall '04:

(1.) $f:[a,b]\to\mathbb{R}$ continuous. Show: $\lim_{n\to\infty}\|f\|_{L^n} = \sup_{x\in[a,b]}|f(x)| = \|f\|_\infty$

For $\varepsilon>0$, let $S_\varepsilon := \{|f| \geq \|f\|_\infty - \varepsilon\} \subseteq [a,b]$

Hence, $\|f\|_{L^n} = \left(\int_a^b |f(x)|^n dx\right)^{1/n} \geq \left(\int_{S_\varepsilon} |f(x)|^n dx\right)^{1/n} \geq \left(\int_{S_\varepsilon}(\|f\|_\infty-\varepsilon)^n dx\right)^{1/n}$

$= \left((\|f\|_\infty-\varepsilon)^n \mu(S_\varepsilon)\right)^{1/n} = (\|f\|_\infty-\varepsilon)\mu(S_\varepsilon)^{1/n}$

$\Rightarrow \liminf_{n\to\infty}\|f\|_n \geq \liminf_{n\to\infty}(\|f\|_\infty-\varepsilon)\mu(S_\varepsilon)^{1/n} = (\|f\|_\infty-\varepsilon)\lim_{n\to\infty}\mu(S_\varepsilon)^{1/n}$ $\boxed{=1 \text{ since } \mu(S_\varepsilon)\geq 0}$

$= (\|f\|_\infty - \varepsilon)$

$\Rightarrow \liminf_{n\to\infty}\|f\|_n \geq \|f\|_\infty - \varepsilon$

Hence letting $\varepsilon\to 0$ we get $\liminf_{n\to\infty}\|f\|_n \geq \|f\|_\infty$ (1)

On the other hand, $|f(x)|\leq\|f\|_\infty$ for all x, and for $p>q$,

$\|f\|_{L^p} = \left(\int|f(x)|^p dx\right)^{1/p} = \left(\int|f(x)|^{p-q}|f(x)|^q d\mu\right)^{1/p}$

$\leq \left(\|f\|_\infty^{p-q}\int|f(x)|^q d\mu\right)^{1/p}$

$= \|f\|_\infty^{\frac{p-q}{p}}\|f\|_{L^q}^{q/p}$

$(n>q)$

$\Rightarrow \limsup_{n\to\infty}\|f\|_{L^n} \leq \limsup_{n\to\infty}\left(\|f\|_\infty^{\frac{n-p}{n}}\|f\|_{L^p}^{\frac{p}{n}}\right) = \limsup_{n\to\infty}\|f\|_\infty\left(\frac{\|f\|_{L^p}}{\|f\|_\infty}\right)^{p/n}$ (2)

$= \|f\|_\infty$ since $\|f\|_{L^p}/\|f\|_\infty \geq 0$

Hence: $\|f\|_\infty \overset{(1)}{\leq} \liminf_{n\to\infty}\|f\|_{L^n} \leq \lim_{n\to\infty}\|f\|_{L^n} \leq \limsup_{n\to\infty}\|f\|_{L^n} \underset{(2)}{\leq} \|f\|_\infty$

$\Rightarrow \lim_{n\to\infty}\|f\|_{L^n} = \|f\|_\infty$

(2) $f: \mathbb{R} \to \mathbb{R}$ integrable. Prove that $g(x) = \sum_{n=1}^{\infty} f(2^n x + \frac{1}{n})$ is integrable and $\int_{-\infty}^{\infty} g(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$

Let $g_k(x) = \sum_{n=1}^{k} |f(2^n x + \frac{1}{n})|$.

Now:

$$\int_{\mathbb{R}} |g(x)|\,dx = \int \left|\sum_{n=1}^{\infty} f(2^n x + \tfrac{1}{n})\right| dx = \int \lim_{k\to\infty} \left|\sum_{n=1}^{k} f(2^n x + \tfrac{1}{n})\right| dx$$

$$\leq \int \lim_{k\to\infty} \sum_{n=1}^{k} |f(2^n x + \tfrac{1}{n})| = \int \lim_{k\to\infty} g_k(x) \quad (*)$$

Now, note that $g_k(x) \leq g_{k+1}(x)$ since $|f(2^n x + \frac{1}{n})| \geq 0$, for all n.

Hence since the g_k are measurable, we may apply MCT.

$$(*) = \lim_{k\to\infty} \int g_k(x) = \lim_{k\to\infty} \int \sum_{n=1}^{k} |f(2^n x + \tfrac{1}{n})|\,dx = \lim_{k\to\infty} \sum_{n=1}^{k} \int |f(2^n x + \tfrac{1}{n})|\,dx$$

$$= \sum_{n=1}^{\infty} \int \frac{|f(u)|\,du}{2^n} = \int |f(u)|\,du \left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right) = \int |f(u)|\,du$$

$$\begin{cases} u = 2^n x + \frac{1}{n} \\ du = 2^n dx \end{cases}$$

(3.) f Leb. integrable on $[0,4]$; $\int_E f = 0$ $\forall E$ with $m(E) = \pi$

Show $f = 0$ a.e. :

Consider the sets $E^+ = \{f > 0\}$, $E^- = \{f < 0\}$, $E^0 = \{f = 0\}$

These are all measurable since f is measurable.

Now, $\mu(E^+ \cup E^0) < \pi$ and $\mu(E^- \cup E^0) < \pi$, otherwise they will have a subset F that has $\int_F f = 0$, which is impossible.

Since $\mu(E^+ \cup E^0) < \pi$, $\exists F^- \subseteq E^-$ s.t. $\mu(E^+ \cup E^0 \cup F^-) = \pi$

$$\Rightarrow \int_{E^+ \cup E^0 \cup F^-} f = 0 \Rightarrow \boxed{\int_{E^+ \cup E^0} f = a} \text{ and } \boxed{\int_{F^-} f = -a}$$

Now, on the other hand, since $\mu(E^- \cup E^0) < \pi$, $\exists F^+ \subseteq E^+$ s.t. $\mu(F^+ \cup E^0 \cup E^-) = \pi$.

But : $-b := \int_{E^-} f < \int_{F^-} f = -a$ since $F^- \subseteq E^-$ and f only negative on E^-.

and $\int_{F^+} f < \int_{E^+} f = a$ since $F^+ \subseteq E^+$.

Hence :

$$\int_{F^+ \cup E^0 \cup E^-} f = \int_{F^+} f + \int_{E^-} f < \int_{E^+} f + \int_{E^-} f < \int_{E^+} f + \int_{F^-} f = a - a = 0$$

Hence $\int_{F^+ \cup E^0 \cup E^-} f \neq 0$, but $\mu(F^+ \cup E^0 \cup E^-) = \pi$, hence contradiction

④ $f(x) = x^2\sin\left(\frac{1}{x^2}\right)$, $g(x) = x^2\sin\left(\frac{1}{x}\right)$ for $x \neq 0$
and $f(0) = g(0) = 0$.

(i) Show f, g diff everywhere: clearly f, g differentiable for all $x \neq 0$ since they are clearly continuous for all $x \neq 0$.

Now:

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2\sin(\frac{1}{x^2})}{x} = \lim_{x\to 0} x\sin\left(\frac{1}{x^2}\right) = 0$$ by squeeze thm.

$$g'(0) = \lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \frac{x^2\sin(\frac{1}{x})}{x} = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$$ by squeeze thm

Hence f', g' exist everywhere.

(ii) Show f is not of bdd variation on $[-1,1]$:

Consider a partition of $[-1,1]$ with $x_0 = -1$, $x_{2N+2} = 1$, and $0 \leq n \leq N$

$$x_{2n} = \frac{-1}{\sqrt{2\pi n + \frac{\pi}{2}}}, \quad x_{2n+1} = \frac{-1}{\sqrt{2\pi n + \frac{3\pi}{2}}}$$

Then:

$$|f(x_{2n+1}) - f(x_{2n})| = \left| \frac{1}{2\pi n + \frac{3\pi}{2}} \sin\left(2\pi n + \frac{3\pi}{2}\right) - \frac{1}{2\pi n + \frac{\pi}{2}} \sin\left(2\pi n + \frac{\pi}{2}\right) \right|$$

$$= \left| \frac{-1}{2\pi n + \frac{3\pi}{2}} + \frac{-1}{2\pi n + \frac{\pi}{2}} \right| = \left| \frac{1}{2\pi n + \frac{3\pi}{2}} + \frac{1}{2\pi n + \frac{\pi}{2}} \right|$$

$$\geq \left| \frac{2}{2\pi n + \frac{\pi}{2}} \right|$$

Hence $$\sum_{P=\{x_i\}} |f(x_i) - f(x_{i-1})| \geq \sum_{n=0}^{N} |f(x_{2n+1}) - f(x_{2n})| \geq \sum_{n=0}^{N} \left| \frac{2}{2\pi n + \frac{\pi}{2}} \right| \to \infty$$ as $N \to \infty$,

Hence, $$\sup_{P=\{x_i\}} \left\{ \sum_P |f(x_i) - f(x_{i-1})| \right\} = \infty.$$

(iii) Show g is of bdd variation on $[-1,1]$.

See that $\lim_{x\to 0} g(x) = \lim_{x\to 0} x^2\sin(\frac{1}{x}) = 0$ hence continuous everywhere,

and that $g'(x) = 2x\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \leq 3$,

hence $g'(x)$ is bounded; then apply MVT:

$$|f(x_i) - f(x_{i-1})| = |f'(c)|\,|x_i - x_{i-1}| \leq M|x_i - x_{i-1}|$$

hence $\sum^{n} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{n} M|x_i - x_{i-1}| = M\cdot 2$ since $\{x_i\}$ a partition of $[-1,1]$.

Incomplete

#1 (ii), (iii)

REAL ANALYSIS GRADUATE EXAM
SPRING 2005

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. A function $f : [a, b] \to \mathbb{R}$ is said to be Hölder continuous of order α if there is a constant L such that

$$|f(x) - f(y)| \leq L|x - y|^\alpha \qquad \text{for all } x, y \in [a, b].$$

(i) Show that if f is Hölder continuous of order 1 then it has bounded variation.
(ii) Let $\alpha \in (0, 1)$. Give an example of a function which is Hölder continuous of order α and does not have bounded variation.
(iii) Give an example of a function of bounded variation which is not Hölder continuous for any $\alpha > 0$.

2. Let $f \in L^1([0, 1])$. For $k = 1, 2, \dots$ let f_k be the step function defined on $[0, 1]$ by

$$f_k(x) = k \int_{j/k}^{(j+1)/k} f(t)\, dt \quad \text{for } \frac{j}{k} \leq x < \frac{j+1}{k}.$$

Show that f_k tends to f in L^1 as $k \to \infty$.
(Hint: Treat first the case when f is continuous, and use approximation.)

3. Suppose that f_n, $n = 1, 2, \dots$, and f are complex valued measurable functions on a measure space $(X, \mathcal{M}, \mu)$. Define the terms
(i) f_n converges in measure to f,
(ii) f_n is Cauchy in measure.
Show that if f_n is Cauchy in measure there exists a subsequence n_k and a measurable function g such that f_{n_k} converges to g almost everywhere, and f_n converges to g in measure.

4. Let $a_1, a_2, \dots > 0$. Prove that $\sum_{i=1}^{\infty} a_i = \infty$ is a necessary and sufficient condition that there exists an enumeration of the rationals

$$\mathbb{Q} = \{r_1, r_2, \dots\}$$

such that

$$\bigcup_{i=1}^{\infty} (r_i - a_i, r_i + a_i) = \mathbb{R}.$$

Catalina t
37 +4 51

Spring 05:

1. $f:[a,b]\to\mathbb{R}$ Hölder cont of order α iff $\exists L$ s.t.

$$|f(x)-f(y)|\le L|x-y|^\alpha \text{ for all } x,y\in[a,b].$$

(i) Show f Hölder cont order $1 \Rightarrow$ Bounded variation

$|f(x)-f(y)|\le L|x-y|$ for all $x,y\in[a,b]$; consider (x_i) partition of $[a,b]$.

Then $\sum_{j=1}^{n}|f(x_j)-f(x_{j-1})|\le L\sum_{j=1}^{n}|x_j-x_{j-1}|=L\mu([a,b])=L(b-a)$

Taking the supremum over partitions, we have:

$$T_a^b(x)=\sup_{P=\{x\}}\left\{\sum_{j=1}^{n}|f(x_j)-f(x_{j-1})|\right\}\le L(b-a).$$

hence bounded variation

(ii) + (iii): - Find f s.t. f Hölder cont of order $\alpha\in(0,1)$ and $f\notin BV$

- Find f s.t. $f\in BV$ but f not Hölder cont for any α.

(2.) $f \in L^1([0,1])$. For $k = 1, 2, \ldots$ define

$$f_k(x) = k \int_{j/k}^{j+1/k} f(t)\,dt \quad \text{for} \quad \frac{j}{k} \le x \le \frac{j+1}{k}$$

for $j = 0, \ldots, k-1$.

Show $f_k \to f$ in L^1

See that, if f is continuous, we have:

$$\int_0^1 |f_k(x) - f(x)|\,dx \le \sum_{0}^{k-1} \int_{j/k}^{j+1/k} |f_k(x) - f(x)|\,dx = \int_{j/k}^{j+1/k} \left| \left(k \int_{j/k}^{j+1/k} f(t)\,dt\right) - f(x) \right|$$

$$= k \int_{j/k}^{j+1/k} \left| \left(\int_{j/k}^{j+1/k} f(t)\,dt \right) - \frac{f(x)}{k} \right| dx$$

$$= k \int_{j/k}^{j+1/k} \left| \int_{j/k}^{j+1/k} (f(t) - f(x))\,dt \right| dx$$

$$\le k \int_{j/k}^{j+1/k} \int_{j/k}^{j+1/k} |f(t) - f(x)|\,dt\,dx.$$

now for $\varepsilon > 0$, $x \in [0,1]$, $\exists\, k$ s.t. $|t - x| < \frac{1}{k} \Rightarrow |f(t) - f(x)| < \varepsilon$

Hence:

$$< k \int_{j/k}^{j+1/k} \int_{j/k}^{j+1/k} \varepsilon\,dt\,dx = k \int_{j/k}^{j+1/k} \frac{\varepsilon}{k}\,dx$$

$$= \frac{\varepsilon}{k} \longrightarrow 0 \text{ as } k \to \infty.$$

Hence, since cont fns are dense in L^1, $\exists$ cont g s.t. $\int_0^1 |f - g| < \varepsilon$ and $\int_0^1 |f_k - g_k| < \varepsilon$ for each k. (since $f_k \in L^1$ as well),

$$\int |f_k(x)| \le \int_{j/k}^{j+1/k} |f_k(x)|$$

$$= \int_{j/k}^{j+1/k} \left| k \int_{j/k}^{j+1/k} f_k(t)\,dt \right| dx$$

$$= \left| k \int_{j/k}^{j+1/k} f_k(t)\,dt \right| \cdot \frac{1}{k}$$

$$= \left| \int_{j/k}^{j+1/k} f_k(x)\,dt \right| \le \int_{j/k}^{j+1/k} |f_k(x)|\,dt < \infty$$

Hence:

$$\int_0^1 |f_k - f| = \int_0^1 |f_k - g_k + g_k - g + g - f|$$

$$\le \int_0^1 |f_k - g_k| + |f - g| + |g_k - g|\,dx$$

$$< \varepsilon + \varepsilon + \int |g_k - g| = 2\varepsilon$$

for large k;

Hence let $\varepsilon \to 0$ and result follows

(3) (f_n), f $\mathbb{C}$-valued measurable on $(X, \mathcal{M}, \mu)$
Show that if f_n Cauchy in measure, then $\exists$ subseq $f_{n_k} \to g$ a.e. and $f_n \to g$ in measure

---

→ Recall that f_n Cauchy in measure if for $\varepsilon > 0$,

$\mu\{|f_n - f_m| \geq \varepsilon\} \to 0$ as $m, n \to \infty$.

So choose subsequence $g_j = f_{n_j}$ of f_n such that

$\mu\{|g_j - g_{j+1}| \geq 2^{-j}\} \leq 2^{-j}$

Let $F_k = \bigcup_{j=k}^{\infty} \{|g_j - g_{j+1}| \geq 2^{-j}\} \Rightarrow \mu(F_k) \leq \sum_{j=k}^{\infty} 2^{-j} = 2^{1-k}$

→ Now, if $x \notin F_k$, then for $i, j \geq k$ we have

(*) $|g_j(x) - g_i(x)| \leq \sum_{l=j}^{i-1} |g_{l+1}(x) - g_l(x)| \leq \sum_{l=j}^{i-1} 2^{-l} \leq 2^{1-j}$

Hence g_i is pointwise Cauchy on F_k^c

→ Let $F = \bigcap_{k=1}^{\infty} F_k$; then $\mu(F) \leq \mu(F_k)$ for all k, hence

$\mu(F) \leq 2^{1-k}$ for all k, hence $\mu(F) = 0$.

Then let $f(x) = \lim_{j\to\infty} g_j(x)$ for $x \notin F$, and $f(x) = 0$ for $x \in F$.

Hence $g_j \to f$ a.e. (on F^c, which is full measure since F is null,

---

Inequality (*) gives us $|g_j(x) - f(x)| \leq 2^{1-j}$ for $x \notin F_k$, $j \geq k$,

hence for $x \in F_k$, $|g_j(x) - f(x)| \geq 2^{1-j}$, hence:

$\mu\{|g_j - f| \geq 2^{1-j}\} \leq \mu(F_k) \to 0$ as $k \to \infty$.

hence $g_j \to f$ in measure. (2)

But now, $\{|f_n - f| \geq \varepsilon\} \subseteq \{|f_n - g_j| \geq \frac{\varepsilon}{2}\} \cup \{|g_j - f| \geq \frac{1}{2}\varepsilon\}$

and $\mu\{|g_j - f| \geq \frac{1}{2}\varepsilon\} \to 0$ since $g_j \to f$ in measure and

$\mu\{|f_n - g_j| \geq \frac{\varepsilon}{2}\} = \mu\{|f_n - f_{n_j}| \geq \frac{\varepsilon}{2}\} \to 0$ since Cauchy in meas

$\to$ so $f_n \to f$ in meas

(4.) Show $\sum_{i=1}^{\infty} a_i = \infty \iff \exists \{r_1, \dots\} = \mathbb{Q}$ s.t. $\bigcup_{i=1}^{\infty} (r_i - a_i, r_i + a_i) = \mathbb{R}$

($\Leftarrow$) Suppose $\bigcup_{i=1}^{\infty} (r_i - a_i, r_i + a_i) = \mathbb{R}$

$m((r_i - a_i, r_i + a_i)) = 2a_i$

Then: $\infty = m(\mathbb{R}) = m\left(\bigcup_{i=1}^{\infty} (r_i - a_i, r_i + a_i)\right) \le \sum m(r_i - a_i, r_i + a_i)$
$= \sum 2a_i$

$\Rightarrow \sum_{i=1}^{\infty} a_i = \infty$

($\Rightarrow$) Suppose $\sum a_i = \infty$:

First, if $\bigcup_{i=1}^{\infty} (r_i - a_i, r_i + a_i) = \mathbb{R}$; then $m\left(\bigcup_{i=1}^{\infty} (r_i - a_i, r_i + a_i)\right)$ must be infinite; but:

$$m\left(\bigcup_{i=1}^{\infty} (r_i - a_i, r_i + a_i)\right) \le \sum m(r_i - a_i, r_i + a_i) = \sum 2a_i$$

hence the measure may be infinite since $\sum a_i = \infty$

→ Now, we want to show that, for each $x \in \mathbb{R}$, $\exists i$ s.t. $x \in (r_i - a_i, r_i + a_i)$:

Since rationals are dense, for $x \in \mathbb{R}$,

$\exists r_i$ such that $|x - r_i| < a_i$, hence $x \in (r_i - a_i, r_i + a_i)$

So $\bigcup_i (r_i - a_i, r_i + a_i) \supseteq \mathbb{R}$, hence $=$.

check!

REAL ANALYSIS GRADUATE EXAM
Fall, 2005

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

check (1) Let $(X, \mathcal{B}, \mu)$ be a measure space with μ finite. For all $A \subset X$ define

$$\mu_1(A) = \sup\{\mu(B) : B \in \mathcal{B}, B \subset A\}, \qquad \mu_2(A) = \inf\{\mu(B) : B \in \mathcal{B}, B \supset A\}.$$

(a) Show that $\mu_1(A^c) + \mu_2(A) = \mu(X)$ for all $A \subset X$.

(b) Let $\mathcal{A} = \{A \subset X : \mu_1(A) = \mu_2(A)\}$. Show *directly from (a) and/or the definitions* that $\mathcal{A}$ is a σ-algebra.

(2) Suppose μ is a measure on $[0, \infty)$ satisfying

$$\int_{[0,\infty)} e^{ax}\, \mu(dx) < \infty \quad \text{for some } a \in \mathbb{R}.$$

Show that the function

see 23-26

$$\psi(t) = \int_{[0,\infty)} e^{tx}\, \mu(dx)$$

is infinitely differentiable on $(-\infty, a)$.

check (3) Suppose $(X, \mathcal{B}, \mu)$ is a measure space, $0 \le f < \infty$ is a measurable function, and $d\nu = f\,d\mu$.

(a) Suppose μ is σ-finite. Show that ν is σ-finite. HINT: You must deal with the fact that f is not assumed bounded or integrable.

(b) A measure ρ is called *semifinite* if for every measurable set E with $\rho(E) > 0$, there is a measurable $F \subset E$ with $0 < \rho(F) < \infty$. Show that if $0 < f < \infty$ and ν is semifinite, the μ is semifinite. (We do not keep the assumption made in (a) that μ is σ-finite.)

(4) Suppose $(X, \mathcal{B}, \mu)$ is a measure space with $\mu(X)$ finite, $\{f_n\}$ are integrable functions, and $f_n \to f$ in L^1.

(a) Show that $f_n \to f$ in measure.

(b) Show that the measures $d\nu_n = |f_n|\, d\mu$ are uniformly absolutely continuous with respect to μ. NOTE: This means that for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $\nu_n(E) < \epsilon$ for all n. Absolute continuity of each ν_n *individually* is a standard result which you may make use of.

$t_n \to t$; $t_n, t \in (-\infty, a)$

$\Rightarrow e^{tx}, e^{t_n x} < e^{ax}$

$e^{yx}(1-e^{(t-y)x})$ $\quad \frac{d}{dt}e^{tx} = xe^{tx}$

$$\frac{e^{yx}-e^{tx}}{y-t} \le \sup_{s\in(y,t)} |xe^{sx}| = xe^{tx}$$

$\cancel{x>0}$

$\cancel{t<a}$

$$\frac{e^{t_n x}-e^{tx}}{t_n - t} \le \sup_{y\in(t_n,t)} xe^{yx}$$

$y<a$

$x>0$

Fall 2005:

(1.) $(X, \mathcal{B}, \mu)$ meas. space w/ $\mu(X) < \infty$. For $A \subseteq X$, define:

$$\mu_1(A) = \sup_{B \in \mathcal{B}} \{\mu(B) : B \subseteq A\}, \quad \mu_2(A) = \inf_{B \in \mathcal{B}} \{\mu(B) : B \supseteq A\}$$

(a) Show $\mu_1(A^c) + \mu_2(A) = \mu(X)$

(b) $\mathcal{A} = \{A \subseteq X : \mu_1(A) = \mu_2(A)\}$ a σ-algebra

---

(a):
$$\begin{aligned}
\mu_1(A^c) &= \sup_{B \in \mathcal{B}} \{\mu(B) : B \subseteq A^c\} \\
&= \sup_{B \in \mathcal{B}} \{\mu(B) : B^c \supseteq A\} \\
&= \sup_{B \in \mathcal{B}} \{\mu(X) - \mu(B^c) : B^c \supseteq A\} \\
&= \mu(X) - \inf_{B \in \mathcal{B}} \{\mu(B^c) : B^c \supseteq A\} \\
&= \mu(X) - \inf_{C \in \mathcal{B}} \{\mu(C) : C \supseteq A\} \\
&= \mu(X) - \mu_2(A)
\end{aligned}$$

since $\mathcal{B}$ closed under complements

$\Rightarrow \mu(X) = \mu_1(A^c) + \mu_2(A)$.

(b) Let $(A_i) \subseteq \mathcal{A}$, so $\mu_1(A_i) = \mu_2(A_i)$; first see that

$\mu_1(A) = \sup_{B \subseteq A} \{\mu(B)\} \leq \mu(A) \leq \inf_{A \subseteq B} \{\mu(B)\} = \mu_2(A)$.

Hence need only show $\mu_1(\cup A_i) \geq \mu_2(\cup A_i)$

$$\begin{aligned}
\mu_2(\cup A_i) &= \inf_{B \in \mathcal{B}} \{\mu(B) : B \supseteq \cup A_i\} \\
&\leq \inf_{B \in \mathcal{B}} \{\mu(B) : B \supseteq \cup A_i,\ B = \cup B_i,\ A_i \subset B_i\} \\
&\leq \sum_i \inf_{B} \{\mu(B_i) : A_i \subset B_i\} = \sum_i \mu_2(A_i) = \sum_i \mu_1(A_i) \\
&= \sum_i \sup_{C \in \mathcal{B}} \{\mu(C_i) : C_i \subseteq A_i\} \\
&\leq \sup_{C \in \mathcal{B}} \{\mu(C) : C \subseteq \cup A_i\} = \mu_1(\cup A_i)
\end{aligned}$$

(2) μ meas. on $[0,\infty)$ s.t. $\int_{[0,\infty)} e^{ax}\,d\mu < \infty$ for $a \in \mathbb{R}$.
Show that $\psi(t) = \int_{[0,\infty)} e^{tx}\,d\mu$ is infinitely diff. on $(-\infty, a)$.

Let $f(x,t) = e^{tx}$ and consider $t \in (-\infty, a-\varepsilon)$.

Then $\left|\frac{\partial f}{\partial t}(x,t)\right| = \left|x e^{tx}\right| \leq x e^{(a-\varepsilon)x} = \frac{x e^{ax}}{e^{\varepsilon x}} \leq M e^{ax}$

since $\frac{x}{e^{\varepsilon x}}$ is a bounded function on $[0,\infty)$ (with bd M)

Clearly $M e^{ax}$ is in $L^1(\mu)$, hence we may apply the corollary to DCT.

(3.) $(X, \mathcal{B}, \mu)$ meas space, $0 \le f < \infty$ meas, $d\nu = f\,d\mu$

(a) Suppose μ σ-finite. Show ν σ-finite:

Since μ σ-finite, we may take the Lebesgue decomposition of ν w.r.t. μ: $\exists \lambda$ meas and $g \in L'(\mu)$ s.t. $\lambda \perp \mu$ and $d\nu = d\lambda + g\,d\mu$

Then $f\,d\mu = d\lambda - g\,d\mu \Rightarrow f\,d\mu - g\,d\mu = d\lambda$
$\Rightarrow (f-g)\,d\mu = d\lambda$.

Since $\lambda \perp \mu$, $\exists$ E s.t. $\lambda(E) = 0$ and $\mu(E^c) = 0$, so:

$$\lambda(A) = \lambda((A \cap E) \cup (A \cap E^c))$$
$$= \lambda(A \cap E) \cup \lambda(A \cap E^c) = 0 + \int_{A \cap E^c} (f-g)\,d\mu = 0 + 0$$

since $\mu(E^c) = 0$.

So λ identically 0; hence $f\,d\mu = g\,d\mu$,

Now since μ σ-finite, $\exists X = \bigcup X_i$ s.t. $\mu(X_i) < \infty$.

But now: $\nu(X_i) = \int_{X_i} g\,d\mu < \infty$ since $g \in L'(\mu)$

(b) Show that if $0 < f < \infty$ and ν semi-finite, then μ semifinite

Recall that if ν semifinite then for $\nu(E) > 0$, $\exists F \subseteq E$ s.t. $0 < \nu(F) < \infty$; namely:

$$\nu(E) = \int_E f\,d\mu > 0 \Rightarrow \exists F \subseteq E \text{ s.t. } \nu(F) = \int_F f\,d\mu < \infty$$

But since $f > 0$, $\nu(E) = \int_E f\,d\mu > 0$ for all E with $\mu(E) > 0$;

so if $\mu(E) > 0$, $\exists F \subseteq E$ w/ $0 < \nu(F) = \int_F f\,d\mu < \infty$

$\Rightarrow 0 < \mu(F) < \infty$ since $f > 0$ on all of F
(otherwise, integral will diverge)

hence μ semifinite

(4.) $(X, \mathcal{B}, \mu)$ with $\mu(X) < \infty$, f_n integrable, $f_n \to f$ in L^1

(a) Show $f_n \to f$ in measure

(b) Show $d\nu_n = |f_n| d\mu$ are unif. abs. cont. w.r.t μ.

(a) See that $\int_{\{|f_n - f| \geq \varepsilon\}} |f_n - f| d\mu \geq \int_{\{|f_n - f| \geq \varepsilon\}} \varepsilon \, d\mu = \varepsilon \mu \{|f_n - f| \geq \varepsilon\}$.

$\Rightarrow \mu\{|f_n - f| \geq \varepsilon\} \leq \frac{1}{\varepsilon} \int_{\{|f_n - f| \geq \varepsilon\}} |f_n - f| d\mu < \frac{1}{\varepsilon} \int_X |f_n - f| d\mu$ $(*)$

Now, for $\varepsilon\varepsilon' > 0$, $\exists M$ s.t. $n \geq M \Rightarrow \int_X |f_n - f| < \varepsilon\varepsilon'$,

hence for large n, $(*) < \frac{1}{\varepsilon} \varepsilon\varepsilon' = \varepsilon'$,

hence for large n, $\mu\{|f_n - f| \geq \varepsilon\} < \varepsilon'$, hence conv. in measure.

(b) [Recall unif. abs. cont. w.r.t. μ means: for $\varepsilon > 0$, $\exists \delta > 0$ s.t. $\mu(E) < \delta \Rightarrow \nu_n(E) < \varepsilon$ for all n]

By definition, $\nu_n \ll \mu$, hence for $\frac{\varepsilon}{2^n}$, $\exists \delta > 0$ such that $\mu(E) < \delta \Rightarrow \nu_n(E) < \frac{\varepsilon}{2^n}$.

Therefore, for all n (since ν_n is positive)

$$\nu_n(E) \leq \sum_n \nu_n(E) < \sum_n \frac{\varepsilon}{2^n} < \varepsilon$$

REAL ANALYSIS GRADUATE EXAM
Spring 2006

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and in $L^1(\mathbb{R})$. For each of (i) and (ii) give a proof or a counterexample.

(i) Is it true that f is bounded on $\mathbb{R}$? FALSE

(ii) Is it true that $f(x) \to 0$ as $x \to \infty$? FALSE

How do the results for (i) and (ii) change under the additional assumption that f' exists everywhere and is bounded?

(2) For $y > 0$ define

$$G(y) = \int_0^\infty \frac{1 - e^{-yx^2}}{x^2}\, dx.$$

(a) Show that this integral is finite for all $y > 0$.

(b) Show that G is differentiable, and find an explicit formula for $G'(y)$ and $G(y)$. HINT: You may take as given that $\int_0^\infty e^{-s^2}\, ds = \sqrt{\pi}/2$.

(3) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let f_n, f be real-valued measurable functions and suppose $f_n \to f$ a.e. Then there exists a partition of X into disjoint measurable sets $E_0, E_1, E_2, \ldots$ with $\mu(E_0) = 0$ and with $f_n \to f$ uniformly on E_i for each $i \geq 1$. HINT: Egoroff's Theorem requires a finite measure space.

(4) A function $g : \mathbb{R} \to \mathbb{R}$ is said to be *lower semi-continuous* if

$$\liminf g(x_n) \geq g(x) \quad \text{whenever } x_n \to x.$$

(a) Suppose that $f_k, k = 1, 2, 3, \ldots$ is a sequence of continuous functions, and $f(x) = \sup_{k \geq 1} f_k(x)$ is finite for all x. Show that f is lower semi-continuous.

(b) Show that a lower semi-continuous function is measurable.

Spring 06

(1.) $f : \mathbb{R} \to \mathbb{R}$ continuous, L^1. Proof or counterex:

(i) f bdd

(ii) $f(x) \to 0$ as $x \to \infty$

what if f' bdd?

(i) False: Let $g(x) = \begin{cases} e^{\frac{-1}{1-x^2}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$; note that $e^{\frac{-1}{1-x^2}} \leq e^{-1}$ for $x \in (-1,1)$

So $\int |g(x)| = \int_{-1}^{1} e^{\frac{-1}{1-x^2}} = \int_{-1}^{1} \frac{1}{e^{\frac{1}{1-x^2}}} < \int_{-1}^{1} \frac{1}{e}\,dx < \infty$, so

Then let $g_n(x) = n\, g((x-n)n^3)$; then we have:

$y = (x-n)n^3$
$dy = n^3 dx$

$$\int g_n(x) = \int n\, g((x-n)n^3)\, dx = \int \frac{n\, g(y)\, dy}{n^3} = \frac{\|g\|}{n^2}$$

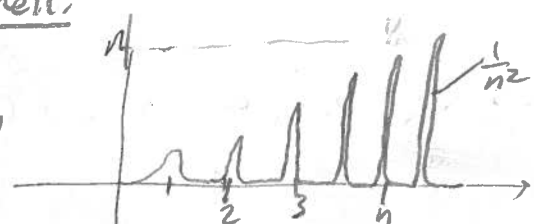
Now note that $g_n(n) = n\, g(0) = n e^{\frac{-1}{1}} = n e^{-1}$

and define $G(x) = \sum_{k \geq 3} g_k(x)$; then $G(n) = \sum_{k \geq 3} g_k(n) \geq n e^{-1}$, hence as $n \to \infty$, $G(n) \to \infty$, hence unbdd, but:

$$\int |G(x)| = \int \Big|\sum_{k \geq 3} g_k(x)\Big| \leq \int \sum_{k \geq 3} |g_k(x)| \underset{\text{Tonelli}}{=} \sum_{k \geq 3} \int |g_k(x)| = \sum_{k \geq 3} \frac{\|g\|}{n^2} < \infty,$$

hence $G \in L^1$.

(ii) False: Consider the function where $f(n) \equiv n$, and



$f \equiv 0$ outside the intervals $(n - \frac{1}{2n^3}, n + \frac{1}{2n^3})$, and $f(x) \leq n$ on.

Then: $\int_{n-\frac{1}{2n^3}}^{n+\frac{1}{2n^3}} f(x) \leq n \cdot \frac{1}{n^3} = \frac{1}{n^2}$, hence integrable, but $f \not\to 0$.

If $|f'(x)| \leq M$, by MVT, $|f(x) - f(y)| \leq M|x-y|$ for $x, y \in \mathbb{R}$,

i.e. f uniformly continuous, hence f bdd since $f(x) \to$

(2) $y>0$; $G(y) = \int_0^\infty \frac{1-e^{-yx^2}}{x^2}dx$

(a) Show $G(y)<\infty \ \forall y>0$

(b) Find $G'(y)$

(a) See that $1-e^{-yx^2} = 1 - \sum_{i=0}^\infty \frac{(-yx^2)^i}{i!} = -\sum_{i=1}^\infty \frac{(-1)^i y^i x^{2i}}{i!}$

$$\Rightarrow \frac{1-e^{-yx^2}}{x^2} = -\sum_{i=1}^\infty \frac{(-1)^i y^i x^{2i-2}}{i!} = -\sum_{i=0}^\infty \frac{(-1)^{i+1} y^{i+1} x^{2i}}{i!} = y\sum_{i=0}^\infty \frac{(-1)^i y^i x^{2i}}{i!} = ye^{-yx^2}$$

hence $\int_0^\infty \frac{1-e^{-yx^2}}{x^2}dx = \int_0^\infty ye^{-yx^2}dx < \infty$

(b) $G'(y) = \lim_{t\to y} \frac{1}{y-t}\int_0^\infty \left[\left(\frac{1-e^{-yx^2}}{x^2}\right) - \left(\frac{1-e^{-tx^2}}{x^2}\right)\right]dx$

$= \lim_{t\to y}\int_0^\infty \frac{e^{-tx^2}-e^{-yx^2}}{y-t}\cdot\frac{dx}{x^2} = \lim_{t\to y} -\int_0^\infty \frac{e^{-yx^2}-e^{-tx^2}}{y-t}\frac{dx}{x^2}$ (*) $f(x,t)$

$\longrightarrow$ Notice that $\lim_{t\to y}\frac{e^{-yx^2}-e^{-tx^2}}{y-t} = \frac{d}{dt}\left(e^{-tx^2}\right)\Big|_{t=y} = -x^2e^{-yx^2}$ (**)

Now, by the MVT, for some $t=c$:

$$\left|\frac{e^{-yx^2}-e^{-tx^2}}{y-t}\right| = \left|\frac{\partial}{\partial t}(f(x,t))_{t=c}\right| \le \sup_t\left|\frac{\partial}{\partial t}f(x,t)\right|$$

Notice that $\frac{\partial}{\partial t}f(x,t) = -x^2e^{-tx^2} = \frac{-x^2}{1+tx+\frac{t^2x^2}{2}+\frac{t^3x^3}{6}+\frac{t^4x^4}{24}+\cdots}$

$\le \frac{-x^2}{\frac{t^4x^4}{24}} = \frac{-24}{t^4x^2} < \frac{-24}{\varepsilon^4x^2}$ for $t\in(\varepsilon,1) \subseteq (0,1)$

So $\left|\frac{e^{-yx^2}-e^{-tx^2}}{y-t}\right| \le \frac{-24}{\varepsilon^2x^2}$, so apply DCT:

$$(*) = -\int_0^\infty \lim_{t\to y}\frac{e^{-yx^2}-e^{-tx^2}}{y-t}\frac{dx}{x^2} \overset{\text{By }(**)}{=} -\int_0^\infty \frac{-\cancel{x^2}e^{-yx^2}}{\cancel{x^2}}dx = \int_0^\infty e^{-yx^2}dx$$

let $x=\frac{u}{\sqrt{y}}$; then $\int_0^\infty e^{-yx^2}dx = \int_0^\infty \frac{e^{-u^2}du}{\sqrt{y}} = \frac{\sqrt{\pi}}{2\sqrt{y}} = \frac{1}{2}\sqrt{\frac{\pi}{y}}$

(3.) $(X, \mathcal{M}, \mu)$ σ-finite, f_n, f $\mathbb{R}$-valued measurable
Suppose $f_n \to f$ a.e.. Show $\exists$ partition of X into disjoint meas. sets $E_0, E_1, \dots$ with $\mu(E_0) = 0$ and $f_n \to f$ unif on E_i, $i \geq$

Since X is σ-finite, we have $X = \bigcup_{i=1}^{\infty} X_i$ for $\mu(X_i) < \infty$.

Let $A_n = \bigcup_{i=1}^{n} X_i \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(X)$ by continuity from below since $A_n \subseteq A_{n+1}$. $(A_0 = \phi)$

Let $B_n = A_{n+1} \setminus A_n$, $B_0 = A_0 = \phi$; then the B_n are disjoint and $\mu(X) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \Rightarrow \lim_{n\to\infty} \mu(B_n) = 0$

By Egoroff's theorem, since $\mu(B_n) < \infty$, for each $\frac{\varepsilon}{2^n}$, $\exists F_n \subseteq B_n$ such that $\mu(F_n) < \frac{\varepsilon}{2^n}$ and $f_n \to f$ unif on $B_n \setminus F_n$.

Now, $\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$

Since the B_n are disjoint, we have that $\bigcup_{n=1}^{\infty} B_n \setminus F_n = \bigcup_{n=1}^{\infty} B_n \setminus \bigcup_{n=1}^{\infty} F_n$,

hence: $\mu(\bigcup B_n \setminus F_n) = \mu(\bigcup B_n \setminus \bigcup F_n) = \mu(\bigcup B_n) - \mu(\bigcup F_n) > \mu(X) - \varepsilon$

letting $\varepsilon \to 0$; we have $\mu(\bigcup B_n \setminus F_n) \geq \mu(X)$, hence $\mu(\bigcup B_n \setminus F_n) = \mu(X)$; and $\mu\left(\bigcup_{n=1}^{\infty} F_n\right) = 0$

Hence let $E_n = B_n \setminus F_n$ for $n \geq 1$ and $E_0 = \bigcup F_n$.

Then $E_0 \cup \left(\bigcup_{n=1}^{\infty} E_n\right) = X$

(4) $g: \mathbb{R} \to \mathbb{R}$ is l.s.c. if $\liminf_{n\to\infty} g(x_n) \geq g(x)$ for $x_n \to x$.

(a) f_k continuous, $f(x) = \sup_k f_k(x) < \infty$. Show f l.s.c.

(b) Show l.s.c. fn. is measurable.

(a) $\liminf_{n\to\infty} f(x_n) = \lim_{n\to\infty} \inf_{k\geq n} f(x_k) = \lim_{n\to\infty}\left(\inf_{k\geq n}\left(\sup_{j\geq 0} f_j(x_k)\right)\right)$

$\geq \lim_{n\to\infty}\left(\inf_{k\geq n} f_j(x_k)\right)$ for all j

$= \liminf_{n\to\infty} f_j(x_n)$ for all j

$= f_j(x)$ for all j

$\Rightarrow \sup_{j\geq 0}\left(\liminf_{n\to\infty} f(x_n)\right) \geq \sup_{j\geq 0} f_j(x) \Rightarrow \liminf_{n\to\infty} f(x_n) \geq f(x)$

(b) Consider the sets $f^{-1}([a,\infty)) = \{f \geq a\}$; f is Borel measurable if each of these sets is Borel measurable.

$\rightarrow$ [L.S.C. at x if for $\varepsilon>0$, $\exists \delta>0$ s.t. $|x-y|<\delta \Rightarrow f(y) \geq f(x)-\varepsilon$]

Consider $x \in f^{-1}([a,\infty))$ and choose $\varepsilon>0$.

Since f is l.s.c., $\exists\, \delta>0$ s.t. $|x-y|<\delta \Rightarrow f(y) \geq f(x)-\varepsilon$,

hence $y \in (x-\delta, x+\delta) \Rightarrow f(y) \geq f(x)-\varepsilon \geq a-\varepsilon$

Letting $\varepsilon \to 0$, $\exists\, \delta'>0$ s.t. $y \in (x-\delta', x+\delta') \Rightarrow f(y) \geq a$, hence $(x-\delta', x+\delta') \subseteq f^{-1}([a,\infty))$.

Hence an arbitrary $x \in f^{-1}([a,\infty))$ has an open nbhd contained in $f^{-1}([a,\infty))$, hence $f^{-1}([a,\infty))$ is open, hence Borel.

$\longrightarrow$ So f is measurable

Real analysis, Graduate Exam Fall 2006

Answer all four questions. Partial credit will be given to partial solutions.

$\|fg\|_1 \le \|f\|_2 \|g\|_2$

1. Find necessary and sufficient conditions for a subset $X \subset \mathbb{R}$ to belong to the σ-algebra generated by all one-point subsets of $\mathbb{R}$.

2. Let (X, μ) be a measure space. Which of the following implications are true?

a. $\mu(X) < \infty$ and $f \in L^2(\mu)$ implies $f \in L^1(\mu)$.
b. $\mu(X) = \infty$ and $f \in L^2(\mu)$ implies $f \in L^1(\mu)$. false
c. $\mu(X) < \infty$ and $f \in L^1(\mu)$ implies $f \in L^2(\mu)$.
d. $\mu(X) = \infty$ and $f \in L^1(\mu)$ implies $f \in L^2(\mu)$.

Give proof or counter-example in each case.

3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} e^{-(x-y)} & \text{if } x > y, \\ 0 & \text{if } x = y, \\ -e^{-(y-x)} & \text{if } x < y. \end{cases}$$

a. Is f Lebesgue integrable?
b. Is it true that

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x,y)\,dx \right) dy = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x,y)\,dy \right) dx \ ?$$

4. Let $f \in L^1(\mathbb{R})$. Show that for each $n = 1, 2, 3, \ldots$, the function

$$f_n(x) = f(x)(\sin x)^n$$

also belongs to $L^1(\mathbb{R})$ and that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = 0.$$

Fall 06 :

(1.) Find necessary and sufficient conditions for $X \subseteq \mathbb{R}$ to belong to σ-algebra gen. by 1-pt subsets of $\mathbb{R}$

Let $\mathcal{A} = \{E \subseteq X ;\ E$ countable or E^c countable$\}$.

- $\mathcal{A}$ is a σ-algebra :

  1. Countable additivity : let $(E_i) \subseteq \mathcal{A}$.
  Consider $E = \bigcup_{i=1}^{\infty} E_i$. If all the E_i are countable, then $E \in \mathcal{A}$. Suppose that some E_j is cocountable.
  Then $E^c = \left(\bigcup_{i=1}^{\infty} E_i\right)^c = \bigcap_{i=1}^{\infty} E_i^c \subseteq E_j^c$, which is countable,
  hence E^c is countable, hence $E \in \mathcal{A}$.

  2. Complement : Consider $E, F \in \mathcal{A}$. If both countable, then $E \setminus F \subseteq E$ countable, hence $E \setminus F \in \mathcal{A}$. Suppose E countable and F cocountable. Then $E \setminus F \subseteq E$ countable $\Rightarrow E \setminus F \in \mathcal{A}$, and $(F \setminus E)^c = (F \cap E^c)^c = F^c \cup E$, countable, hence $F \setminus E \in \mathcal{A}$.
  Suppose both cocountable ; then $F \setminus E = F \cap E^c \subseteq E^c$ hence countable, hence $F \setminus E \in \mathcal{A}$.

- $P = \{$singleton sets$\} \subseteq \mathcal{P}(\mathbb{R})$; show $\mathcal{M}(P) = \mathcal{A}$.
$\mathcal{M}(P)$ → σ-algebra generated by P

($\subseteq$) $P \subseteq \mathcal{A} \Rightarrow \mathcal{M}(P) \subseteq \mathcal{A}$.

($\supseteq$) $E \in \mathcal{A}$;

If E countable, then $E = \bigcup_{n=1}^{\infty} \{x_i\} \in \mathcal{M}(P)$

If E cocountable, then E^c countable, then $E^c = \bigcup_{n=1}^{\infty} \{y_i\} \in \mathcal{M}(P)$ and $E = (E^c)^c \in \mathcal{M}(P)$.

So $\mathcal{A} \subseteq \mathcal{M}(P)$.

Therefore, $\mathcal{M}(P) = \mathcal{A}$

Hence $X \in \mathcal{M}(P) \iff X$ countable or cocountable

(2.) (X,μ) measure space. True or false?

(a) $\mu(X)<\infty$ and $f\in L^2(\mu) \Rightarrow f\in L^1(\mu)$: TRUE

$\|f\|_1 = \|f\cdot 1\|_1 \le \|f\|_2 \|1\|_2$ by Hölder's inequality

$f\in L^2(\mu) \Rightarrow \|f\|_2<\infty$ and $\mu(X)<\infty \Rightarrow \|1\|_2<\infty$.

Hence $\|f\|_1<\infty$, hence $f\in L^1(\mu)$.

(b) $\mu(X)=\infty$ and $f\in L^2(\mu) \Rightarrow f\in L^1(\mu)$: FALSE

$X=[1,\infty)$, $\mu(X)=\infty$, $f(x)=\frac{1}{x}$.

Then $f\in L^2$ since $\int_1^\infty \frac{1}{x^2}dx<\infty$, but $f\notin L^1$ since $\int_1^\infty \frac{1}{x}dx=\infty$

(c) $\mu(X)<\infty$ and $f\in L^1(\mu) \Rightarrow f\in L^2(\mu)$: FALSE

$X=[0,1]$, $\mu(X)<\infty$, $f(x)=\frac{1}{\sqrt{x}}$.

Then $\int_0^1 |f|dx<\infty$, but $\int_0^1 |f|^2 dx = \int_0^1 \frac{1}{x}dx = \infty$.

(d) $\mu(X)=\infty$ and $f\in L^1(\mu) \Rightarrow f\in L^2(\mu)$ FALSE

$X=[0,\infty)$, $\mu(X)=\infty$; $f(x)=\frac{1}{\sqrt{x}}\chi_{(0,1]}$.

Then $\int_0^\infty |f|dx = \int_0^1 \frac{1}{\sqrt{x}}dx<\infty$, but $\int_0^\infty |f|^2dx = \int_0^1 \frac{1}{x}dx = \infty$

Alt proof of (a) w/o Hölder:

$\mu(X)<\infty$ and $f\in L^2(\mu)$. Let $E=\{|f|>1\}$.

Then $\|f\|_1 = \int_X |f|d\mu = \int_E |f|d\mu + \int_{X\setminus E}|f|d\mu$

$\le \int_E |f|^2 d\mu + \int_{X\setminus E} d\mu < \infty$

since $f\in L^2(\mu)$ and $\mu(X\setminus E)<\infty$ since $\mu(X)<\infty$

(3.) $f:\mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = \begin{cases} e^{-(x-y)} & \text{if } x>y \\ 0 & \text{if } x=y \\ -e^{-(y-x)} & \text{if } x<y \end{cases}$

(a) Is f Lebesgue integrable?
(b) Is it true that: $\iint f(x,y)\,dx\,dy = \iint f(x,y)\,dy\,dx$

(a) See that $|f(x,y)| = |f(y,x)|$, so we have:

$$\iint |f|\,dm = 2\iint_{\{x>y\}} |f|\,dm = 2\int_{-\infty}^{\infty}\left[\int_{-\infty}^{x} e^{y-x}\,dy\right]dx \quad (*)$$

Now, see that $\int_{-\infty}^{x} e^{y-x}\,dy = \int_{-\infty}^{0} e^{u}\,du$ (under substitution $u=y-x$)

$$= e^{0} - \lim_{u\to-\infty} e^{u} = 1-0 = 1$$

So:

$(*) = 2\int_{-\infty}^{\infty} 1\,dx = \infty$, hence f is not Lebesgue integrable.

(b) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\,dx\,dy = \int_{-\infty}^{\infty}\left(\int_{\{x>y\}} e^{-(x-y)}dx + \int_{\{x<y\}} -e^{-(y-x)}dx\right)dy$

$= \int_{-\infty}^{\infty}\left(\int_{\{x>y\}} e^{-(x-y)}dx - \int_{\{x<y\}} e^{-(y-x)}dx\right)dy = \int_{-\infty}^{\infty}\left(\underbrace{\int_{\{x>y\}} e^{-(x-y)}dx - \int_{\{x>y\}} e^{-(x-y)}}_{0}\right)dy$

$= \int_{-\infty}^{\infty} 0\,dy = 0$, and same for the other side,

hence the equality holds

(4.) $f \in L^1(\mathbb{R})$. Show $f_n(x) = f(x)\sin^n x \in L^1(\mathbb{R})$ for all n, and that $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx = 0$

See that $|f_n(x)| = |f(x)\sin^n x| = |f|\,|\sin x|^n \leq |f| \in L^1$, hence $f_n \in L^1$.

Now, $\lim_{n\to\infty} \int |f_n|\,dx = \int \lim_{n\to\infty} |f_n|\,dx$ by DCT since $|f_n| \leq |f|$ for all n and $f \in L^1$.

Let $x \in \mathbb{R} \setminus \{\frac{(2k+1)\pi}{2}\}_{k\in\mathbb{Z}}$; then for all such x, $|\sin x| < 1$, hence $\lim_{n\to\infty} |\sin x|^n = 0$ on $\mathbb{R} \setminus \{\frac{(2k+1)\pi}{2}\}$,

hence $\lim_{n\to\infty} |f(x)\sin^n x| = |f(x)| \lim_{n\to\infty} |\sin x|^n = 0$

for all such x, hence $\lim_{n\to\infty} \int |f_n|\,dx = 0$ for all such x.

Now, the set for which this not true $\left(\{\frac{(2k+1)\pi}{2}\}_{k\in\mathbb{Z}}\right)$ is countable, hence measure 0,

hence $\lim_{n\to\infty} \int |f_n|\,dx = 0$ almost everywhere.

REAL ANALYSIS GRADUATE EXAM
SPRING 2007

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $\{\mu_n\}$ be a sequence of measures on $(X, \mathcal{M})$ with $\mu_1(E) \leq \mu_2(E) \leq \ldots$ for all $E \in \mathcal{M}$. Let $\mu(E) = \lim_n \mu_n(E)$. Show that μ is a measure.

(2) Suppose $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) < \infty$, and $f \in L^1(\mu)$ is strictly positive. Let $0 < \alpha < \mu(X)$.

(a) Show that

$$\inf\left\{\int_E f\, d\mu : \mu(E) \geq \alpha\right\} > 0.$$

(b) Show that (a) can be false if we remove the assumption $\mu(X) < \infty$.

(3) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. The *graph* of f is $\{(x, f(x)) : x \in [0,1]\}$. Show that the graph has two-dimensional Lebesgue measure 0.

(4) Let $n \geq 1$. Show that the function

$$g(u) = \int_{-\infty}^{\infty} \frac{x^n e^{ux}}{e^x + 1}\, dx, \qquad u \in (0,1),$$

Differentiation is local!

is differentiable in $(0,1)$.

Spring 07:

1. (μ_n) seq. of measures on $(X, \mathcal{M})$ with $\mu_1(E) \le \mu_2(E) \le \cdots$ for all $E \in \mathcal{M}$; Let $\mu(E) = \lim_{n\to\infty} \mu_n(E)$; show μ is a measure

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Clearly $\mu(\emptyset) = \lim_{n\to\infty} \mu_n(\emptyset) = \lim_{n\to\infty} 0 = 0$, hence must only verify countable additivity; let $A = \bigcup_{k=1}^{\infty} A_k$, $A_k \in \mathcal{M}$ disjoint

- Case $\mu(A) < \infty$: $\mu(A) = \lim_{n\to\infty} \mu_n(A) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \mu_n(A_k)$
$\le \sum_{k=1}^{\infty} \mu(A_k)$ since $\mu_n(A_k) < \mu(A_k)$ for all n.

Now choose $N < \infty$; Then:

$$\sum_{k=1}^{N} \mu(A_k) = \sum_{k=1}^{N} \lim_{n\to\infty} \mu_n(A_k) = \lim_{n\to\infty} \sum_{k=1}^{N} \mu_n(A_k) \le \lim_{n\to\infty} \sum_{k=1}^{\infty} \mu_n(A_k)$$
$$= \lim_{n\to\infty} \mu_n\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right).$$

So we have $\sum_{k=1}^{N} \mu(A_k) \le \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$; letting $N \to \infty$, we get that $\sum_{k=1}^{\infty} \mu(A_k) \le \mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu(A)$; hence $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$

- Case $\mu(A) = \infty$: $\infty = \mu(A) = \lim_{n\to\infty} \mu_n(A)$
$$= \lim_{n\to\infty} \sum_{k=1}^{\infty} \mu_n(A_k) \le \sum_{k=1}^{\infty} \mu(A_k) \Rightarrow \sum_{k=1}^{\infty} \mu(A_k) = \infty$$
$$\Rightarrow \mu(A) = \infty = \sum_{k=1}^{\infty} \mu(A_k).$$

Alternative: $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{n\to\infty} \mu_n\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \mu_n(A_k)$

$\mu_n(A_k) \le \mu_{n+1}(A_k)$ for all n, k, and all positive measures, hence apply MCT:

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \mu_n(A_k) = \sum_{k=1}^{\infty} \lim_{n\to\infty} \mu_n(A_k) = \sum_{k=1}^{\infty} \mu(A_k)$$

(2.) $\mu(X)<\infty$, $f\in L^1(\mu)$, $f>0$, $0<\alpha<\mu(X)$.

(a) Show $\inf\{\int_E f\,d\mu : \mu(E)\geq\alpha\}>0$

(b) Show (a) is false with $\mu(X)=\infty$

(a) Let $E_n=\{f\geq\frac{1}{n}\}$; clearly $E_n\subseteq E_{n+1}$, and since $f>0$,

$X=\bigcup_{n=1}^{\infty}E_n$; so by continuity from below, $\lim_{n\to\infty}\mu(E_n)=\mu(X)$.

Therefore, $\exists N$ such that $\mu(E_N)>\mu(X)-\alpha$

Now choose $E\subseteq X$ such that $\mu(E)\geq\alpha$ (note that N does not depend on the choice of E). So we have:

$$\mu(X)<\mu(E_N)+\alpha\leq\mu(E_N)+\mu(E)$$

hence $\mu(E\cap E_N)>0$, and since $E\cap E_N\subseteq E$, we have now:

$$\int_E f\,d\mu\geq\int_{E\cap E_N}f\,d\mu\geq\frac{\mu(E\cap E_N)}{N}$$

But, $\mu(E\cap E_N)=\mu(E)+\mu(E_N)-\mu(E\cup E_N)$

$$>\alpha+(\mu(X)-\alpha)-\mu(X)=0$$

Hence $\mu(E\cap E_N)>0$, hence $\int_E f\,d\mu\geq\int_{E\cap E_N}f\,d\mu\geq\frac{\mu(E\cap E_N)}{N}>0$,

so $\inf\{\int_E f\,d\mu : \mu(E)\geq\alpha\}>0$

(b) Let $X=[1,\infty)$; $f(x)=\frac{1}{x^2}$; $\alpha=1$; $E_n=[n,n+\alpha]$

Then $\int_{E_n}f\,dm=\int_{E_n}\frac{dx}{x^2}=\frac{1}{n}-\frac{1}{n+\alpha}=\frac{\alpha}{n(n+\alpha)}\longrightarrow 0$ as $n\to\infty$,

Recall that $m(E_n)=\alpha$ for all n; thus:

$$\inf\{\int_E f\,d\mu : \mu(E)\geq\alpha\}\leq\inf\{\int_{E_n}f\,d\mu : E_n\}=0$$

$$\Rightarrow \inf\{\int_E f\,d\mu : \mu(E)\geq\alpha\}=0$$

(3.) $f:[0,1]\to\mathbb{R}$ continuous, $G(f)=\{(x,f(x)) : x\in[0,1]\}$.
Show that $m(G(f))=0$

$f:[0,1]\to\mathbb{R}$ continuous and $[0,1]$ compact, hence f is uniformly continuous; i.e. for $\varepsilon>0$; $\exists\,\delta>0$ such that $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$ for all $x,y\in[0,1]$, hence we may take $\delta<1$.

Now, let $n=\min\{n\in\mathbb{N} : n\delta>1\}$ and then let $0=x_0<x_1<\cdots<x_n=1$ in $[0,1]$ such that $|x_i-x_{i-1}|<\delta$

So for $z\in[x_{i-1},x_i]$, we have $|z-x_i|<\delta \Rightarrow |f(z)-f(x_i)|<\varepsilon$

$$\Rightarrow G(f)\subseteq \bigcup_{i=1}^{n}[x_{i-1},x_i]\times[f(x_i)-\varepsilon, f(x_i)+\varepsilon]$$

$$\Rightarrow m(G(f)) \leq \sum_{i=1}^{n} m([x_{i-1},x_i]\times[f(x_i)-\varepsilon, f(x_i)+\varepsilon])$$
$$< \sum_{i=1}^{n}\delta\cdot 2\varepsilon = 2\varepsilon n\delta$$

Since n is the smallest s.t. $n\delta>1$, we must have that $(n-1)\delta\leq 1 \Rightarrow n\delta-\delta\leq 1 \Rightarrow n\delta\leq\delta+1$

Hence $2\varepsilon n\delta \leq 2\varepsilon(\delta+1)<2\varepsilon(2)=4\varepsilon$ since $\delta<1$.

Thus $\underline{m(G(f))<4\varepsilon}$; letting $\varepsilon\to 0$, we get our result.

④ $n \geq 1$; show that $g(u) = \int_{-\infty}^{\infty} \frac{x^n e^{ux}}{e^x+1} dx$ for $u \in (0,1)$.

Let $f(x,u) = \frac{x^n e^{ux}}{e^x+1}$; recall that the statement is true if $\left|\frac{\partial}{\partial u} f(x,u)\right| \leq g(x)$ some $g \in L^1$, for all x, u. Recall that differentiation is <u>local</u>, hence we may restrict to $u \in (0, 1-\varepsilon) \subsetneq (0,1)$.

∴ For $u \in (0, 1-\varepsilon)$, we have:
for $e^x > 1$, $e^{ux} < e^{(1-\varepsilon)x}$ and for $e^x < 1$, $e^{ux} < 1$.
Hence for $u \in (0, 1-\varepsilon)$, $e^{ux} < 1 + e^{(1-\varepsilon)x}$

Then: $\left| \frac{x^{n+1} e^{ux}}{e^x+1} \right| \leq \left| \frac{x^{n+1}(e^{(1-\varepsilon)x} + 1)}{e^{x+1}} \right|$

and clearly $\frac{x^{n+1}}{e^{x+1}}$ is integrable, as is
$$\frac{x^{n+1} e^{(1-\varepsilon)x}}{e^{x+1}} = \frac{x^{n+1}}{e^{\varepsilon x+1}} \text{ integrable}$$

Hence we may take the derivative inside the integral:
$$g'(u) = \int_{-\infty}^{\infty} \frac{x^{n+1} e^{ux}}{e^x+1} dx$$

Incomplete
#2(b)

REAL ANALYSIS GRADUATE EXAM
FALL 2007

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. In questions 3 and 4, $|A|$ is used to denote the Lebesgue measure of a measurable subset $A \subseteq \mathbb{R}^d$.

1. By differentiating the equation

$$\int_{-\infty}^{\infty} e^{-tx^2}\,dx = \sqrt{\frac{\pi}{t}} \qquad t > 0$$

show that

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2}\,dx = \frac{(2n)!\sqrt{\pi}}{4^n n!}$$

for $n \geq 1$. You should be careful to justify your calculations.

2. (a) Construct a sequence f_n of Lebesgue measurable functions on $(0,1)$ such that $f_n(x) \to 0$ as $n \to \infty$ for each $x \in (0,1)$ and

$$\int_0^1 |f_n(x)|\,dx \to \infty$$

as $n \to \infty$.

(b) Give an example of a continuous function $F : [a,b] \to \mathbb{R}$ which is differentiable almost everywhere in $[a,b]$ with F' Lebesgue integrable on $[a,b]$ and such that

$$F(b) - F(a) \neq \int_a^b F'(t)\,dt.$$

3. Let g_k, $k = 1, 2, \ldots$, be a sequence of nonnegative measurable functions on a measurable subset E of $\mathbb{R}^d$. Suppose that

$$|\{x \in E : g_k(x) > 1/2^k\}| < 1/2^k \quad \text{for each } k \geq 1.$$

Prove that $\sum_{k=1}^{\infty} g_k$ converges almost everywhere on E.

4. Let $g \in L^p(\mathbb{R}^d)$ and define

$$\mu(t) = |\{x \in \mathbb{R}^d : |g(x)| > t\}| \quad \text{for } t \geq 0.$$

Show that

$$\int_{\mathbb{R}^d} |g(x)|^p\,dx = -\int_0^{\infty} t^p\,d\mu(t).$$

Fall 2007:

① Differentiate $\longrightarrow \int_{-\infty}^{\infty} e^{-tx^2}\,dx = \sqrt{\frac{\pi}{t}}$

to show that $\int_{-\infty}^{\infty} x^{2n} e^{-tx^2}\,dx = \frac{(2n)!\sqrt{\pi}}{4^n n!}$

Consider $f(x,t) = e^{-tx^2}$ and let $t \in (\varepsilon, \infty) \rightarrow$ $\boxed{\text{differentiation in land}}$

Then $\left|\frac{\partial}{\partial t} f(x,t)\right| = \left|x^2 e^{-tx^2}\right| \le \left|\frac{x^2}{e^{\varepsilon x^2}}\right| \in L^1$

and $\left|\frac{\partial^n}{\partial t^n} f(x,t)\right| = \left|x^{2n} e^{-tx^2}\right| \le \left|\frac{x^{2n}}{e^{\varepsilon x^2}}\right| \in L^1$ for all n.

Hence apply the corollary to DCT:

$$\sqrt{\pi}\frac{d^n}{dt^n}\left(t^{-1/2}\right) = \frac{d^n}{dt^n}\int_{-\infty}^{\infty} e^{-tx^2}\,dx = \int_{-\infty}^{\infty} (-1)^n x^{2n} e^{-tx^2}\,dx$$

and $$\frac{d^n}{dt^n}\left(t^{-1/2}\right) = \left((-1)^n \frac{2n-1 \cdots 7\cdot 5\cdot 3}{2^n} t^{-(2n+1)/2}\right)$$

$$= \left((-1)^n \frac{(2n)!}{4^n n!} t^{-(2n+1)/2}\right)$$

Hence let $t = 1$,

and then:

$$\int_{-\infty}^{\infty} x^{2n} e^{-tx^2} = \frac{2n!}{4^n n!}\sqrt{\pi}$$

(2.) (a) Construct a sequence f_n of Leb-measurable functions on $(0,1)$ s.t. $f_n(x) \to 0$ as $n \to \infty$ for each $x \in (0,1)$ and $\int_0^1 |f_n(x)|\,dx \to \infty$

Consider $f_n(x) = \chi_{(0,\frac{1}{n}]}(x)\, n^2$

Then $\int_0^1 |f_n(x)|\,dx = \int_0^1 \chi_{[0,\frac{1}{n}]} n^2 = \int_0^{1/n} n^2\,dx = n^2 \frac{1}{n} = n \to \infty$ as $n \to \infty$

But $f_n(x) = \chi_{[0,\frac{1}{n}]}(x)\, n^2 \to 0$ as $n \to \infty$.

(b) Give an example of continuous fn. $F: [a,b] \to \mathbb{R}$ which is differentiable a.e. in $[a,b]$, F' Lebesgue integrable, but $F(b) - F(a) \neq \int_a^b F'(t)\,dt$

(3.) $(g_k) \subseteq L^+(E)$ where $E \subseteq \mathbb{R}^d$.

Suppose $|\{x \in E : g_k(x) > \frac{1}{2}^k\}| < \frac{1}{2^k}$ for each $k \geq 1$.

Prove that $\sum_{k=1}^{\infty} g_k$ converges almost everywhere on E

→ Let $E_k = \{x \in E : g_k(x) > \frac{1}{2^k}\}$; then $m(E_k) < \frac{1}{2^k}$

and furthermore,

$$m\left(\bigcup_{k \geq n} E_k\right) \leq \sum_{k=n}^{\infty} \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{n-1} \qquad (*)$$

Hence $m\left(\bigcup_{k \geq n} E_k\right) \longrightarrow 0$ as $n \longrightarrow \infty$

→ Now define $B_n = \bigcup_{k \geq n} E_k$ and consider $B = \bigcap_{n=1}^{\infty} B_n$

Note that $B_1 \supseteq B_2 \supseteq \cdots$ (and $m(B_1) < \infty$), hence by cont. from above, $m(B) = m\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n \to \infty} m(B_n) = 0$ by $(*)$

→ See now that $g_k(x) > \frac{1}{2^k}$ for infinitely many k

$\iff x \in E_k$ for infinitely many k.

$\iff x \in B_n$ for all n

$\iff x \in B$

So $x \in B$ means $g_k(x) > \frac{1}{2^k}$ for infinitely many k, but B is Lebesgue null, hence <u>for almost every $x \in E$</u>: $g_k(x) > \frac{1}{2^k}$ for only finitely many k, i.e. $\exists N$ such that $g_k(x) \leq \frac{1}{2^k}$ for all $k \geq N$

Hence:

$$\sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{N-1} g_k(x) + \sum_{k=N}^{\infty} g_k(x)$$

$$\leq \sum_{k=0}^{N-1} g_k(x) + \sum_{k=N}^{\infty} \frac{1}{2^k} < \infty$$

hence convergent for a.e. $x \in E$.

④ $g \in L^p(\mathbb{R}^d)$; let $\mu(t) = m(\{x : |g(x)| > t\})$ for $t \geq 0$.
Show that $\int_{\mathbb{R}^d} |g(x)|^p dx = \int_0^\infty t^p d\mu(t)$

Recall integration by parts:

$$\int_0^\infty f dg = fg\Big|_0^\infty - \int_0^\infty g df$$

Hence:

$$\int_0^\infty t^p d\mu(t) = \mu(t)t^p\Big|_0^\infty - \int_0^\infty \mu(t) d(t^p)$$

and see that

$$\int_0^\infty \mu(t) d(t^p) = \int_0^\infty \mu(t) p t^{p-1} dt = \int_0^\infty p t^{p-1} \left(\int_{\mathbb{R}^d} \chi_{\{|g(x)|>t\}} dx\right) dt$$

$$= \int_{\mathbb{R}^d} \int_0^\infty p t^{p-1} \chi_{\{|g(x)|>t\}} dt dx \quad \underline{\text{by Tonelli}}$$

$$= \int_{\mathbb{R}^d} \int_0^{|g(x)|} p t^{p-1} dt dx$$

$$= \int_{\mathbb{R}^d} |g(x)|^p dx$$

Thus we must only show that $\mu(t)t^p\Big|_0^\infty = 0$, i.e. that $\underline{\lim_{t\to\infty} \mu(t)t^p = 0}$. Suppose that $g = \sum a_i \chi_{E_i}$ integrable simple.

Then for sure $M = \max\{a_i\}$, for $t \geq M$ we have:

$\mu(t) = m(\{|g(x)| \geq t\}) = 0$

Hence $\lim_{t\to\infty} \mu(t) t^p = 0$ (since for $t > M$, $\mu(t) t^p = 0$)

Now let ϕ_n be seq of int. simples s.t. $|\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$;

then $\{|\phi_i| \geq t\} \subseteq \{|\phi_{i+1}| \geq t\} \subseteq \{|f| \geq t\}$ for all i,

hence by cont from below, $\lim_{i\to\infty} m\{|\phi_i| \geq t\} = \{|f| \geq t\}, = \mu(t)$

hence $\lim_{t\to\infty} \mu(t) t^p = \lim_{t\to\infty} t^p \lim_{i\to\infty} m\{|\phi_i| \geq t\} = \lim_{i\to\infty} 0 = 0$

REAL ANALYSIS GRADUATE EXAM
SPRING 2008

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let m denote Lebesgue measure on the unit square $E = [0,1] \times [0,1]$. In each case determine whether the integral exists:

$$\text{(i)} \int_E \frac{1}{x-y}\, dm(x,y), \qquad \text{(ii)} \int_E \frac{1}{x+y}\, dm(x,y).$$

2. Let f be a nonnegative measurable function on [0,1] such that $\int_0^1 f(x)\,dx = 1$. Define a measure μ on $[0,1]$ by

$$\mu(A) = \int_A f(x)\,dx, \quad A \in \mathcal{B}([0,1]).$$

Let K be the intersection of all compact subsets E of $[0,1]$ such that $\mu(E) = 1$. Find $\mu(K)$.

3. For a function $f : [0,1] \to \mathbb{R}$ determine whether either of the statements

"f is continuous almost everywhere on $[0,1]$"

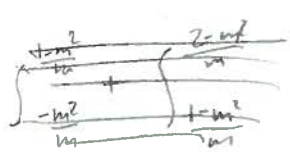
and

"there is a continuous $g : [0,1] \to \mathbb{R}$ such that $f = g$ almost everywhere"

implies the other one. In each case justify your answer with a proof or counterexample.

4. Suppose that $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{m\to\infty} \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} f(x)\,dx \right| = \|f\|_{L^1(\mathbb{R})}.$$

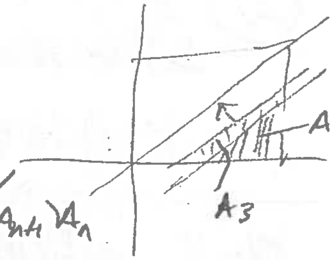


Spring '08:

(1.) m = Leb. meas on $E=[0,1]\times[0,1]$. Determine existence of the integrals: (i) $\int_E \frac{1}{x-y}\,dm$, (ii) $\int_E \frac{1}{x+y}\,dm$

(i) Consider the subsets $A_n=\{y\le x-\frac{1}{n}\}\subseteq E$, $n\ge 1$

Note that $A_n\subseteq A_{n+1}$ and $\bigcup_{n=1}^{\infty} A_n = E\cap\{y\le x\}$.



Therefore: $\chi_{\bigcup_{n=1}^{\infty} A_n} = \chi_{\bigcup_{n=1}^{\infty} A_n\setminus A_{n+1}} = \sum_{n=1}^{\infty}\chi_{A_{n+1}\setminus A_n} = \lim_{k\to\infty}\sum_{n=1}^{k}\chi_{A_{n+1}\setminus A_n}$

$= \lim_{k\to\infty}\left(\sum_{n=1}^{k}(\chi_{A_{n+1}}-\chi_{A_n})\right) = \lim_{k\to\infty}\left(\chi_{A_2}-\chi_{A_1}+\chi_{A_3}-\chi_{A_2}+\dots+\chi_{k+1}-\chi_{A_k}\right)$

$= \lim_{k\to\infty}\left(\chi_{k+1}-\chi_{A_1}\right)$

Now: $\int_{E\cap\{y\le x\}}\frac{dm}{x-y} = \int_{\bigcup_{n=1}^{\infty} A_n}\frac{dm}{x-y} = \int_E \chi_{\bigcup_{n=1}^{\infty} A_n}\frac{dm}{x-y} = \int_E\left(\lim_{k\to\infty}\chi_{A_{k+1}}-\chi_{A_1}\right)\frac{dm}{x-y}$

$= \int_E \lim_{k\to\infty}\chi_{A_{k+1}}\frac{dm}{x-y} - \underbrace{\cancel{\int_E \chi_{A_1}\frac{1}{x-y}\,dm}}_{0}$ since $m(A_1)=0$

And since $\frac{1}{x-y}$ is positive on all the A_n's and $\frac{1}{x-y}\le n$ on each A_n, the sequence $\chi_{A_n}\frac{1}{x-y}$ is increasing, so we apply MCT:

$\int_E \lim_{k\to\infty}\chi_{k+1}\frac{dm}{x-y} = \lim_{k\to\infty}\int\chi_{A_{k+1}}\frac{dm}{x-y} = \lim_{k\to\infty}\int_{A_{k+1}}\frac{dm}{x-y}$

But now we see that:

$\int_{A_n}\frac{dm}{x-y} = \int_{1/n}^{1}\int_0^{x-\frac{1}{n}}\frac{dy\,dx}{x-y} = -\int_{1/n}^{1}\ln|x-y|\Big]_0^{x-\frac{1}{n}}dx = -\int_{1/n}^{1}\left(\ln|\tfrac{1}{n}|-\ln|x|\right)dx$

$= -\int_{1/n}^{1}\ln|\tfrac{1}{nx}|\,dx = \int_{1/n}^{1}\ln|nx|\,dx = x\ln|nx|\Big]_{1/n}^{1} - \int_{1/n}^{1}x\left(\frac{n\,dx}{nx}\right) = \Big[x\ln|nx|-x\Big]_{1/n}^{1}$

$= [(1)\ln|n|-1]-[\tfrac{1}{n}\overset{0}{\ln|1|}-\tfrac{1}{n}] = \ln|n|+\tfrac{1}{n}-1 \longrightarrow \infty$ as $n\to\infty$

Hence $\int_{E\cap\{y\le x\}}\frac{dm}{x-y}=\infty$, hence $\int_E\frac{dm}{x-y}=\infty$.

(ii) By symmetry of $\frac{1}{x+y}$ on E, we have:

$$\int_E \frac{1}{x+y}\,dm = 2\int_{E\cap\{y\le x\}} \frac{1}{x+y}\,dm = 2\int_0^1\int_0^x \frac{dy\,dx}{x+y} = 2\int_0^1 (\ln|2x| - \ln|x|)\,dx$$

$$= 2\int_0^1 \ln\left|\frac{2x}{x}\right| dx = 2\int_0^1 \ln|2|\,dx = \underline{2\ln|2|}$$

---

(2) $f \ge 0$ measurable on $[0,1]$ s.t. $\int_0^1 f(x)\,dx = 1$

Define the measure μ on $[0,1]$ by $\mu(A) = \int_A f(x)\,dx$, $A \in \mathcal{B}[0,1]$

Let $K = \bigcap\{E \subseteq [0,1] \text{ cpt}, \mu(E) = 1\}$; $\mu(K) = ?$

<u>Step 1: $\mu(E_1 \cap E_2) = 1$ for E_1, E_2 cpt, meas 1</u>

Let $E_1, E_2 \subseteq [0,1]$ cpt, $\mu(E_1) = \mu(E_2) = 1$. Now $E_1 \cap E_2$, $E_1 \cup E_2 \subseteq [0,1]$, hence $\mu(E_1 \cap E_2) \le 1$, $\mu(E_1 \cup E_2) \le 1$, and $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2) - \mu(E_1 \cap E_2)$

$$\Rightarrow \mu(E_1 \cap E_2) = \mu(E_1) + \mu(E_2) - \mu(E_1 \cup E_2)$$

$$= 1 + 1 - \mu(E_1 \cup E_2) \ge 2 - 1 = 1$$

Hence $\mu(E_1 \cap E_2) = 1$

<u>Step 2: $K = \bigcap_\alpha E_\alpha$ where E_α cpt, $\mu(E_\alpha) = 1$ has meas 1</u>

Recall that $\mu(K) = \inf\{\mu(U) : K \subseteq U, U \text{ open}\}$. (*)

Let $K = \bigcap_\alpha E_\alpha \subseteq U$ and let $F_\alpha = E_\alpha \cap U^c$. Since $\bigcap_\alpha E_\alpha \subseteq U$, we have that $\bigcap_\alpha F_\alpha = \emptyset$.

Note that F_α is intersection of two closed sets, hence closed, hence $\{F_\alpha\}$ is a family of closed sets in $[0,1]$ with $\bigcap_\alpha F_\alpha = \emptyset$;

$[0,1]$ is compact, hence every family of closed sets with the finite-intersection prop. has a nonempty intersection; hence $\{F_\alpha\}$ does <u>not</u> have the finite intersection prop., i.e. $\exists$ finite subfamily $\{F_{\alpha_1}, \dots, F_{\alpha_n}\}$ s.t. $\bigcap_{i=1}^n F_{\alpha_i} = \emptyset$. Hence $\exists$ finite collection $\{E_{\alpha_1}, \dots, E_{\alpha_n}\}$ s.t. $\bigcap_{i=1}^n E_{\alpha_i} \subseteq U$, hence $\mu(\bigcap_{i=1}^n E_{\alpha_i}) \le \mu(U)$. By Part 1, a finite int. of cpt, meas 1 sets is meas 1, hence $\mu(U) \ge 1$, hence $\underline{\mu(K) = 1}$ by (*).

(3.) $f:[0,1]\to\mathbb{R}$.

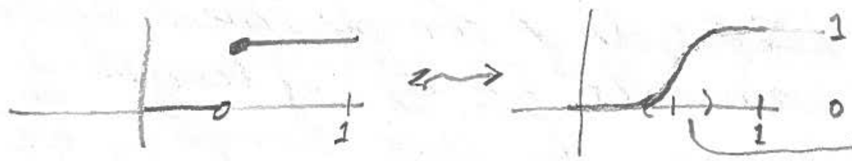
I. f continuous a.e. on $[0,1]$

II. $\exists$ continuous $g:[0,1]\to\mathbb{R}$ s.t. $f=g$ a.e.

Determine if I $\Rightarrow$ II or II $\Rightarrow$ I.

I $\not\Rightarrow$ II : Counterexample: let $f(x)=\chi_{[\frac{1}{2},1]}(x)$

- Set of discontinuities of $f = \{\frac{1}{2}\}$, hence set is measure 0, hence f is cont. a.e.
- Any continuous function approximating f will have to have smoothing near the jump discontinuity w/ non-zero measure:



$(-\varepsilon,\varepsilon)$ has non-zero measure

II $\not\Rightarrow$ I : Counterexample:

$$f(x)=\begin{cases}0 & x\in\mathbb{Q}\\ 1 & x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}\quad ; \quad g(x)\equiv 1$$

$f=g$ a.e., but f is nowhere continuous.

(4.) $f \in L^1(\mathbb{R})$. Prove that $\lim_{m \to \infty} \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{\frac{(k+1)}{m}} f(x)\,dx \right| = \|f\|_{L^1}$

Since $f \in L^1(\mathbb{R})$ with Lebesgue measure, $\exists$ continuous $g \in L^1$ such that $\|f-g\|_{L^1} = \int_{\mathbb{R}} |f-g|\,dx < \varepsilon$ for each $\varepsilon > 0$.

Hence: for $\varepsilon > 0$, $\exists$ continuous g s.t.

(1) $\varepsilon > \int_{\mathbb{R}} |f-g| > \int_{k/m}^{k+1/m} |f-g| > \int_{k/m}^{k+1/m} |f| - \int_{k/m}^{k+1/m} |g|$, and

(2) $\varepsilon > \int_{\mathbb{R}} |g-f| > \int_{k/m}^{k+1/m} |g-f| > \left| \int_{k/m}^{k+1/m} g-f \right| = \left| \int_{k/m}^{k+1/m} g - \int_{k/m}^{k+1/m} f \right|$

$$> \left| \int_{k/m}^{k+1/m} g \right| - \left| \int_{k/m}^{k+1/m} f \right|$$

Now, the left hand side of the desired equality is a sum over intervals $\left[\frac{k}{m}, \frac{k+1}{m}\right]$ of length $\frac{1}{m}$.
Since g is continuous, for large enough m we have that $g \geq 0$ or $g \leq 0$ on each $\left[\frac{k}{m}, \frac{k+1}{m}\right]$.
$\rightarrow$ Hence for $m \to \infty$, $\int_{k/m}^{k+1/m} |g| = \left| \int_{k/m}^{k+1/m} g \right|$

See that (1) $\int_{k/m}^{k+1/m} |f| - \int_{k/m}^{k+1/m} |g| < \varepsilon$ for large m

(2) $\int_{k/m}^{k+1/m} |g| - \left| \int_{k/m}^{k+1/m} f \right| < \varepsilon$

$\rightarrow \int_{k/m}^{k+1/m} |f| - \left| \int_{k/m}^{k+1/m} f \right| < 2\varepsilon$ for large m

Hence $\int_{k/m}^{k+1/m} |f| = \left| \int_{k/m}^{k+1/m} f \right|$ for large m, hence

$$\lim_{m \to \infty} \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{k+1/m} f \right| = \lim_{m \to \infty} \sum_{k=-m^2}^{m^2} \int_{k/m}^{k+1/m} |f| = \int_{-\infty}^{\infty} |f| = \|f\|_{L^1}$$

✓

Incomplete

4(ii)

REAL ANALYSIS GRADUATE EXAM
Fall 2008

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let μ, ν be finite Borel measures on $\mathbb{R}^2$ such that $\mu(B) = \nu(B)$ for every open triangular region B in the plane. Show that $\mu(E) = \nu(E)$ for all Borel sets E. [Note added later: This is a modified version of the problem actually asked, which was inappropriately difficult, with balls in place of triangles.]

(2) Show that

$$\lim_{n\to\infty} \int_0^\infty \frac{1 + nx^2 + n^2x^4}{(1+x^2)^n}\, dx$$

$\leftarrow$ $= 0$ by DCT

exists, and determine its value.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function and let m be Lebesgue measure. Suppose there exist $M > 0$ and $c \in (0,1)$ such that

$$m(\{x : |f(x)| \geq t\}) \leq \frac{M}{t^c} \quad \text{for all } t > 0.$$

Show that f is Lebesgue integrable.

(4) Let $T_0^1(g) = \sup_{0=x_0<x_1<\cdots<x_n=1} \sum_{i=1}^n |g(x_i) - g(x_{i-1})|$ denote the total variation of a function $g : [0,1] \to \mathbb{R}$. Suppose that f_n, f are real-valued with $f_n(x) \to f(x)$ for all $x \in [0,1]$.

(i) Show that $T_0^1(f) \leq \liminf_{n\to\infty} T_0^1(f_n)$.

(ii) If we also assume each f_n is absolutely continuous and $T_0^1(f_n) \leq 1$ for all n, is it necessarily true that $T_0^1(f) = \lim_{n\to\infty} T_0^1(f_n)$? Justify.

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

$$\sum_{i=0}^{\infty} x^n = \frac{1}{1-x}$$

Fall '08:

(1.) μ, ν finite Borel on $\mathbb{R}^2$ such that $\mu(T) = \nu(T)$ for every open triangular region $T \subseteq \mathbb{R}^2$?
Show $\mu(E) = \nu(E)$ for all Borel sets $E \in \mathcal{B}_{\mathbb{R}^2}$

- Recall that $\mathcal{B}_{\mathbb{R}^2} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ (Corollary 1.6), and by Prop 1.4, since $\mathcal{B}_{\mathbb{R}}$ is generated by open intervals $\{(a,b)\} = \mathcal{E}$, we have that $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ is generated by $\mathcal{F} = \{(a,b) \times (c,d) : (a,b), (c,d) \in \mathcal{E}\}$
Hence $\mathcal{B}_{\mathbb{R}^2}$ is generated by open rectangles $(a,b) \times (c,d)$.

- Now, note that any open rectangle R can be subdivided into two disjoint open triangles $T_1 \cup T_2 = R$.
Therefore:

$$\mu(R) = \mu(T_1 \cup T_2) = \mu(T_1) + \mu(T_2) = \nu(T_1) + \nu(T_2)$$

$= \nu(T_1 \cup T_2) = \nu(R)$, hence μ and ν agree on a generating set for $\mathcal{B}_{\mathbb{R}^2}$, hence they must agree on all of $\mathcal{B}_{\mathbb{R}^2}$.

(2.) Show $\lim_{n\to\infty} \int_0^\infty \frac{1+nx^2+n^2x^4}{(1+x^2)^n}\,dx$ exists, and compute

Recall that $(1+x^2)^n = \sum_{k=0}^{n} \binom{n}{k} x^{2k}$

Then we have that:

$$\frac{1+nx^2+n^2x^4}{(1+x^2)^n} = \frac{1+nx^2+n^2x^4}{1+nx+\binom{n}{2}x^4+\binom{n}{3}x^6+\cdots} \leq \frac{1+nx^2+n^2x^4}{\binom{n}{3}x^6}$$

$$= \frac{6(1+nx^2+n^2x^4)}{n(n-1)(n-2)x^6} = \frac{6n^2(\frac{1}{n^2}+\frac{1}{n}x^2+x^4)}{n(n-1)(n-2)x^6}$$

$$\leq \frac{6n(1+x^2+x^4)}{(n-1)(n-2)x^6}$$

Now, $\frac{d}{dn}\left[\frac{n}{(n-1)(n-2)}\right] = \frac{2-n^2}{(n-1)^2(n-2)^2} < 0$ for all $n \geq 3$,

hence $\frac{n}{(n-1)(n-2)} \leq \frac{3}{(3-1)(3-2)} = \frac{3}{2}$

so $\frac{6n(1+x^2+x^4)}{(n-1)(n-2)x^6} \leq \frac{9(1+x^2+x^4)}{x^6}$ for $n \geq 3$

and see that $\frac{9(1+x^2+x^4)}{x^6} \in L^1(m)$, hence we may apply DCT:

$$\lim_{n\to\infty} \int_0^\infty \frac{1+nx^2+n^2x^4}{(1+x^2)^n}\,dx = \int_0^\infty \lim_{n\to\infty} \frac{1+nx^2+n^2x^4}{(1+x^2)^n}\,dx$$

Now, note that $1+x^2>1$ for $x\in(0,\infty)$, hence $\lim_{n\to\infty}(1+x^2)^n = \infty$,

and then:

$$\lim_{n\to\infty} \frac{1}{(1+x^2)^n} = 0,$$

$$\lim_{n\to\infty} \frac{nx^2}{(1+x^2)^n} = \lim_{n\to\infty} \frac{x^2}{\ln(1+x^2)e^{n\ln(1+x^2)}} = 0$$

$$\lim_{n\to\infty} \frac{n^2x^4}{(1+x^2)^n} = \lim_{n\to\infty} \frac{2nx^4}{e^{n\ln(1+x^2)}\ln(1+x^2)} = \lim_{n\to\infty} \frac{2x^4}{e^{n\ln(1+x^2)}\ln(1+x^2)^2} = 0$$

by L'Hôpital's rule

Hence $\int_0^\infty \lim_{n\to\infty} \frac{1+nx^2+n^2x^4}{(1+x^2)^n}\,dx = \int_0^\infty 0 = 0$

(3.) $f: \mathbb{R} \to \mathbb{R}$ bounded, measurable, and m Lebesgue, and $\exists M > 0$, $c \in (0,1)$ s.t. $m\{|f(x)| \geq t\} \leq \frac{M}{t^c}$ for all $t > 0$.

Show f is Lebesgue integrable:

f bdd, hence $|f(x)| \leq N$ for some N.

Now, $m\{|f| \geq t\} = \int_{\mathbb{R}} \chi_{\{|f| \geq t\}} \, dm$, and we can see that:

$$\int_0^N m\{|f| \geq t\} \, dt \leq \int_0^N \frac{M}{t^c} \, dt < \infty \text{, so now:}$$

$$\begin{aligned}
\int_0^N m\{|f| \geq t\} \, dt &= \int_0^N \left(\int_{\mathbb{R}} \chi_{\{|f| \geq t\}}(y,t) \, dy \right) dt \quad \text{Tonelli} \\
&= \int_{\mathbb{R}} \int_0^N \chi_{\{|f| \geq t\}}(y,t) \, dt \, dy \quad \text{since } \chi_{\{|f| \geq t\}} = 1 \text{ for } 0 \leq t \leq |f(y)| \\
&= \int_{\mathbb{R}} \int_0^{|f(y)|} dt \, dy \\
&= \int_{\mathbb{R}} |f(y)| \, dy
\end{aligned}$$

Hence $\int_{\mathbb{R}} |f| \, dm < \infty$, hence Lebesgue integrable

④ $T_0^1(g) = \sup\{ \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| : 0 = x_0 < x_1 < \cdots < x_n = 1 \}$ partitions

Suppose f_n, f real valued s.t. $f_n(x) \to f(x)$ for all $x \in [0,1]$

(i) Show $T_0^1(f) \le \liminf_{n\to\infty} T_0^1(f_n)$

(ii) If f_n abs. cont. and $T_0^1(f_n) \le 1$ $\forall n$, then is $T_0^1(f) = \lim_{n\to\infty} T_0^1(f_n)$ → NO!

(i) $\boxed{\text{Proof \#1}}$ $f_n(x) \to f(x)$, i.e. for $x \in \mathbb{R}$, $\exists N_x$ s.t. $n \ge N \Rightarrow |f_n(x) - f(x)| < \varepsilon$

Now: $T_0^1(f) = \sup_{P=\{x_i\}} \{ \sum |f(x_i) - f(x_{i-1})| \}$

$= \sup_P \{ \sum |f(x_i) - f(x_{i-1}) + f_n(x_i) - f_n(x_i) + f_n(x_{i-1}) - f_n(x_{i-1})| \}$

$\le \sup_P \{ \sum |f(x_i) - f_n(x_i)| + |f(x_{i-1}) - f_n(x_{i-1})| + |f_n(x_i) - f_n(x_{i-1})| \}$

$\le \sup_P \{ \sum |f(x_i) - f_n(x_i)| \} + \sup_P \{ \sum |f(x_{i-1}) - f_n(x_{i-1})| \} + \sup_P \{ \sum |f_n(x_i) - f_n(x_{i-1})| \}$

[For x_i, let $n \ge N^{\varepsilon}_{x_i} \Rightarrow |f(x_i) - f_n(x_i)| \le \frac{\varepsilon}{2^i}$
Then let $n \ge \max_i \{N_{x_i}\}$ and we have!]

$\to \le \sup_{k \ge 0} \{ \sum_{i=0}^{k} \frac{\varepsilon}{2^i} \} + \sup_{k\ge 0} \{ \sum_{i=0}^{k} \frac{\varepsilon}{2^i} \} + T_0^1(f_n)$

$\le \varepsilon + \varepsilon + T_0^1(f_n) = 2\varepsilon + T_0^1(f_n)$

Hence $T_0^1(f) \le 2\varepsilon + \liminf T_0^1(f_n)$, hence, letting $\varepsilon \to 0$, we

get $T_0^1(f) \le \liminf T_0^1(f_n)$!

$\boxed{\text{Proof \#2}}$ Fix part $P = \{x_0, \ldots, x_n\}$. Then

$\sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| = \lim_{n\to\infty} \sum_{i=1}^{m} |f_n(x_i) - f_n(x_{i-1})| = \liminf_{n\to\infty} \sum_{i=1}^{m} |f_n(x_i) - f_n(x_{i-1})|$

$\le \liminf_{n\to\infty} T_0^1(f_n)$.

⇒ hence, taking sup over P, get

$T_0^1(f) \le \liminf_{n\to\infty} T_0^1(f_n)$

Incomplete back
1(b), 2(c)

REAL ANALYSIS GRADUATE EXAM
Spring 2009

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1)(a) Let $(X, \mathcal{B}, \mu)$ be a measure space, with μ finite, and let $\mathcal{A} \subset \mathcal{B}$ be an algebra. A set $E \in \mathcal{B}$ is called *approximable from inside by* $\mathcal{A}$ if for every $\epsilon > 0$ there exists $A \in \mathcal{A}$ with $A \subset E, \mu(E \backslash A) < \epsilon$. Show that $\mathcal{C} = \{E \in \mathcal{B} : E \text{ is approximable from inside by } \mathcal{A}\}$ is closed under countable unions.

(b) Find an example which shows $\mathcal{C}$ need not be closed under complements. HINT: Consider the rationals and irrationals in an interval.

(2) Let f, g be absolutely continuous on $[a, b]$.

(a) Show that fg is absolutely continuous.

(b) Show that the integration by parts formula is valid:

$$\int_{[a,b]} fg' \, dx = f(b)g(b) - f(a)g(a) - \int_{[a,b]} f'g \, dx.$$

(c) Show by example that the integration by parts formula need not be valid if we only assume f, g differentiable a.e. (that is, we do not assume they are absolutely continuous.)

(3) Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable and $f = 0$ outside $[-1, 1]$. Define $f_n(x) = f(x + \frac{1}{n})$. Must $f_n \to f$ in measure? Justify your answer. HINT: What about convergence in L^1? Also, first consider a special subclass of the specified functions f.

(4) Let $\mu(X) < \infty$, and suppose $f \geq 0$ is measurable on X. Prove: f is μ-integrable iff

$$\text{need: } \int f \, d\mu < \infty \Rightarrow \sum_{n=0}^{\infty} 2^n \mu(\{x : f(x) \geq 2^n\}) < \infty.$$

Spring 09:

(1.) $(X, \mathcal{B}, \mu)$ meas. space, μ finite, $\mathcal{A} \subseteq \mathcal{B}$ algebra.

[$E \in \mathcal{B}$ is approx from inside by $\mathcal{A}$ if for $\varepsilon > 0$, $\exists A \in \mathcal{A}$ s.t. $\mu(E \setminus A) < \varepsilon$]

(a) <u>Show</u>: $\mathcal{C} = \{E \in \mathcal{B} : E \text{ approx inside by } \mathcal{A}\}$ is closed under countable unions

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Let $(E_i) \subseteq \mathcal{C}$ and let $A_i \in \mathcal{A}$ be such that $\mu(E_i \setminus A_i) < \frac{\varepsilon}{2^i}$

<u>Finite unions</u>: $\mu\left(\bigcup_{i=1}^k E_i \setminus \bigcup_{i=1}^k A_i\right) \leq \mu\left(\bigcup_{i=1}^k E_i \setminus A_i\right)$
$\leq \sum_{i=1}^k \mu(E_i \setminus A_i) = \sum_{i=1}^k \frac{\varepsilon}{2^i} < \varepsilon$; since $\mathcal{A}$ is an algebra, $\bigcup_{i=1}^k A_i \in \mathcal{A}$, hence $\bigcup_{i=1}^k E_i \in \mathcal{C}$.

<u>Countable unions</u>: Let $B_n = A_n \setminus \left(\bigcup_{j=1}^{n-1} A_j\right)$; hence $\bigcup B_i = \bigcup A_i$

$$\bigcup_{i=1}^\infty E_i \setminus \bigcup_{j=1}^\infty A_j \subseteq \bigcup_{i=1}^\infty E_i \setminus \bigcup_{j=1}^k A_j$$
$$= \left(\bigcup_{i=1}^k E_i \setminus \bigcup_{j=1}^k A_j\right) \cup \left(\bigcup_{i=k+1}^\infty E_i \setminus \bigcup_{j=1}^k A_j\right)$$
$$\subseteq \left(\bigcup_{i=1}^k E_i \setminus A_i\right) \cup \left(\bigcup_{i=k+1}^\infty E_i \setminus \bigcup_{j=1}^k A_j\right)$$

and $\left(\bigcup_{i=k+1}^\infty E_i \setminus \bigcup_{j=1}^k A_j\right) = \left[\bigcup_{i=k+1}^\infty E_i \setminus \bigcup_{j=1}^\infty A_j\right] \cup \left[\left(\bigcup_{i=k+1}^\infty E_i \setminus \bigcup_{i=1}^k A_i\right) \cap \left(\bigcup_{i=k+1}^\infty A_i\right)\right]$

since $A_i \subseteq E_i$ $\Rightarrow$ $= \left[\bigcup_{i=k+1}^\infty \left(E_i \setminus \bigcup_{j=1}^\infty A_j\right)\right] \cup \left[\left(\bigcup_{i=k+1}^\infty A_i\right) \setminus \left(\bigcup_{i=1}^k A_i\right)\right]$
$$\subseteq \left[\bigcup_{i=k+1}^\infty (E_i \setminus A_i)\right] \cup \left[\bigcup_{i=k+1}^\infty B_i\right]$$

Now, the measure is finite, hence $\infty > \mu\left(\bigcup_{i=1}^\infty A_i\right) = \mu\left(\bigcup_{i=1}^\infty B_i\right) = \sum_{i=1}^\infty \mu(B_i)$ since B_i's are disjoint, hence $\mu(B_i) \to 0$ as $i \to \infty$

So now:
$\mu\left(\bigcup E_i \setminus \bigcup A_j\right) \leq \mu\left(\bigcup_{i=1}^k E_i \setminus A_i\right) + \mu\left(\bigcup_{i=k+1}^\infty E_i \setminus A_i\right) + \sum_{i=k+1}^\infty \mu(B_i) <$

So now:

$$\mu\left(\bigcup_{i=1}^{\infty} E_i \setminus \bigcup_{i=1}^{k} A_i\right) \leq \mu\left(\bigcup_{i=1}^{k} E_i \setminus A_i\right) + \mu\left(\bigcup_{k+1}^{\infty} E_i \setminus \bigcup_{i=1}^{k} A_i\right)$$

$$\leq \mu\,\varepsilon + \mu\left(\bigcup_{i=k+1}^{\infty} E_i \setminus A_i\right) + \mu\left(\bigcup_{i=k+1}^{\infty} B_k\right)$$

$$\leq \varepsilon + \sum_{i=k+1}^{\infty} \mu(E_i \setminus A_i) + \sum_{i=k+1}^{\infty} B_k$$

$$\leq \varepsilon + \sum_{i=k+1}^{\infty} \frac{\varepsilon}{2^i} + \sum_{i=k+1}^{\infty} B_k$$

$$\leq 2\varepsilon + \sum_{i=k+1}^{\infty} B_k \longrightarrow 2\varepsilon + 0$$

as $k \to \infty$.

Hence for k large enough,

$\mu\left(\bigcup_{i=1}^{\infty} E_i \setminus \bigcup_{i=1}^{k} A_i\right) < \varepsilon$, hence $\bigcup_{i=1}^{\infty} E_i \in \mathcal{C}$

since $\bigcup_{i=1}^{k} A_i \in \mathcal{A}$ for all k.

(b) Find an example where $\mathcal{C}$ not closed under complements:

(2.) f, g abs cont on $[a,b]$.

(a) Show fg is abs. cont.

(b) Show $\int_a^b fg'\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'g\,dx$

(c) Show (b) invalid if f, g only diff. a.e. (not abs. cont.)

(a) f, g abs cont, hence for $\varepsilon > 0$, $\exists\, \delta_1, \delta_2$ s.t.

$\sum_{i=1}^n |y_i - x_i| < \min(\delta_1, \delta_2) := \delta \Rightarrow \sum_{i=1}^n |f(y_i) - f(x_i)| < \varepsilon$ and $\sum_{i=1}^n |g(y_i) - g(x_i)| < \varepsilon$.

Now, since f, g are abs. cont on closed interval $[a,b]$ they are also bdd, ie. $\exists M$ s.t. $f(x) \le M$, $g(x) \le M$ for all x.

Now suppose $\sum |y_i - x_i| < \delta$ and then:

$$\sum_{i=1}^n |f(y_i)g(y_i) - f(x_i)g(x_i)|$$
$$= \sum_{i=1}^n |f(y_i)g(y_i) - f(y_i)g(x_i) + f(y_i)g(x_i) - f(x_i)g(x_i)|$$
$$\le \sum_{i=1}^n |f(y_i)g(y_i) - f(y_i)g(x_i)| + |f(y_i)g(x_i) - f(x_i)g(x_i)|$$
$$= \sum_{i=1}^n \left(|f(y_i)|\,|g(y_i) - g(x_i)| + |g(x_i)|\,|f(y_i) - f(x_i)|\right)$$
$$< \sum_{i=1}^n M|g(y_i) - g(x_i)| + M|f(y_i) - f(x_i)| = M\left(\sum_{i=1}^n |g(y_i) - g(x_i)| + \sum_{i=1}^n |f(y_i) - f(x_i)|\right)$$
$$= M(2\varepsilon)$$

Hence let $\delta' > 0$ be s.t. $\sum |y_i - x_i| < \delta' \Rightarrow \sum |f(y_i) - f(x_i)| < \frac{\varepsilon}{2M}$ and $\sum |g(y_i) - f(x_i)| < \frac{\varepsilon}{2M}$

$$\Rightarrow \sum |fg(y_i) - fg(x_i)| < 2M\left(\frac{\varepsilon}{2M}\right) = \varepsilon$$, hence fg abs. cont.

(b) Since fg is absolutely continuous, we have:

$$\int_a^b (fg)'\,dx = fg(b) - fg(a)$$
$$\Rightarrow \int_a^b (f'g + g'f)\,dx = f(b)g(b) - f(a)g(a)$$
$$\Rightarrow \int_a^b fg'\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'g\,dx$$

(3.) $f: \mathbb{R} \to \mathbb{R}$ int, $f = 0$ outside $[-1,1]$.
Define $f_n(x) = f(x + \frac{1}{n})$. Must $f_n \to f$ in measure?

Recall that convergence in $L^1 \Rightarrow$ convergence in measure.

- Continuous case: Suppose f is continuous, i.e. for $\varepsilon > 0$, $\exists N$ s.t. $|x-y| < \frac{1}{N} \Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{3}$. Now $n \geq N \Rightarrow \frac{1}{n} < \frac{1}{N}$, hence

$$|x - (x+\tfrac{1}{n})| = \tfrac{1}{n} < \tfrac{1}{N} \Rightarrow |f(x) - f(x+\tfrac{1}{n})| < \tfrac{\varepsilon}{3}$$

Then we have:

$$\|f-f_n\|_{L^1} = \int_{\mathbb{R}} |f-f_n|\,dx = \int_{\mathbb{R}} |f(x)-f(x+\tfrac{1}{n})|\,dx$$

$$\stackrel{*}{=} \int_{-1-\frac{1}{n}}^{1} |f(x)-f(x+\tfrac{1}{n})| < \int_{-1-\frac{1}{n}}^{1} \frac{\varepsilon}{3}\,dx = \frac{\varepsilon}{3}\left(1-(-1-\tfrac{1}{n})\right) = \frac{\varepsilon}{3}(2+\tfrac{1}{n}) \leq \varepsilon$$

for all $n \geq N$

$*$ since $f(x)=0$ for $x \notin [-1,1]$

Hence $n > N \Rightarrow \|f-f_n\|_{L^1} < \varepsilon$, i.e. $f_n \to f$ in L^1

- General case: Since the measure is Lebesgue, we have that continuous functions with bdd support are dense in L^1.
So for f integrable, $\exists$ cont. g s.t. $\int|g-f| < \frac{\varepsilon}{3}$

Then, for $n \geq N$:

$$\begin{aligned}\|f-f_n\|_{L^1} = \int|f-f_n|\,dx &= \int|f-g+g-g_n+g_n-f_n|\,dx \\ &\leq \int(|f-g|+|g-g_n|+|g_n-f_n|)\,dx \\ &= \int|f-g|\,dx + \int|g-g_n|\,dx + \int|g_n-f_n|\,dx \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \int|g_n-f_n|\,dx\end{aligned}$$

continuous case

Now recall that Lebesgue measure is translation invariant:

$$\int|g_n-f_n| = \int|g(x+\tfrac{1}{n})-f(x+\tfrac{1}{n})|\,dx = \int|g(x)-f(x)|\,d(x+\tfrac{1}{n}) = \int|g-f|$$

Hence:

$$\|f-f_n\|_{L^1} < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\int|g_n-f_n|\,dx = \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\int|g-f|\,dx < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon$$

Hence $\|f-f_n\|_{L^1}<\varepsilon \Rightarrow f_n \to f$ in $L^1 \Rightarrow f_n \to f$ in measure.

(4.) $\mu(X) < \infty$, $f \geq 0$ measurable

Show: f μ-integrable $\iff \sum_{n=0}^{\infty} 2^n \mu(\{f(x) \geq 2^n\}) < \infty$

$(\Leftarrow)$ Suppose $\sum_{n=1}^{\infty} 2^n \mu(\{f(x) \geq 2^n\}) < \infty$:

Let $A_n = \{2^n \leq f(x) < 2^{n+1}\}$, $A_{-1} = \{0 \leq f(x) < 1\}$

Since $f \geq 0$, $\bigcup_{i=-1}^{\infty} A_i = X$; then:

$$\int f\,d\mu = \int_{A_{-1}} f\,d\mu + \sum_{n=0}^{\infty} \int_{A_n} f\,d\mu \leq \int_{A_{-1}} d\mu + \sum_{n=0}^{\infty} \int_{A_n} 2^n d\mu$$

$$= \mu(A_{-1}) + \sum_{n=0}^{\infty} 2^n \mu(A_n) = \mu(A_{-1}) + \sum_{n=0}^{\infty} 2^n \mu(\{2^n \leq f(x) < 2^{n+1}\})$$

$$\leq \mu(A_{-1}) + \sum_{n=0}^{\infty} 2^n \mu(\{2^n \leq f(x)\}) < \infty$$

finite since $\mu(X) < \infty$

finite by hypothesis

$(\Rightarrow)$ Suppose $\int_X f\,d\mu \leq \infty$

Then we have $\infty > \int_X f\,d\mu = \int_{\bigcup A_n} f\,d\mu = \int_X \left(\sum_{n=-1}^{\infty} \chi_{A_n} f\right) d\mu$

$$\stackrel{(=)}{=} \sum_{n=-1}^{\infty} \int_{A_n} f\,d\mu \geq \sum_{n=-1}^{\infty} 2^n \mu(A_n) \Rightarrow \sum_{n=-1}^{\infty} 2^n \mu(A_n) < \infty$$

by Tonelli since all positive (or MCT)

Now consider the sets $B_n = \{2^n \leq f(x)\}$

Clearly $B_n = \bigcup_{j=n}^{\infty} A_j$

Now: $\sum_{n=0}^{\infty} 2^n \mu(\{f(x) \geq 2^n\}) = \sum_{n=0}^{\infty} 2^n \mu(B_n) = \sum_{n=0}^{\infty} 2^n \mu\left(\bigcup_{j=n}^{\infty} A_j\right)$

$$\stackrel{(=)}{=} \sum_{n=0}^{\infty} 2^n \left(\sum_{j=n}^{\infty} \mu(A_j)\right) = 2^0(A_0 + A_1 + \cdots) + 2^1(A_1 + A_2 + \cdots) + 2^2(A_2 + \cdots)$$

A_j disjoint

$$= A_0 + (1+2)A_1 + (1+2+2^2)A_2 + \cdots$$

$$= \sum_{j=0}^{\infty} \left(\sum_{n=0}^{j} 2^n \mu(A_j)\right)$$

Lemma: let $s = \sum_{n=0}^{j} 2^n \Rightarrow 2s = \sum_{n=0}^{j} 2^{n+1}$ and $s = 2s - s = \sum_{n=0}^{j} 2^{n+1} - \sum_{n=0}^{j} 2^n$

$= (2 + 2^2 + \cdots 2^{j+1}) - (1 + 2 + \cdots + 2^j) = 2^{j+1} - 1$

So now $\sum_{j=0}^{\infty} \left(\sum_{n=0}^{j} 2^n \mu(A_j)\right) = \sum_{j=0}^{\infty} (2^{j+1} - 1)\mu(A_j) = \sum_{j=0}^{\infty} 2^{j+1} \mu(A_j) - \sum_{j=0}^{\infty} \mu(A_j)$

Recall that $\int_X f\,d\mu < \infty \Rightarrow \sum_{j=0}^{\infty} 2^{j+1} \mu(A_j) < \infty$ and $\mu(X) < \infty$,

$$\Rightarrow \sum_{j=0}^{\infty} \mu(A_j) < \infty \Rightarrow \sum_{n=0}^{\infty} 2^n \mu(\{f \geq 2^n\}) = \sum_{j=0}^{\infty} 2^{j+1} \mu(A_j) - \sum_{j=0}^{\infty} \mu(A_j) < \infty$$

A_j disjoint

check

REAL ANALYSIS GRADUATE EXAM

Fall ~~Spring~~ 2009

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let f be a bounded function on $\mathbb{R}^n$ and for $\epsilon > 0$ let $M_\epsilon(x) = \sup_{y:|y-x|<\epsilon} f(y)$.
(a) Show that $M(x) = \lim_{\epsilon\to 0} M_\epsilon(x)$ exists for all x.
(b) Show that M is upper semicontinuous, that is, $\limsup_{y\to x} M(y) \le M(x)$.

(2) Let m be Lebesgue measure on $[0,1]$, suppose $f \in L^1(m)$ and let $F(x) = \int_0^x f(t)\,dt$. Suppose φ is a Lipshitz function, that is, for some M, $|\varphi(x) - \varphi(y)| \le M|x-y|$ for all x, y. Show that there exists $g \in L^1(m)$ such that $\varphi(F(x)) = \int_0^x g(t)\,dt$.

(3) Let χ_E denote the indicator function of a set E. Suppose $E \subset \mathbb{R}$ has finite Lebesgue measure and define

$$f(x) = \int_{\mathbb{R}} \chi_E(y)\chi_E(y-x)\,dy.$$

Show that f is continuous.

(4) Let m be Lebesgue measure on $\mathbb{R}$ and let $f_n, f \in L^1(m)$. Suppose there is a constant C such that $||f_n - f||_1 \le \frac{C}{n^2}$ for all $n \ge 1$. Show that $f_n \to f$ a.e. HINT: Consider the sets

$$\{x : |f_n(x) - f(x)| > \epsilon \text{ for some } n \ge N\}.$$

REAL ANALYSIS QUALIFYING EXAM

Fall [illegible]

Answer all four questions. Partial credits will be awarded, but it is important that you not only solve a problem, you should state clearly what it is you have done and what you have left out. Irrelevant "dumped" formulas, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let f be a bounded function on $\mathbb{R}^n$ and for $\varepsilon > 0$ let $M_\varepsilon(x) = \sup_{|y-x|<\varepsilon} f(y)$.

(a) Show that $M(x) = \lim_{\varepsilon \to 0} M_\varepsilon(x)$ exists for all x.

(b) Show that M is upper semicontinuous, that is, $\limsup_{y \to x} M(y) \le M(x)$.

2. Let m be Lebesgue measure on $[0,1]$, suppose $f \in L^1(m)$ and let $F(x) = \int_0^x f(t)\,dt$. Suppose F is a Lipschitz function, that is, for some M, $|F(x) - F(y)| \le M|x-y|$ for all x, y. Show that there exists $g \in L^\infty(m)$ such that $F(x) = \int_0^x g(t)\,dt$.

3. Let χ_E denote the indicator function of a set E. Suppose $E \subset \mathbb{R}$ has finite Lebesgue measure and define

$$\phi(t) = \int_{\mathbb{R}} \chi_E(y)\,\chi_E(y+t)\,dy.$$

Show that ϕ is continuous.

4. Let m be Lebesgue measure on $[0,1]$ and let $f_n, f \in L^2(m)$. Suppose there is a constant C such that $\|f_n\|_2 \le C$ for all n. [illegible] Consider the case

$$[illegible]$$

Fall 09

(1) f bounded on $\mathbb{R}^n$, $M_\varepsilon(x) = \sup_{|y-x|<\varepsilon} f(y)$

(a) Show $M(x) := \lim_{\varepsilon\to 0} M_\varepsilon(x)$ exists for all x

(b) Show $M(x)$ is u.s.c., namely $\limsup_{y\to x} M(y) \le M(x)$

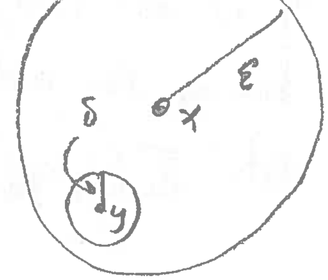
(a) See that, for $\varepsilon_1 < \varepsilon_2$,

$$M_{\varepsilon_1}(x) = \sup_{|y-x|<\varepsilon_1} f(y) = \sup_{y\in B_{\varepsilon_1}(x)} f(y) \le \sup_{y\in B_{\varepsilon_2}(x)} f(y) = M_{\varepsilon_2}(x)$$

since $B_{\varepsilon_1}(x) \subseteq B_{\varepsilon_2}(x)$; hence $M_\varepsilon(x)$ is nonincreasing as $\varepsilon\to 0$, but $M_\varepsilon(x) = \sup_{|y-x|<\varepsilon} f(y) \ge f(x)$, hence $M_\varepsilon(x)$ is bdd below since $f(x)$ is bdd. Thus $\lim_{\varepsilon\to 0} M_\varepsilon(x)$ is convergent

(b) See that $M(x) = \lim_{\varepsilon\to 0} \sup_{|y-x|<\varepsilon} f(y) = \limsup_{y\to x} f(y)$

Fix $\varepsilon > 0$, and consider $B_\varepsilon(x)$: $\longrightarrow$



Clearly, $\sup_{y\in B_\varepsilon(x)} f(y) \ge \sup_{t\in B_\delta(y)} f(t)$ since $B_\delta(y) \subseteq B_\varepsilon(x)$,

hence $\sup_{y\in B_\varepsilon(x)} f(y) \ge \sup_{t\in B_\delta(y)} f(t) \ge \lim_{\delta\to 0}\left(\sup_{t\in B_\delta(y)} f(t)\right)$

$$\Longrightarrow \sup_{y\in B_\varepsilon(x)} f(y) \ge \sup_{y\in B_\varepsilon(x)}\left(\lim_{\delta\to 0}\left(\sup_{t\in B_\delta(y)} f(t)\right)\right)$$

Now let $\varepsilon\to 0$, hence:

$$\lim_{\varepsilon\to 0}\left(\sup_{y\in B_\varepsilon(x)} f(y)\right) \ge \lim_{\varepsilon\to 0}\left(\sup_{y\in B_\varepsilon(x)}\left(\lim_{\delta\to 0}\left(\sup_{t\in B_\delta(y)} f(t)\right)\right)\right)$$

$$\limsup_{y\to x} f(y) \qquad\qquad \limsup_{y\to x}\left(\lim_{\delta\to 0}\left(\sup_{t\in B_\delta(y)} f(t)\right)\right) = \limsup_{y\to x}\left(\limsup_{t\to y} f(t)\right)$$

$$\Rightarrow \limsup_{y\to x} f(y) \ge \limsup_{y\to x}\left(\limsup_{t\to y} f(t)\right)$$

$$\Rightarrow \underline{M(x) \ge \limsup_{y\to x}(M(y))}$$

(2.) m Leb. meas on $[0,1]$; $f \in L^1(m)$, $F(x) = \int_0^x f(t)\,dt$.

Suppose φ is Lipschitz

Show : $\exists g \in L^1(m)$ s.t. $\varphi(F(x)) = \int_0^x g(t)\,dt$

- Claim : φ Lipschitz $\Rightarrow$ φ absolutely continuous :

See that $\sum_{k=1}^{n} |\varphi(x_k) - \varphi(y_k)| \le M \sum_{k=1}^{n} |x_k - y_k|$, hence let $\delta = \frac{\varepsilon}{M}$.

Hence $\sum_{k=1}^{n} |x_k - y_k| < \frac{\varepsilon}{M} \Rightarrow M \sum_{k=1}^{\infty} |x_k - y_k| < \varepsilon \Rightarrow \sum_{k=1}^{n} |\varphi(x_k) - \varphi(y_k)|$

$\le M \sum_{k=1}^{m} |x_k - y_k| < \varepsilon$,

here φ is absolutely continuous

- Claim : $\varphi \circ F$ is absolutely continuous

By construction, F is absolutely cont. Choose $\varepsilon > 0$;

φ is abs. cont, hence $\exists \delta > 0$ s.t. $\sum_{k=1}^{n} |x_k - y_k| < \delta \Rightarrow \sum_{k=1}^{n} |\varphi(x_k) - \varphi(y_k)| < \varepsilon$,

but on the other hand, F is abs. cont, hence given $\delta > 0$, $\exists \delta' > 0$

s.t. $\sum (x_k - y_k) < \delta' \Rightarrow \sum |F(x_k) - F(y_k)| < \delta$

$\Rightarrow \sum | \varphi(F(x_k)) - \varphi(F(y_k)) | < \varepsilon$

Hence $\varphi \circ F$ is absolutely continuous, i.e. $\exists g \in L^1$ s.t.

$\varphi \circ F(x) = \int_0^x g(t)\,dt$.

Recall : FTC Lebesgue :

TFAE ① F abs cont.

② $\exists g \in L^1$ s.t. $F(x) - F(a) = \int_a^x g(t)\,dt$

3. $E \subseteq \mathbb{R}$ has finite Lebesgue measure; show that
$$f(x) = \int_{\mathbb{R}} \chi_E(y)\chi_E(y-x)\,dy$$

Since we have Lebesgue measure, we know that $m(E) = m(E+x) = m(E+p)$, hence
$$\int \chi_E(y)\,dy < \infty, \quad \int \chi_{E+x}(y)\,dy < \infty, \quad \int \chi_{E+p}(y)\,dy < \infty,$$

Since χ_E is integrable, we have, for each $\varepsilon > 0$, an int. step function ϕ s.t. $\int |\chi_E(y) - \phi(y)| < \varepsilon$.
Since m is Lebesgue, $\phi(y) = \sum_{i=1}^{k} a_i \chi_{E_i}$ where E_i are finite unions of open intervals. Since $\chi_E = 0$ or 1, $a_i = 1$ $\forall i$.

Now, let $\varepsilon > 0$ and then:
$$|f(x) - f(p)| = \left| \int_{\mathbb{R}} \chi_E(y)\chi_E(y-x)\,dy - \int_{\mathbb{R}} \chi_E(y)\chi_E(y+p)\,dy \right|$$
$$\leq \int_{\mathbb{R}} |\chi_E(y)|\,|\chi_E(y-x) - \chi_E(y+p)|\,dy$$
$$\leq \mu(E) \int_{\mathbb{R}} |\chi_{E+x}(y) - \chi_{E+y}(y)|\,dy. \quad (*)$$

Now, $\int |\chi_E(y) - \phi(y)| < \varepsilon$, hence $\int |\chi_E(y-x) - \phi(y-x)| < \varepsilon$ and $\int |\chi_E(y-p) - \phi(y-p)| < \varepsilon$ by substitution (and integrability).
Let $\phi_p = \phi(y-p)$ and $\phi_x = \phi(y-x)$; now we have:

$$(*) = \mu(E) \int_{\mathbb{R}} |\chi_{E+x} - \phi_x + \phi_x - \phi_p + \phi_p - \chi_{E-p}|\,dy$$
$$\leq \mu(E) \int_{\mathbb{R}} (|\chi_{E+x} - \phi_x| + |\chi_{E+p} - \phi_p| + |\phi_x - \phi_p|)\,dy$$
$$< \mu(E)\left(\varepsilon + \varepsilon + \int_{\mathbb{R}} |\phi_x - \phi_p|\,dy\right)$$

Now, recall that $\phi(y) = \sum_{i=1}^{k} \chi_{E_i}$ where $E_i = \bigcup_{j=1}^{n} (a_j, b_j)$

(we'll use the same n and let some intervals be empty)

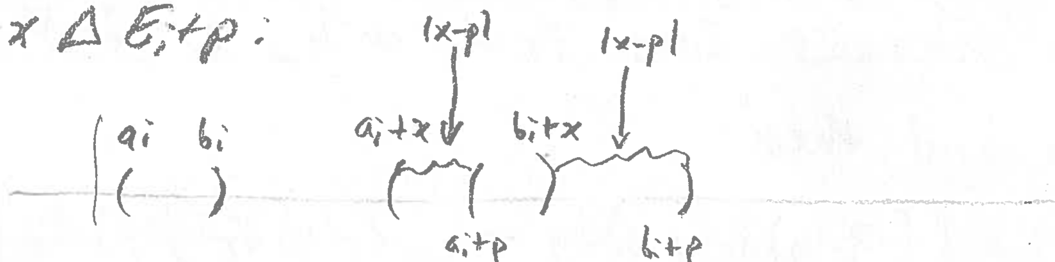
Then: $\int |\phi_x - \phi_p| \, dy = \int |\phi(y-x) - \phi(y-p)| \, dy$

$$= \int \left| \sum_{i=1}^{k} \chi_{E_i}(y-x) - \sum_{i=1}^{k} \chi_{E_i}(y-p) \right| dy$$

$$= \int \left| \sum_{i=1}^{k} \chi_{E_i+x}(y) - \sum_{i=1}^{k} \chi_{E_i+p}(y) \right| dy$$

$$= \int \left| \sum_{i=1}^{k} \chi_{E_i+x \,\Delta\, E_i+p}(y) \right| dy = \sum_{i=1}^{k} \int \chi_{E_i+x \,\Delta\, E_i+p}(y) \, dy$$

Recall that E_i is a union of intervals, hence consider $E_i + x \,\Delta\, E_i + p$:



$$m\left[\left(\bigcup_{j=1}^{n}(a_j,b_j)+x\right)\Delta\left(\bigcup_{j=1}^{n}(a_j,b_j)+p\right)\right] = m\left[\bigcup_{j=1}^{n}(a_j+x, b_j+x)\,\Delta\,\bigcup_{j=1}^{n}(a_j+p, b_j+p)\right]$$

$$\leq 2n|x-p|$$

Hence $m(E_i + x \,\Delta\, E_i + p) \leq 2n|x-p|$,

hence $\sum_{i=1}^{k} \int \chi_{E_i+x\,\Delta\,E_i+p}(y) \, dy = \sum_{i=1}^{k} \mu(E_i+x \,\Delta\, E_i+p) \leq 2kn|x-p|$

Hence:

$$|f(x)-f(p)| < \mu(E)\left(\varepsilon + \varepsilon + \int_{\mathbb{R}} |\phi_x - \phi_p| \, dy\right)$$

$$\leq \mu(E)\left(\varepsilon + \varepsilon + 2kn|x-p|\right)$$

Hence if $|x-p| < \frac{1}{kn}\left(\frac{\varepsilon}{2\mu(E)} - \varepsilon\right)$, then $|f(x)-f(p)| < \varepsilon$,

hence f is continuous.

(4.) m Lebesgue measure on $\mathbb{R}$, $f_n, f \in L^1(m)$, $\|f_n - f\|_{L^1} \le \frac{C}{n^2}$ for all $n \ge 1$. Show $f_n \to f$ a.e.

Recall: $f_n(x) \to f(x)$ if for $\varepsilon > 0$, $\exists N > 0$ s.t. $n \ge N \Rightarrow |f_n(x) - f(x)| < \varepsilon$,
hence $f_n(x) \not\to f(x)$ if $\exists \varepsilon > 0$ s.t. $\forall N > 0$, $\exists n \ge N$ s.t. $|f_n(x) - f(x)| > \varepsilon$

Hence we'll show that:

$A = \{x : \exists \varepsilon > 0$ s.t. for all $N > 0$, $\exists n \ge N \Rightarrow |f_n(x) - f(x)| > \varepsilon\}$ is meas 0.

Consider the sets $A_{\varepsilon, N} = \{x : |f_n(x) - f(x)| \ge \varepsilon$ for some $n \ge N\}$

Then $A = \bigcup_{\varepsilon > 0} \{x :$ for all $N > 0$, $|f_n(x) - f(x)| > \varepsilon$ for some $n \ge N\}$

$$= \bigcup_{\varepsilon > 0} \bigcap_{N=1}^{\infty} \{x : |f_n(x) - f(x)| > \varepsilon \text{ for some } n \ge N\}$$

$$= \bigcup_{\varepsilon > 0} \bigcap_{N=1}^{\infty} A_{\varepsilon, N} \quad ; \text{ let } \bigcap_{N=1}^{\infty} A_{\varepsilon, N} = A_\varepsilon$$

→ The A_ε are an increasing sequence as $\varepsilon \to 0$:

$$\varepsilon_1 > \varepsilon_2 \Rightarrow A_{\varepsilon_1} = \{x \in \mathbb{R} : |f_n(x) - f(x)| > \varepsilon_1 > \varepsilon_2$$
$$\subseteq \{x \in \mathbb{R} : |f_n(x) - f(x)| > \varepsilon_2\} = A_{\varepsilon_2}$$

Hence apply continuity from below:

$$m\left(\bigcup_{\varepsilon > 0} A_\varepsilon\right) = \lim_{\varepsilon \to 0} m(A_\varepsilon) = \lim_{\varepsilon \to 0} m\left(\bigcap_{N=1}^{\infty} A_{\varepsilon, N}\right)$$

→ The $A_{\varepsilon N}$ are a decreasing sequence as $N \to \infty$

$$N_1 > N_2 \Rightarrow A_{\varepsilon, N_1} = \{x \in \mathbb{R} : |f_n(x) - f(x)| > \varepsilon \text{ some } n \ge N_1 > N_2\}$$
$$\supseteq \{x \in \mathbb{R} : |f_n(x) - f(x)| > \varepsilon \text{ some } n \ge N_2\} = A_{\varepsilon, N_2}$$

(since if $|f_n(x) - f(x)| > \varepsilon$ for $n \ge N_2$, then $|f_n(x) - f(x)| > \varepsilon$ for $n \ge N_1 > N_2$)

and see that $m(A_{\varepsilon,1}) = m(\{x : |f_n(x) - f(x)| > \varepsilon$ for some $n \ge 1\})$,

so:
$$\varepsilon\, m(A_{\varepsilon,1}) = \int \varepsilon \chi_{A_{\varepsilon,1}}\, dm = \int_{A_{\varepsilon,1}} \varepsilon\, dm < \int_{A_{\varepsilon,1}} |f_n - f|\, dm \le \int_{\mathbb{R}} |f_n - f|\, dm \le \frac{C}{n^2}$$

$\Rightarrow m(A_{\varepsilon,1}) < \infty$ so now we may apply continuity from above:

$$\lim_{\varepsilon\to 0} m\left(\bigcap_{N=1}^{\infty} A_{\varepsilon,N}\right) = \lim_{\varepsilon\to 0}\lim_{N\to\infty} m(A_{\varepsilon,N})$$

Now:

$$\varepsilon m(A_{N,\varepsilon}) = \int_{A_{N,\varepsilon}} \varepsilon \, dm < \int_{A_{N,\varepsilon}} |f_n - f| \, dm < \int_{\mathbb{R}} |f_n - f| \, dm \le \frac{C}{n^2}$$

for $n \geq N$, hence $n^2 \geq N^2$, hence $\frac{1}{n^2} \le \frac{1}{N^2}$

hence $m(A_{N,\varepsilon}) \le \frac{C}{N^2\varepsilon}$

$$\Rightarrow m(A) = \lim_{\varepsilon\to 0}\lim_{N\to\infty} m(A_{\varepsilon,N}) \le \lim_{\varepsilon\to 0}\lim_{N\to\infty} \frac{C}{N^2\varepsilon} = \lim_{\varepsilon\to 0} 0 = 0$$

$\Rightarrow m(A) = 0 \Rightarrow$ <u>$f_n \to f$ a.e.</u>

REAL ANALYSIS GRADUATE EXAM
Spring 2010

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

Check

(1) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *upper semicontinuous* (or *u.s.c.*) if for all $x \in \mathbb{R}$ and all $\epsilon > 0$ there exists $\delta > 0$ such that $f(y) < f(x) + \epsilon$ whenever $|y - x| < \delta$.

(i) Show that every u.s.c. function is Borel measurable. HINT: Consider $\{x : f(x) < a\}$.

(ii) Suppose μ is a finite measure on $\mathbb{R}$ and A is a closed subset of $\mathbb{R}$. Using (i) or otherwise, show that the function $x \mapsto \mu(x + A)$ is measurable. Here $x + A = \{x + y : y \in A\}$.

f meas, $\mu(X)<\infty$ show $\lim_{N\to\infty} \mu\{|f| \geq N\} = 0$

(2) Suppose $\{f_n\}$ and f are measurable functions on $(X, \mathcal{M}, \mu)$ and $f_n \to f$ in measure. Is it necessarily true that $f_n^2 \to f^2$ in measure if:

(a) $\mu(X) < \infty$

(b) $\mu(X) = \infty$. $\longrightarrow$ counterex: $f_n(x) = \sqrt{x^4 + \frac{x}{n}}$

In each case, prove or give a counterexample.

(3) Suppose $f : [0, 1] \to \mathbb{R}$ is a strictly increasing absolutely continuous function. Let m denote Lebesgue measure. If $m(E) = 0$ show that $m(f(E)) = 0$. $\rightarrow$ use thm 1.18

(4) For $n \geq 1$ define h_n on $[0, 1]$ by

$$h_n = \sum_{j=1}^{n} (-1)^j \chi_{(\frac{j-1}{n}, \frac{j}{n}]}.$$

Here χ_E denotes the characteristic function of E. If f is Lebesgue integrable on $[0, 1]$, show that

$$\lim_{n\to\infty} \int_{[0,1]} f h_n \, dm = 0.$$

HINT: First consider f in a suitably smaller function space.

Approx. w/ Thm 2.6

Spring '10 :

(1.) [$f:\mathbb{R}\to\mathbb{R}$ u.s.c if for all $x\in\mathbb{R}$, $\varepsilon>0$, $\exists\delta>0$ s.t
$|y-x|<\delta \Rightarrow f(y)<f(x)+\varepsilon$]

(i) Show every u.s.c. function is Borel measurable
(ii) μ finite measure on $\mathbb{R}$, $A\subseteq\mathbb{R}$ closed; show $x\mapsto\mu(x+A)$ is measurable :

(i) Recall that a function $f:\mathbb{R}\to\mathbb{R}$ is Borel measurable if every set $f^{-1}((-\infty,a])=\{f\leq a\}$ is a Borel set.

- Consider $x\in f^{-1}((-\infty,a])=\{f\leq a\}$ and choose $\varepsilon>0$:
  f is u.s.c., hence $\exists\delta>0$ such that
  $|y-x|<\delta \Rightarrow f(y)<f(x)+\varepsilon$, ie.:
  $y\in(x-\delta,x+\delta) \Rightarrow f(y)<f(x)+\varepsilon<a+\varepsilon$
- The inequalities are strict, hence letting $\varepsilon\to0$, $\exists\delta'>0$ s.t. $y\in(x-\delta',x+\delta')\Rightarrow f(y)\leq a$, here $(x+\delta',x+\delta')\subseteq\{f\leq a\}$
  hence x is contained in an open set contained in $f^{-1}((-\infty,a])$;
  x was arbitrary, hence $f^{-1}((-\infty,a])$ is open, hence Borel.

(ii) part (i) tells us that u.s.c. $\Rightarrow$ measurable, hence we want to show that $f(x)=\mu(x+A)$ is u.s.c.

- f u.s.c $\Leftrightarrow \limsup_{n\to\infty} f(x_n)\leq f(x)$ for $x_n\to x$: Let $x_n\to x$; for $\delta>0$,
  $\exists M$ s.t. $n\geq M\Rightarrow |x_n-x|<\delta \Rightarrow f(x_n)<f(x)+\varepsilon$
  $\Rightarrow \limsup_{n\to\infty} f(x_n)<f(x)+\varepsilon \Rightarrow \limsup_{n\to\infty} f(x_n)\leq f(x)$ as $\varepsilon\to0$ $\Big\}(\Rightarrow)$
- On the other hand, suppose $\limsup_{n\to\infty} f(x_n)\leq f(x)$, then
  $\lim_{k\to\infty}\left(\sup_{n\geq k} f(x_n)\right)\leq f(x)$, hence for k large (say M), we have
  $\sup_{n\geq M} f(x_n)<f(x)+\varepsilon$, hence $f(x_n)<f(x)+\varepsilon$ for $n\geq M$,
  hence f is u.s.c.

So we want to show $\limsup_{n\to\infty} f(x_n) \le f(x)$, for $x_n \to x$.

Let $f(x) = \mu(x+A)$; now:

$$\limsup_{n\to\infty} f(x_n) = \limsup_{n\to\infty} \mu(x_n+A) \quad (\ast)$$

$\limsup_{n\to\infty} \mu(A_n) \le \mu(\limsup_{n\to\infty} A_n)$:

$$\mu(\limsup_{n\to\infty} A_n) = \mu\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n\right) = \mu\left(\bigcap_{k=1}^{\infty} B_k\right)$$

$B_k = \bigcup_{n=k}^{\infty} A_n$, $B_{k+1} \subseteq B_k$

$\to \mu$ finite so may apply cont from above

$$= \lim_{k\to\infty} \mu(B_k) = \lim_{k\to\infty} \mu\left(\bigcup_{n=k}^{\infty} A_n\right) \ge \lim_{k\to\infty} \left(\sup_{n\ge k} \mu(A_n)\right)$$
$$= \limsup_{n\to\infty} \mu(A_n)$$

Hence $\limsup_{n\to\infty} \mu(x_n+A) \le \mu(\limsup_{n\to\infty} x_n + A)$;

Suppose $y \in \limsup_{n\to\infty} (x_n+A) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} (x_k+A)$

$\Rightarrow y \in \bigcup_{k=n}^{\infty} (x_k+A)$ for all n

$\Rightarrow y = x_{k_j} + a_j$ for some $a_j \in A$, for all j

Now, letting $j \to \infty$, we get $y = x + \lim_{j\to\infty} a_j$

Since A is closed, $\lim_{j\to\infty} a_j \in A$, hence $y \in x+A$

so: $\limsup_{n\to\infty} (x_n+A) \subseteq x+A$

$\Rightarrow \mu(\limsup_{n\to\infty}(x_n+A)) \le \mu(x+A)$

$\Rightarrow \limsup_{n\to\infty} \mu(x_n+A) \le \mu(x+A) \Rightarrow \limsup_{n\to\infty} f(x_n) \le f(x)$,

hence u.s.c.,

hence measurable

(2.) $(f_n), f$ measurable on (X,μ), $f_n \to f$ in measure.
Is it true that $f_n^2 \to f^2$ in measure if:
(a) $\mu(X) < \infty$, (b) $\mu(X) = \infty$

(a) f is measurable, hence $A_N = \{|f| > N\}$ is measurable.
See that $A_{N+1} = \{N+1 < |f|\} \subseteq \{N < |f|\} = A_N$, and
$\mu(A_0) = \mu(X) < \infty$, hence we may use continuity from above.
First, let $B_N = \{N < |f| \leq N+1\}$; then $A_N = \bigcup_{i=N}^{\infty} B_i$
and $\sum_{i=0}^{\infty} \mu(B_i) = \mu\left(\bigcup_{i\geq 0} B_i\right) = \mu(X) < \infty$. hence
Now:
$$\lim_{N\to\infty} \mu(A_N) = \mu\left(\bigcap_{N\geq 0} A_N\right)$$
$$\leq \mu(A_k) \text{ for all } k$$
$$= \mu\left(\bigcup_{i=k}^{\infty} B_i\right) = \sum_{i=k}^{\infty} \mu(B_i) \text{ for all } k.$$
Hence let $k \to \infty$ and then $\sum_{i=k}^{\infty} \mu(B_i) \to 0$ since the tail of a convergent series must go to zero.
Hence $\lim_{N\to\infty} \mu(A_N) = 0$. hence we have that, for $\varepsilon > 0$, $\exists M \geq 0$ such that $\underline{\mu\{|f_n| \geq M\} < \varepsilon,\ \mu\{|f| \geq M\} < \varepsilon}$.

- Now see the following: $|f_n^2 - f^2| = |f_n^2 - ff_n + ff_n - f^2|$
$$\leq |f_n^2 - ff_n| + |ff_n - f^2|$$
$$= |f_n| \cdot |f_n - f| + |f|\,|f_n - f|$$

$\leq + (.$

Now:

$$\{|f_n^2 - f^2| \geq \varepsilon\} \subseteq \{|f_n||f_n - f| + |f||f_n - f| \geq \varepsilon\}$$
$$\subseteq \{|f_n||f_n-f| \geq \tfrac{\varepsilon}{2}\} \cup \{|f||f_n - f| \geq \tfrac{\varepsilon}{2}\}$$
$$\subseteq \{|f_n| \geq M\} \cup \{|f_n - f| \geq \tfrac{\varepsilon}{2M}\} \cup \{|f| \geq M\} \cup \{|f_n$$
$$\cup \{|f_n - f| \geq \tfrac{\varepsilon}{2M}\}$$

Hence:

$$\mu\{|f_n^2 - f^2| \geq \varepsilon\} \leq \mu\{|f_n| \geq M\} + \mu\{|f_n - f| \geq \tfrac{\varepsilon}{2M}\} + \mu\{|f| \geq M\} + \mu\{|f_n - f| \geq \tfrac{\varepsilon}{2M}\}$$
$$< \varepsilon + \varepsilon + \mu\{|f_n - f| \geq \tfrac{\varepsilon}{2M}\} + \mu\{|f_n - f| \geq \tfrac{\varepsilon}{2M}\}$$

$\longrightarrow$ by part 1

$$< \varepsilon + \varepsilon + \varepsilon + \varepsilon$$

for n large enough $\longrightarrow$ since $f_n \to f$ converges in measure

Therefore, $f_n^2 \to f^2$ in measure.

(b) $\mu(X) = \infty$; let $X = \mathbb{R}$, $\mu = m$ Lebesgue.

Let $f(x) = x^2$, $f_n(x) = x^2 + \frac{1}{n}$; then, for $x \in [n, \infty)$,

we have: $|f_n(x)^2 - f(x)^2| = |(x^2 + \frac{1}{n})^2 - x^4|$

$$= |x^4 + \frac{2x^2}{n} + \frac{1}{n^2} - x^4|$$
$$= |\frac{2x^2}{n} + \frac{1}{n^2}| = \frac{2x^2}{n} + \frac{1}{n^2}$$

since all positive

So for $x \in [n, \infty)$,

$|f_n^2(x) - f^2(x)| = \frac{2x^2}{n} + \frac{1}{n^2} \geq \frac{2x^2}{n} \geq 2n \geq 2$,

hence $[n, \infty) \subseteq \{|f_n^2 - f^2| \geq 2\}$, hence

$m([n,\infty)) \leq m\{|f_n^2 - f^2| \geq 2\}$, but $\lim_{n \to \infty} m([n, \infty)) = \infty$

since m is an infinite measure and $m([n, \infty)) = \infty$.

Therefore $f_n^2 \not\to f^2$ in measure.

③ $f:[0,1]\to\mathbb{R}$ strictly increasing, absolutely continuous. ~~m,~~ Lebesgue measure; Show $m(E)=0 \Rightarrow m(f(E))=0$

---

f strictly increasing, hence 1-1, hence f^{-1} is well-defined. Now:

$$m(f(E)) = \int_{\mathbb{R}} \chi_{f(E)}\,dx = \int \chi_E(f^{-1}(x))\,dx$$

Now let $u=f^{-1}(x) \Rightarrow du=(f^{-1})'(x)\,dx$

But recall $(f\circ f^{-1})(x)=x$, hence $(f\circ f^{-1})'(x)=1$

$$\Rightarrow f'(f^{-1}(x))(f^{-1})'(x)=1 \Rightarrow (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(u)}$$

Hence $du = \dfrac{dx}{f'(f^{-1}(x))} \Rightarrow f'(u)\,du = dx$

So $\int \chi_E(f^{-1}(x))\,dx = \int \chi_E(u)\,f'(u)\,du = \int_E f'(u)\,du = 0$

since $m(E)=0$, hence $\underline{m(f(E))=0}$.

---

④ $n\geq 1$; define h_n on $[0,1]$ by $h_n = \sum_{j=1}^{n}(-1)^j \chi_{[\frac{j-1}{n},\frac{j}{n}]}$

Show $\lim_{n\to\infty}\int_{[0,1]} f h_n\,dm = 0$ for $f\in L^1(\mathbb{R})$

- Case $f=\chi_{(a,b)}$: $\int_{[0,1]} f h_n\,dm = \int_{(a,b)} h_n\,dm$

Now let $I_j = [\frac{j-1}{n},\frac{j}{n}]$; clearly $[0,1]=\bigcup_{j=1}^{n} I_j$, hence let $I_j, I_{j+1},\dots,I_{j+(k-1)}$ be the k intervals completely contained in (a,b). (clearly, k may be 0).

Finally, let J_1, J_2 be the I_i s.t. $J_1\cap(a,b)=(a,a+\delta_1)$ and $J_2\cap(a,b)=(b-\delta_2,b)$ for some $\delta_1,\delta_2\geq 0$.

Then: $(a,b) = J_1\cup J_2\cup\left(\bigcup_{i=0}^{k-1} I_{j+i}\right)$ since the I_i are disjoint.

So now:

$$\int_{(a,b)} h_n\,dm = \int_{J_1} h_n\,dm + \int_{J_2} h_n\,dm + \sum_{i=0}^{k-1} \int_{I_{j+i}} h_n\,dm$$

Since for all $x \in I_j$, $h_n(x) = (-1)^j$, we have that

$$\int_{I_j} h_n\,dm = \frac{(-1)^j}{n}, \text{ and}$$

hence: $-\frac{1}{n} < \int_{J_i} h_n < \frac{1}{n}$ for J_1, J_2

and $\displaystyle\sum_{i=0}^{k-1} \int_{I_{j+i}} h_n\,dm = \sum_{i=0}^{k-1} \frac{(-1)^{j+i}}{n} = \begin{cases} 0 & \text{if } k \text{ even} \\ \pm\frac{1}{n} & \text{if } k \text{ odd.} \end{cases}$

hence $-\frac{1}{n} < \displaystyle\sum_{i=0}^{k-1} \int_{I_{j+i}} h_n\,dm < \frac{1}{n}$ also

Therefore: $-\frac{3}{n} < \int_{(a,b)} h_n\,dm < \frac{3}{n}$, hence

$\lim_{n\to\infty} \int_{(a,b)} h_n\,dm = 0$ by the squeeze theorem, hence

$\lim_{n\to\infty} \int f h_n\,dm = 0$ for $f = \chi_{(a,b)}$.

• Case $f = \sum c_i \chi_{(a_i,b_i)}$, i.e. an int. simple fn. with E_i = interval

Then: $\lim_{n\to\infty} \int f h_n\,dm = \lim_{n\to\infty} \int \left(\sum_{i=1}^{n} c_i \chi_{(a_i,b_i)}\right) h_n\,dm$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} c_i \int \chi_i h_n\,dm = \sum_{i=1}^{n} c_i \left(\lim_{n\to\infty} \int \chi_i h_n\,dm\right) = \sum_{i=1}^{n} c_i(0) = 0$$

• general case: for $\varepsilon > 0$, $\exists$ integrable simple fn. with E_i = finite unions of intervals ϕ s.t. $\int |f-\phi| < \varepsilon$ (since Lebesgue

Then: $\varepsilon > \int |f-\phi| = \int |h_n||f-\phi| \geq \left|\int h_n f - h_n\phi\right| \geq \left|\int h_n f\right| - \left|\int h_n \phi\right|$

hence as $n \to \infty$, $\left|\int h_n f\right| < \varepsilon$, but since ε was arbitrary,

$\left|\int h_n f\right| = 0$ as $n \to \infty$.

SPRING 2010 $\quad \sum (f(x_i) - f(y_i)) < \delta$

(3.) $f : [0,1] \to \mathbb{R}$ strictly increasing, abs. cont.
Show $m(E) = 0 \Rightarrow m(f(E)) = 0$

Since $E \subseteq [0,1]$ is measure 0, $\exists$ countable sequence of intervals (x_i, y_i) s.t. $E \subseteq \bigcup_{i=1}^{\infty} (x_i, y_i)$
and $m(\bigcup_{i=1}^{\infty} (x_i, y_i)) \leq \sum_{i=1}^{\infty} |y_i - x_i| < \delta$, for any given $\delta > 0$.

Now, choose $\varepsilon > 0$; since f is abs. cont., $\exists \delta > 0$ s.t. $\sum_{i=1}^{n} |y_i - x_i| < \delta \Rightarrow \sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon$.

See that: $f(E) \subseteq f(\bigcup_{i=1}^{\infty} (x_i, y_i)) \subseteq \bigcup_{i=1}^{\infty} (f(x_k), f(y_k))$
since f is continuous and strictly increasing.
Then we have:
$$m(f(E)) \leq m\left(\bigcup_{i=1}^{\infty} (f(x_k), f(y_k))\right) \leq \sum_{i=1}^{\infty} |f(y_k) - f(x_k)| < \varepsilon$$
since we chose intervals s.t. $\sum |y_k - x_k| < \delta$

$m(E) = 0$, $\exists$ countable seq $\bigcup_{i=1}^{\infty} (x_i, y_i) \supseteq E$
s.t.
$m(\bigcup (x_i, y_i) \setminus E) < \varepsilon$
$\quad \|$
$m(\bigcup (x_i, y_i)) - m(E)$

$(x_i, y_i) \subseteq \bigcup (x_i, y_i)$
$\to m((x_i, y_i)) \leq m(\bigcup (x_i, y_i)) < \varepsilon$
$m((x_i, y_i)) < \varepsilon$

check

REAL ANALYSIS GRADUATE EXAM
Fall 2010

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $\mathcal{A}$ be a collection of pairwise disjoint subsets of a σ-finite measure space, and suppose each set in $\mathcal{A}$ has strictly positive measure. Show that $\mathcal{A}$ is at most countable.

* true for clear fns, true for L by DCT.

(2) (a) Let m denote Lebesgue measure on $\mathbb{R}$ and let f be an integrable function. Show that for $a > 0$,

$$\int f(ax)\, m(dx) = \frac{1}{a} \int f(x)\, m(dx).$$

HINT: Consider a restricted class of functions f first.

(b) Let F be a measurable function on $\mathbb{R}$ satisfying $|F(x)| \leq C|x|$ for all x, and suppose F is differentiable at 0. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{nF(x)}{x(1+n^2x^2)}\, m(dx) = \pi F'(0).$$

HINT: Use (a).

check (3) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let f be a measurable function with $|f| < 1$. Prove that

$$\lim_{n\to\infty} \int_X (1 + f + \cdots + f^n)\, d\mu$$

exists (it may be ∞.) HINT: First consider $f \geq 0$.

Review → (4) Let $\{F_j\}$ be a sequence of nonnegative nondecreasing right-continuous functions on $[a, b]$ and suppose $F(x) = \sum_{j=1}^{\infty} F_j(x)$ is finite for all $x \in [a, b]$. Show that

Folland 3-39

$$F'(x) = \sum_{j=1}^{\infty} F_j'(x) \quad \text{for } m\text{-a.e. } x \in [a, b].$$

HINT: Consider the corresponding measures μ_F and μ_{F_j}.

Fall -2010:

① $\mathcal{A}$ collection of pairwise-disjoint subsets of σ-finite measure space. Suppose every $E \in \mathcal{A}$ has $\mu(E) > 0$. Show $\mathcal{A}$ is at most countable

σ-finite $\Rightarrow X = \bigcup_{i=1}^{\infty} X_i$ where $\mu(X_i) < \infty \ \forall i$.

Then $A = \bigcup_{i=1}^{\infty} (A \cap X_i)$ for each $A \in \mathcal{A}$, but $\mu(A) > 0$, hence $\sum_{i=1}^{\infty} \mu(A \cap X_i) \geq \mu(A) > 0$, hence $\mu(A \cap X_i) > 0$ for some i.

Now suppose that $\mathcal{A}$ is uncountable:

since there are only countably many X_i, this means there must be some X_i s.t. there are uncountably many $A \in \mathcal{A}$ with $A \cap X_i \neq \phi$ and $\mu(A \cap X_i) > 0$.

Call this family $\mathcal{C} \subseteq \mathcal{A}$.

The $A \in \mathcal{C}$ are disjoint since $\mathcal{C} \subseteq \mathcal{A}$, hence the $A \cap X_i$ are also disjoint. Now see that:

$$X_i \supseteq \bigcup_{A \in \mathcal{C}} A \cap X_i$$

since disjoint — uncountable sum of positives

$$\Rightarrow \mu(X_i) \geq \mu\Big(\bigcup_{A \in \mathcal{C}} A \cap X_i\Big) = \sum_{A \in \mathcal{C}} \mu(A \cap X_i) = \infty$$

which is a contradiction since $\mu(X_i) < \infty$. So $\mathcal{A}$ is at most countable

② (a) m Leb. on $\mathbb{R}$, f integrable. Show for $a > 0$:

$$\int f(ax)\,dm = \frac{1}{a}\int f(x)\,dm$$

Case: simple functions: let $f = \chi_{[b,c]}$ be a characteristic function.

Then: $\int f(ax)\,dm = \int \chi_{[b,c]}(ax)\,dm = \int \chi_{[\frac{b}{a},\frac{c}{a}]}(x)\,dm = \frac{c}{a} - \frac{b}{a}$

$= \frac{1}{a}(c-b) = \frac{1}{a}\int \chi_{[b,c]}(x)\,dm = \frac{1}{a}\int f(x)\,dm.$

Hence also true for simple functions (linear comb. of char. fns)

Recall that f is integrable; thus $\exists$ seq of simple functions ϕ_n converging monotonically to f. If $f \geq 0$ so:

Case: f positive: $\int f(ax)\,dm = \int \lim_{n\to\infty} \phi_n(ax)\,dm \overset{MCT}{=} \lim_{n\to\infty} \int \phi_n(ax)\,dm$

$= \lim_{n\to\infty} \frac{1}{a}\int \phi_n(x)\,dm \underset{MCT}{=} \frac{1}{a}\int \lim_{n\to\infty} \phi_n(x)\,dm = \frac{1}{a}\int f(x)\,dm$

General case: $f = f_+ - f_-$

(b.) F meas on $\mathbb{R}$ s.t. $|F(x)| \le C|x|$ $\forall x$. Suppose F diff. at 0.

Show: $\lim_{n\to\infty} \int_{\mathbb{R}} \frac{nF(x)}{x(1+n^2x^2)}\,dm = \pi F'(0)$

Let $f(x) = \frac{nF(x)}{x(1+n^2x^2)}$ and apply part (a):

$$n\int \frac{nF(x)}{x(1+n^2x^2)}\,dm = \int nf(x) = \int f\left(\frac{x}{n}\right) = \int \frac{nF(\frac{x}{n})}{(\frac{x}{n})(1+n^2(\frac{x}{n})^2)}$$

$$= \int \frac{n^2F(\frac{x}{n})}{x(1+x^2)} \Rightarrow \int \frac{nF(x)}{x(1+n^2x^2)}\,dm = \int \frac{nF(\frac{x}{n})}{x(1+x^2)}\,dm$$

Now: $\left|\frac{nF(\frac{x}{n})}{x(1+x^2)}\right| \le \frac{nC|\frac{x}{n}|}{|x(1+x^2)|} = \frac{C}{(1+x^2)} \in L^1(m)$

$\to$ so we may apply DCT:

$$\lim_{n\to\infty} \int \frac{nF(\frac{x}{n})}{x(1+x^2)}\,dm = \int \left(\lim_{n\to\infty} \frac{nF(\frac{x}{n})}{x(1+x^2)}\right)dm = \int \left(\frac{1}{(1+x^2)} \lim_{n\to\infty} \frac{F(\frac{x}{n})}{(\frac{x}{n})}\right)dm$$

Now, $|F(0)| \le C|0| = 0$, hence $F(0)=0$, and so:

$$\int \left[\frac{1}{(1+x^2)}\left(\lim_{n\to\infty} \frac{F(\frac{x}{n})-F(0)}{\frac{x}{n}-0}\right)\right]dm = \int \left[\frac{1}{(1+x^2)}F'(0)\right]dm = F'(0)\pi$$

Hence: $\lim_{n\to\infty} \int \frac{nF(x)}{x(1+n^2x^2)}\,dm = \lim_{n\to\infty} \int \frac{nF(\frac{x}{n})}{x(1+x^2)}\,dm = \pi F'(0)$

(3.) (X,μ) meas, $\mu(X)<\infty$, f meas, $|f|<1$.

Show: $\lim_{n\to\infty}\int_X (1+f+\cdots+f^n)\,d\mu$ exists

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- case $f\geq 0$: let $g_n=\sum_{i=0}^{n} f^i$. Then $g_n=\frac{1-f^{n+1}}{1-f}$ (since $|f|<1$)

and $\lim_{n\to\infty} g_n=\frac{1}{1-f}$; clearly g_n increases monotonically since $f\geq 0$, so apply MCT

$$\lim_{n\to\infty}\int_X (1+\cdots+f^n)\,d\mu=\lim_{n\to\infty}\int_X g_n\,d\mu \overset{\text{MCT}}{=}\int_X \lim_{n\to\infty} g_n\,d\mu$$

$=\int_X \frac{1}{1-f}\,d\mu<\infty$ since $\frac{1}{1-f(x)}<\infty$ for all x since $|f|<1$

- general case: let $f=f_+-f_-$ where f_+, f_- nonnegative and such that one is always zero:

Then $(f_+-f_-)^k=f_+^k+(-1)^k f_-^k+\cancel{(\text{cross-terms})}^{=0}$ (since either $f_+(x)=0$ or $f_-(x)=0$ $\forall x$)

So we have:

$$\lim_{n\to\infty}\int_X (1+\cdots+f^n)\,d\mu=\lim_{n\to\infty}\int_X (1+\cdots+(f_+-f_-)^n)\,d\mu$$

$$=\lim_{n\to\infty}\int_X (1+\cdots+f_+^n+(-1)^n f_-)\,d\mu$$

$$=\lim_{n\to\infty}\left[\int_X (1+f_++\cdots+f_+^n)\,d\mu+\int_X (-f_-+f_-^2+\cdots+(-1)^n f_-^n)\,d\mu\right] \quad (*)$$

Now, note that $-f_-+f_-^2+\cdots+(-1)^n f_-^n\leq 1+f_-+f_-^2+\cdots+f_-^n$ since $f_-\geq 0$

hence by comparison:

$$(*)\leq \lim_{n\to\infty}\left[\int_X (1+\cdots+f_+^n)\,d\mu+\int_X (1+\cdots+f_-^n)\,d\mu\right]$$

$$=\lim_{n\to\infty}\int_X (1+\cdots+f_+^n)\,d\mu+\lim_{n\to\infty}\int_X (1+\cdots+f_-^n)\,d\mu$$

Both of which exist by the $f\geq 0$ case.

(4) (F_j) seq of non-negative, non-decreasing, right-continuous functions on $[a,b]$ and suppose $F(x) = \sum_{j=1}^{\infty} F_j(x)$.

<u>Show</u>: $F'(x) = \sum_{j=1}^{\infty} F_j'(x)$ for m-a.e. $x \in [a,b]$.

Since the F_j are non-decreasing, so is F, hence F is differentiable a.e. Furthermore, the F_j are non-decreasing and right continuous hence so is F, hence the measures μ_F and μ_{F_j} are well-defined. Since F, F_j are rt-cont. on a closed interval they are also bdd, hence NBV, hence

$\mu_F \ll m \iff F$ abs cont.

$\mu_{F_j} \ll m \iff F_j$ abs cont.

Now, suppose $m(E)=0 \Rightarrow 0 = \inf\{\sum (b_j - a_j) : E \subseteq \bigcup (a_j, b_j)\}$

Then: $\mu_F(E) = \inf\{\sum \mu_F((a_j, b_j)) : E \subseteq \bigcup (a_j, b_j)\}$

$= \inf\{\sum F(b_j) - F(a_j) : E \subseteq \bigcup (a_j, b_j)\}$

$= 0$ since the first infemum implies that $a_j = b_j$ for each interval.

$\rightarrow$ Similarly for F_j.

Hence $\mu_F \ll m$, $\mu_{F_j} \ll m$, hence $F_j(b) - F_j(a) = \int_a^b F_j'(x)\,dx$

$F(b) - F(a) = \int_a^b F'(x)\,dx$,

thus:

by Tonelli since all positive

$$\int_a^b \sum F_j'(x) = \sum \int_a^b F_j'(x) = \sum F_j(b) - F_j(a)$$
$$= F(b) - F(a)$$

hence $\sum F_j'(x) = F'(x)$

check

REAL ANALYSIS GRADUATE EXAM
Spring 2011

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $A \subset \mathbb{R}$ and suppose that for each $\epsilon > 0$ there are Lebesgue-measurable sets E, F with $E \subset A \subset F$ and $m(F\backslash E) \le \epsilon$. Show that A is Lebesgue measurable.

(2) Let $f > 0$ be a Lebesgue-integrable function on $[0,1]$. Show that

$$\lim_{\epsilon \searrow 0} \frac{1}{\epsilon} \int_{[0,1]} (f^{\epsilon} - 1)\, dm = \int_{[0,1]} \log f\, dm.$$

Here m denotes Lebesgue measure. HINT: Decompose f (or $\log f$) into two parts.

(3) Suppose $f \in L^1(\mathbb{R})$ is absolutely continuous, and

$$\lim_{h \searrow 0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right| dx = 0.$$

Show that $f = 0$ a.e.

(4)(a) Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$, and suppose $F_1, \ldots, F_7$ are 7 measurable sets with $\mu(F_j) \geq 1/2$ for all j. Show that there exist indices $i_1 < i_2 < i_3 < i_4$ for which $F_{i_1} \cap F_{i_2} \cap F_{i_3} \cap F_{i_4} \neq \phi$.

(b) Let m denote Lebesgue measure on $[0,1]$, and let $f_n \in L^1(m)$ be nonnegative and measurable with

$$\int_{[0,1/n]} f_n\, dm \geq 1/2$$

for all $n \geq 1$. Show that $\int_{[0,1]} [\sup_n f_n(x)]\, m(dx) = \infty$. HINT: Part (b) does not necessarily use part (a).

Spring '11:

(1.) $A \subseteq \mathbb{R}$ and for each $\varepsilon > 0$, $\exists$ Leb-meas. E, F s.t. $E \subseteq A \subseteq F$ and $m(F \setminus E) < \varepsilon$. Show A is Leb-measurable

- For each $\varepsilon > 0$, $\exists$ Leb-meas E, F s.t. $E \subseteq A \subseteq F$ and $m(F \setminus E) < \varepsilon$, i.e. in particular $\exists F_n, E_n$ s.t. $E_n \subseteq A \subseteq F_n$ and $m(F_n \setminus E_n) < \frac{1}{n}$ for each n
- Now let $E = \bigcup E_n$, $F = \bigcap F_n$, hence $E \subseteq A \subseteq F$, and particularly, $F \setminus E \subseteq F_n \setminus E_n$ for each n, hence:

$$m(F \setminus E) \leq \lim_{n \to \infty} m(F_n \setminus E_n) < \lim_{n \to \infty} \frac{1}{n} = 0$$

$\Rightarrow m(F \setminus E) = 0$.

- Since Lebesgue meas is complete every subset of a null set is measurable (and meas. 0)

In particular $A \setminus E \subseteq F \setminus E$, hence $m(A \setminus E) = 0$, here $A = E \cup N$ where N is Lebesgue null, hence A is union of two measurable sets, hence measurable.

(2) $f > 0$ Leb-int on $[0,1]$. Show: $\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^1 (f^\varepsilon - 1)\, dm = \int_0^1 \log f \, dm$

- First see that $\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^1 (f^\varepsilon - 1)\, dm = \lim_{k \to 0} \frac{1}{1/k} \int_0^1 (f^{1/k} - 1)\, dm$.

Now consider the sequence $g_n = \frac{f^{1/n} - 1}{1/n} = n(f^{1/n} - 1)$.

<u>case $f > 1$</u>: this implies that $\log f > 0$; now consider

$$\frac{d}{dn} g_n = \frac{d}{dn}\left(n(f^{1/n} - 1)\right) = (f^{1/n} - 1) + n\left(-\frac{\log f}{n^2} e^{\frac{\log f}{n}}\right)$$
$$= f^{1/n}\left(1 - \frac{\log f}{n}\right) - 1$$
$$\leq \left(1 - \frac{\log f}{n}\right) - 1 = \frac{-\log f}{n} < 0$$

hence $g_n(x)$ is decreasing as $n \to \infty$ for all x.

Since f integrable, $\int g_1 = \int (f - 1) < \infty$, so may apply DCT

since now $|g_n| \leq |g_1| \in L^1$ for all n (since g_n decreasing) ?

case $f<1$: this implies that $\log f<0$, so:

$$\frac{d}{dn} g_n = \underbrace{f^{1/n}}_{\leq 1}\underbrace{\left(1-\frac{\log f}{n}\right)}_{\geq 1} - 1 \geq \left(1-\frac{\log f}{n}\right)-1 = \frac{-\log f}{n} > 0,$$

here $g_n(x)$ is increasing for all x, hence we may apply MCT.

- Apply MCT: $\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^1 (f^{\varepsilon}-1)\,dm = \int_0^1 \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}(f^{\varepsilon}-1)\,dm$

$$= \int_0^1 \frac{d}{d\varepsilon}(f^{\varepsilon})\Big|_{\varepsilon=0} dm = \int_0^1 \log f\, e^{\varepsilon \log f}\Big|_{\varepsilon=0} dm = \int_0^1 \log f\, dm$$

---

3. $f\in L^1(\mathbb{R})$ abs cont and $\lim_{h\downarrow 0}\int_{\mathbb{R}}\left|\frac{f(x+h)-f(x)}{h}\right|dx = 0$. Show $f=0$ a.e.

Recall $\lim_{h\downarrow 0}\frac{f(x+h)-f(x)}{h} = \lim_{n\to\infty}\frac{f(x+\frac{1}{n})-f(x)}{\left(\frac{1}{n}\right)}$

Consider now the sequence $g_n(x) = \frac{f(x+\frac{1}{n})-f(x)}{\left(\frac{1}{n}\right)}$ and:

$$\int |f'(x)|\,dm = \int \left|\liminf_{n\to\infty}\frac{f(x+\frac{1}{n})-f(x)}{\left(\frac{1}{n}\right)}\right|$$

exists by abs. cont. of f

$$= \int \liminf_{n\to\infty}|g_n| \leq \liminf_{n\to\infty}\int |g_n|\,dm \quad \text{(FATOU)}$$

$$= \liminf_{n\to\infty}\int \left|\frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}}\right| dm$$

$$= \liminf_{h\to 0}\int \left|\frac{f(x+h)-f(x)}{h}\right| dm = 0$$

hypothesis

Hence: $\int |f'(x)|\,dm = 0 \Rightarrow |f'(x)| = 0$ a.e.

$\Rightarrow f'(x) = 0$ a.e.

$\Rightarrow f(x) = \text{const. a.e. } x$

$\Rightarrow f(x) \equiv 0$ a.e. x since f is L^1.

(4.) (X,μ) meas. space, $\mu(X)=1$, $F_1,\dots,F_7$ measurable with $\mu(F_j)\geq\frac{1}{2}$ for all $j=1,\dots,7$.

(a) Show $\exists\, i_1<i_2<i_3<i_4$ s.t. $F_{i_1}\cap F_{i_2}\cap F_{i_3}\cap F_{i_4}\neq\emptyset$.

- Suppose $\nexists$ non-empty 4-way intersection, i.e.
$\sum_{i=1}^{7}\chi_{F_i}(x)<4$ for all $x\in X$ (i.e. $\sum_{i=1}^{7}\chi_{F_i}(x)\leq 3$)

- Now consider the integral:

$$3 \geq \int_X \sum_{i=1}^{7}\chi_{F_i}(x)\,d\mu = \sum_{i=1}^{7}\int\chi_{F_i}(x)\,d\mu = \sum_{i=1}^{7}\mu(F_i)$$

$$\geq \sum_{i=1}^{7}\frac{1}{2} = \frac{7}{2}$$

since $\mu(X)=1$

hence $3.5 \leq \int_X\sum_{i=1}^{7}\chi_{F_i}(x)\,d\mu\leq 3$, a contradiction

Therefore a nonempty 4-way intersection must exist.

(b) $f_n\in L^1(m)$ with $f_n\geq 0$, $\int_0^{1/n} f_n\,dm\geq\frac{1}{2}$ $\forall n\geq 1$.

Show that $\int_0^1 \sup_n f_n(x)\,dm=\infty$

- Define the measure $\nu(E)=\int_E \sup_n f_n(x)\,dm$; by definition $\nu\ll m$, hence (Thm 3.5) for $\varepsilon>0$, $\exists\,\delta>0$ s.t. $|m(E)|<\delta\Rightarrow|\nu(E)|<\varepsilon$

- Suppose that $\int_0^1 \sup_n f_n(x)\,dm=\nu([0,1])<\infty$.
Choose $\varepsilon=\frac{1}{2}$, hence $\exists\,\delta>0$ s.t. $m(A)<\delta\Rightarrow\nu(A)<\frac{1}{2}$;
now choose m s.t. $\frac{1}{m}<\delta$. Then, for $A=[0,\frac{1}{m}]$, we have:

$$m(A)<\delta \Rightarrow \nu(A)<\frac{1}{2}\Rightarrow\int_0^{1/m}\sup_n f_n(x)\,dm<\frac{1}{2},$$

but $\int_0^{1/m}\sup_n f_n(x)\,dm\geq\int_0^{1/m} f_m(x)\,dm\geq\frac{1}{2}$, a contradiction.

$$\sum \frac{\mu_k(E)}{\mu_k(X) 2^k}$$

REAL ANALYSIS GRADUATE EXAM
Fall 2011

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $f \geq 0$ and suppose $f \in L^1([0,\infty))$. Find

$$\lim_n \frac{1}{n} \int_0^n x f(x)\, dx.$$

(2) Suppose $f \geq 0$ is absolutely continuous on $[0,1]$ and $\alpha > 1$. Show that f^α is absolutely continuous.

(3)(a) Let $\{\mu_k\}$ be a sequence of finite signed measures. Find a finite positive measure μ such that $\mu_k \ll \mu$ for all k.

(b) Construct an increasing function whose set of discontinuities is $\mathbb{Q}$. (Prove it is a valid example.)

(4) Let m be Lebesgue measure on $\mathbb{R}$. For $f \in L^1_{loc}$ and $x \in \mathbb{R}^n$, define the function $A_r f$ by

$$A_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\, dy,$$

which is the average value of f on the ball $B(x,r)$ of radius r centered at x, and define the function Hf by $Hf(x) = \sup_{r>0} A_r|f|(x)$, $x \in \mathbb{R}^d$.

(a) Show that for $f \in L^1(\mathbb{R}^n)$, $f \neq 0$, there exist $C, C', R > 0$ such that $Hf(x) \geq C|x|^{-n}$ for all $|x| > R$ and

$$m\Big(\{x : Hf(x) > \alpha\}\Big) \geq \frac{C'}{\alpha} \quad \text{for all sufficiently small } \alpha.$$

(b) Define the function $H^* f$ by

$$H^* f(x) = \sup\left\{\frac{1}{m(B)} \int_B |f(y)|\, dy : B \text{ is a ball containing } x\right\}.$$

Show that $Hf \leq H^* f \leq 2^n Hf$. (Note that unlike Hf, in the definition of $H^* f$ the ball B need not be centered at x.)

Fall 2011:

(1.) $f \geq 0$; suppose $f \in L^1([0,\infty))$. Find $\lim_{n\to\infty} \frac{1}{n}\int_0^n x f(x)\,dx$

See that $\lim_{n\to\infty} \frac{1}{n}\int_0^n x f(x)\,dx = \lim_{n\to\infty}\int_0^\infty \chi_{[0,n]}(x)\,\frac{1}{n}\,x\,f(x)\,dx$

So let $|f_n(x)| = |\frac{1}{n}\underbrace{\chi_{[0,n]}(x)\,x}\,f(x)|$

$\leq |\frac{1}{n}\cdot n \cdot f(x)| = |f(x)| = f(x) \in L^1$ non-neg

Therefore $|f_n(x)|$ is bdd by an L^1 fn. for all n, so apply DCT:

$$\lim_{n\to\infty} \frac{1}{n}\int_0^n x f(x)\,dx = \lim_{n\to\infty}\int_0^\infty \chi_{[0,n]}(x)\,\frac{1}{n}\,x f(x)\,dx$$

$$= \int_0^\infty \lim_{n\to\infty} \chi_{[0,n]}(x)\,\frac{1}{n}\,x\,f(x)\,dx$$

$$= \int_0^\infty 0\,dx = 0.$$

(2.) $f \geq 0$ abs. cont on $[0,1]$, $\alpha \geq 1$. Show f^α is abs. cont.

$\mapsto$ we will use the fact that f abs cont, g Lipschitz $\Rightarrow$ $g\circ f$ abs. cont.)

Step 1: $g(x) = x^\alpha$ is Lipschitz: note that g is defined on cpt interval $[0,1]$, hence g and g' are bdd.

By the MVT, for every x,y, $\exists c$ st. $|g(x)-g(y)| = |g'(c)||x-y|$

However, g' is bdd, hence $|g(x)-g(y)| = |g'(c)||x-y| \leq M|x-y|$, hence g is Lipschitz.

Step 2: $g\circ f$ is abs. cont.: f is abs. cont. on $[0,1]$, hence given $\varepsilon > 0$, $\exists \delta > 0$ st. $\sum(y_k - x_k) < \delta \Rightarrow \sum|f(x_k)-f(y_k)| < \varepsilon$; in particular, $\exists \delta_0 > 0$ st. $\sum|f(x_k)-f(y_k)| < \frac{\varepsilon}{M}$; now:

$$\sum|g\circ f(x_k) - g\circ f(y_k)| \leq \sum M|f(x_k)-f(y_k)| < M\left(\frac{\varepsilon}{M}\right) = \varepsilon,$$

hence $g\circ f$ abs. cont.

Other methods?

(3.) (a) (μ_k) sequence of finite, signed measures.
Find positive measure μ s.t. $\mu_k \ll \mu$ $\forall k$.

Let $\mu = \sum_{k=0}^{\infty} \frac{|\mu_k|}{|\mu_k|(X) 2^k}$; positive by definition

- See first that for pairwise disjoint sets (E_n),

$$\mu(\bigcup_n E_n) = \sum_{k=0}^{\infty} \frac{|\mu_k|(\bigcup_n E_n)}{|\mu_k|(X) 2^k} = \sum_{k=0}^{\infty} \frac{\sum_{n=0}^{\infty} |\mu_k|(E_n)}{|\mu_k|(X) 2^k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{|\mu_k|(E_n)}{|\mu_k|(X) 2^k} = \sum_{n=0}^{\infty} \mu(E_n),$$ hence ctbly additive, so a measure

- Furthermore, $\mu = \sum_{k=0}^{\infty} \frac{|\mu_k|}{|\mu_k|(X) 2^k} \leq \sum_{k=0}^{\infty} \frac{1}{2^k} = 2$, hence finite

Now: Let $\mu(E) = 0 \Rightarrow 0 = \sum_{k=0}^{\infty} \frac{|\mu_k|(E)}{|\mu_k|(X) 2^k} \Rightarrow |\mu_k|(E) = 0$ for all k (since all posit

But $0 = |\mu_k|(E) = \mu_k^+(E) + \mu_k^-(E)$ by Jordan decomp, hence $\mu_k^+(E) = 0 = \mu_k^-(E)$ since both positive. Therefore, $\mu_k(E) = \mu_k^+(E) - \mu_k^-(E) = 0 - 0 = 0$, i.e. $\mu_k \ll \mu$ $\forall k$.

(b) Construct increasing function with set of discont $= \mathbb{Q}$

Let $\mathbb{Q} = \{q_i\}_{i \in \mathbb{N}}$ be an enumeration of the rationals and let $\sum_{n=1}^{\infty} a_n$ be a convergent series with $a_i \geq 0$ $\forall i$

Then let $f(x) = \sum_{m=0}^{\infty} \chi_{[q_m, \infty)}(x) \, a_m$

→ cont'd

3(b) cont'd:

→ <u>f is increasing</u>: Suppose $x<y$. $\mathbb{Q}$ is dense, hence $\exists\, q_k$ such that $y < q_k < x$, hence:

$$f(y)-f(x) = \sum_{m\geq 0} \chi_{[q_m,\infty)}(y)\, a_m - \sum_{m\geq 0} \chi_{[q_m,\infty)}(x)\, a_m \geq a_k > 0,$$

→ <u>f is continuous at $\mathbb{R}\setminus\mathbb{Q}$</u>: let $p \notin \mathbb{Q}$.

For $\varepsilon > 0$, choose N such that $\sum_{n=N+1}^{\infty} a_n < \varepsilon$ (possible since $\sum a_n$ is a convergent series)

Let $r_1 = \max(\{q_1,\dots,q_N\} \cap (-\infty, p])$ and $r_2 = \min(\{q_1,\dots,q_N\} \cap [p,\infty))$

Then, for $x_1 \in (r_1, p)$ and $x_2 \in (p, r_2)$, we have

$\{q_1,\dots,q_N\} \cap (x_1, x_2) = \emptyset$, hence since $r_1 < x_1 < p < x_2 < r_2$,

we have: $\left.\begin{array}{l}\chi_{[q_m,\infty)}(x_1) = \chi_{[q_m,\infty)}(x_2) = 1 \text{ for all } q_m < r_1 \\ \text{and } \chi_{[q_m,\infty)}(x_1) = \chi_{[q_m,\infty)}(x_2) = 0 \text{ for all } q_m > r_2\end{array}\right\}$

(where $q_m \in \{q_1,\dots,q_N\}$)

Hence: $f(x_2) - f(x_1) = \sum_{m=0}^{\infty} \chi_{[q_m,\infty)}(x_2)\, a_m - \sum_{m=0}^{\infty} \chi_{[q_m,\infty)}(x_1)\, a_m$

$$\leq \sum_{m=N+1}^{\infty} a_m < \varepsilon$$

$x_1 < p$, so $\lim_{x\to p^-} f(x) = \lim_{x \to p^-} \sum_{m=0}^{\infty} \chi_{[q_m,\infty)}(x)\, q_m \geq f(x_1)$

$x_2 > p$, so $\lim_{x\to p^+} f(x) = \lim_{x\to p^+} \sum_{m=0}^{\infty} \chi_{[q_m,\infty)}(x)\, q_m \leq f(x_2)$

and so:

$\lim_{x\to p^+} f(x) - \lim_{x\to p^-} f(x) \leq f(x_2) - f(x_1) < \varepsilon$, which is true for all ε,

hence letting $\varepsilon \to 0$ we get $\lim_{x\to p^+} f(x) = \lim_{x\to p^-} f(x)$, hence <u>$f$ is cont. at p.</u>

→ <u>f is discontinuous at $\mathbb{Q}$</u>: $p \in \mathbb{Q}$. Then $\exists n$ s.t. $q_n = p$. For all $x \in (p,\infty) = (q_n, \infty)$ we may choose q_k s.t. $q_n = p < q_k < x$, hence:

$$f(x) - f(p) = \sum_{m\geq 0} \chi_{[q_m,\infty)}(x)\, a_m - \sum_{m \geq 0} \chi_{[q_m,\infty)}(p)\, a_m \geq a_k$$

So $f(x) \geq a_k + f(p)$, hence $\lim_{x\to p^+} f(x) \geq f(p) + a_k \underline{\underline{>}} f(p)$, hence discontinuous at p.

④ $A_r f(x) = \frac{1}{m(B_r(x))} \int_{B_r(x)} f(y)\,dy$

$$Hf(x) = \sup_{r\geq 0} A_r|f|(x) = \sup_{r\geq 0}\left\{\frac{1}{m(B_r(x))}\int_{B_r(x)} |f(y)|\,dy\right\}$$

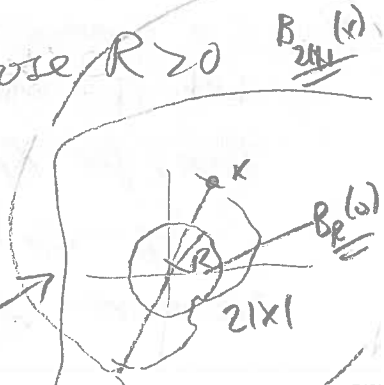
(a) Show for $f\in L^1(\mathbb{R}^n)$, $f\neq 0$, $\exists C, C', R>0$ s.t.

(1) $Hf(x) \geq C|x|^{-n}$ $\forall |x|>R$

(2) $m(\{Hf(x)>\alpha\}) \geq \frac{C'}{\alpha}$

Since $f\neq 0$, we have $\int_{\mathbb{R}^n}|f|\neq 0$, hence choose $R\geq 0$ such that

$$\int_{\underset{\exists}{B_R(0)}} |f| > \alpha > 0 \text{ for some } \alpha.$$



→ Now let $|X|>R$, hence :

$$Hf(x) = \sup_{r\geq 0}\left\{\frac{1}{m(B_r(x))}\int_{B_r(x)}|f(y)|\,dy\right\} \geq \frac{1}{m(B_{2|x|}(x))}\int_{B_{2|x|}(x)}|f(y)|\,dy$$

$$\geq \frac{1}{m(B_{2|x|}(x))}\int_{B_R(0)}|f(y)|\,dy = \frac{1}{C_1(2|x|)^n}\int_{B_R(0)}|f(y)|\,dy > \frac{\alpha}{C_1 2^n|x|^n}$$

$\boxed{\text{since } B_{2|x|}(x) \supseteq B_R(0) \text{ for } |x|>R}$

Let $C = \frac{\alpha}{C_1 2^n}$

Hence $Hf(x) > \frac{C}{|x|^n}$ (1)

→ Now choose α small enough such that $\frac{C}{\alpha} > R^n$:

Then $\left(\frac{C}{\alpha}\right)^{1/n} > R$, hence may choose x with $R<|x|<\left(\frac{C}{\alpha}\right)^{1/n}$.

Then since $R<|x|$, we have $Hf(x)>\alpha$, hence $\{Hf(x)>\alpha\}\supseteq\{R<|x|<\left(\frac{C}{\alpha}\right)^{1/n}\}$

$$\Rightarrow m(\{Hf(x)>\alpha\}) \geq m\left(\left\{R<|x|<\left(\tfrac{C}{\alpha}\right)^{1/n}\right\}\right) = m\left(\left\{|x|<\left(\tfrac{C}{\alpha}\right)^{1/n}\right\}\right) - m(\{|x|\leq R\})$$

$= K\left(\frac{C}{\alpha}\right) - KR^n = K\left(\frac{C}{\alpha} - R^n\right)$. Hence let $C' = CK\left(1-\frac{R^n\alpha}{C}\right)$

(and then $\frac{C'}{\alpha} = \frac{CK}{\alpha}\left(1-\frac{R^n\alpha}{C}\right) = K\left(\frac{C}{\alpha}-R^n\right) \leq m(\{Hf(x)>\alpha\})$)

(6.) $H^*f(x) = \sup\left\{\frac{1}{m(B)}\int_B |f(y)|dy : B \text{ ball w/ } x \in B\right\}$

→ $Hf(x) = \sup_{r>0}\left\{\frac{1}{m(B_r(x))}\int_{B_r(x)} |f(y)|\,dy\right\}$

$H^*f(x) = \sup_{B \ni x}\left\{\frac{1}{m(B)}\int_B |f(y)|\,dy\right\}$

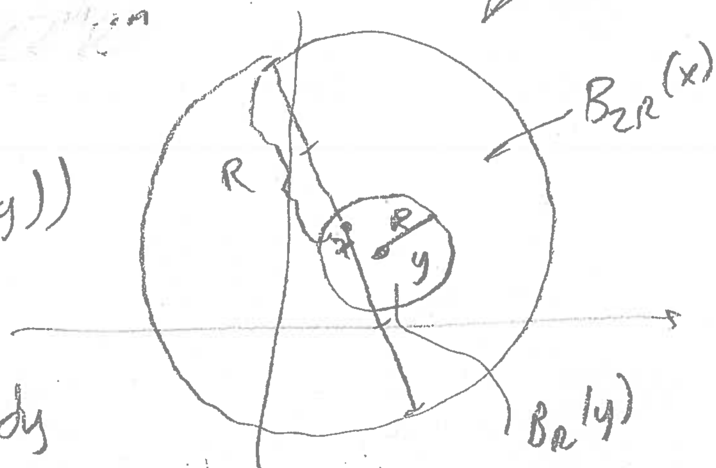
Note that Hf takes a sup over a smaller class of balls, hence $Hf(x) \leq H^*f(x)$

→ On the other hand, if $x \in B = B_R(y)$, then $B_R(y) \subseteq B_{2R}(x)$ and see that

$Vol(B_{2R}(x)) = K(2R)^n$

$Vol(B_R(y)) = KR^n$,

hence $Vol(B_{2R}(x)) = 2^n Vol(B_R(y))$



Hence we have:

$$\frac{1}{m(B)}\int_B |f(y)|dy = \frac{2^n}{m(B_{2R}(x))}\int_B |f(y)|dy$$

$$\leq \frac{2^n}{m(B_{2R}(x))}\int_{B_{2R}(x)} |f(y)|\,dy$$

By $B_R(y) \subseteq B_{2R}(x)$

⇒ inequalities preserved after taking sup's,

so $H^*f(x) \leq 2^n Hf(x)$

REAL ANALYSIS GRADUATE EXAM

Fall 2012

$\int \liminf f_n \leq \int f_n \, d\mu$

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let m be the Lebesgue measure on $X = [0,1]$. If

$$m(\limsup_{n\to\infty} A_n) = 1, m(\liminf_{n\to\infty} B_n) = 1,$$

$m(\limsup (A_n \cap B_n)) \leq 1$

prove that $m\left(\limsup_{n\to\infty}(A_n \cap B_n)\right) = 1$, where

$m(\liminf B_n) \leq \liminf_{n\to\infty} \mu(B_n)$

$$\limsup_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k, \liminf_{n\to\infty} B_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} B_k.$$

2. Assume $f : X \to [0,\infty)$ is measurable. Find

$(1+\frac{\alpha}{n})^n \to n(1+\frac{\alpha}{n})^{n-1} \cdot \frac{-\alpha}{n^2}$

$$\lim_n \int_X n \log\left[1 + \frac{f(x)}{n}\right] d\mu.$$

3. Let $f \in L^1(m)$. For $k = 1, 2, \ldots$ let f_k be the step function defined by

$$f_k(x) = k \int_{j/k}^{(j+1)/k} f(t)dt$$

$$\text{for } \frac{j}{k} < x \leq \frac{j+1}{k}, j = 0, \pm 1, \ldots.$$

Show that f_k converges to f in L^1 as $k \to \infty$.

4. If E is Borel set in $\mathbb{R}^n$ the density $D_E(x)$ of E at x is defined as

$$D_E(x) = \lim_{r\to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))},$$

$\to$ Learn!

whenever the limit exists [Here m denotes the Lebesgue measure and $B(x,r)$ is the open ball with center at x and radius r.]

(a) Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \notin E$.

(b) For $\alpha \in (0,1)$ find an example of E and x such that $D_E(x) = \alpha$.

(c) Find an example of E and x such that $D_E(x)$ does not exist.

$\int_{B(x,r)} \chi_E \, d\mu$

(a) $\lim_{r\to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))} \neq m(\{x^{c_E} \mid D_E(x) \neq 0\}) = 0$

a.e

$m(A\cup B) = \mu(A) + \mu(B) - \mu(A\cap B)$

$$\mu\left(\liminf A_n\right) = \mu\left(\bigcup_{n\ge 0}\bigcap_{k=n}^{\infty} A_n\right)$$

$$= \lim_{n\to\infty} \mu\left(\bigcap_{k=n}^{\infty} A_k\right) \quad \text{for all } n.$$

$$\le \lim_{n\to\infty} \mu(A_n)$$

$$\overbrace{\quad}^{\liminf \mu(A_n)}$$

$$\mu\left(\bigcap_{k=n}^{\infty} A_k\right) \le \mu(A_k) \quad \forall k \ge n. \quad \ge \inf_{n\ge k} \mu(A_n) \quad \text{for all } k$$

$$\bigcap_{k=n}^{\infty} A_k \subseteq A_k \quad \text{for all } k \ge n.$$

$$\mu\left(\bigcap_{k=n}^{\infty} A_k\right) \le \inf_{k\ge n} \mu(A_k)$$

$$\mu\left(\bigcap A_k\right) \le \mu(A_k) \quad \forall k \ge n.$$

$$\mu\left(\liminf A_n\right) \le \inf \mu(A_n)$$

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OFFENDING COMMAND: xshow

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Fall 2012:

(1.) m Lebesgue on $X=[0,1]$. Suppose $m(\limsup_{n\to\infty} A_n)=1$ and $m(\liminf_{n\to\infty} B_n)=1$. Show that $m(\limsup_{n\to\infty}(A_n\cap B_n))=1$

Recall that $\limsup_{n\to\infty} E_n = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} E_n = \{x : x\in E_n \text{ for infinitely many } n\}$

$\liminf_{n\to\infty} E_n = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} E_n = \{x : x\in E_n \text{ for all but finitely many } n\}$.

So: $x\in \limsup_{n\to\infty} A_n \cap \liminf_{n\to\infty} B_n$

$\Rightarrow x\in \limsup_{n\to\infty} A_n$ and $x\in\liminf_{n\to\infty} B_n$

$\Rightarrow$ · $x\in A_n$ for infinitely many n

· $x\in B_n$ for all but finitely many n

$\Rightarrow \exists$ seq. n_j s.t. $x\in A_{n_j}$ for all $j=0,1,2\ldots$

$\exists N$ s.t. $x\in B_n$ for all $n\geq N$

$\Rightarrow x\in A_{n_j}\cap B_{n_j}$ for all $n_j\geq N$, hence $x\in A_n\cap B_n$ for infinitely many n

$\Rightarrow x\in \limsup_{n\to\infty} A_n\cap B_n$.

So $m(\limsup_{n\to\infty} A_n\cap \liminf_{n\to\infty} B_n) \leq m(\limsup_{n\to\infty} A_n\cap B_n)$.

But $m(\limsup_{n\to\infty} A_n\cap\liminf_{n\to\infty} B_n] = m(\limsup A_n)+m(\liminf B_n)$

$-m(\limsup A_n\cup\liminf B_n)$

$= 1+1-\mu(\limsup A_n\cup\liminf B_n)$

$\geq 2-1=1$

Hence $m(\limsup A_n\cap\liminf B_n)=1$

② $f: X \to [0,\infty)$ measurable.
Find $\lim_{n\to\infty} \int_X n \log\left(1 + \frac{f(x)}{n}\right) d\mu$.

Consider the sequence of fns $n\log\left(1+\frac{f(x)}{n}\right)$

Then consider $\frac{d}{dn}\left(n\log\left(1+\frac{f(x)}{n}\right)\right) = \log\left(1+\frac{f(x)}{n}\right) + n\left(\frac{-f(x)n^{-2}}{1+\frac{f(x)}{n}}\right)$

$= \log\left(1+\frac{f(x)}{n}\right) - \frac{f(x)}{n+f(x)} = \log\left(1+\frac{f(x)}{n}\right) - \frac{\frac{f(x)}{n}}{1+\frac{f(x)}{n}} \quad (*)$

→ Now consider the following for $y \geq 0$:

for $y=0$, $\frac{y}{y+1} = 0 = \log(1+y)$; and:

$\frac{d}{dy}\left(\frac{y}{y+1}\right) = \frac{-y}{(1+y)^2} + \frac{1}{(1+y)} = \frac{1}{(1+y)^2}$

$\frac{d}{dy}\left(\log(1+y)\right) = \frac{1}{1+y}$;

the two functions start in the same place, but one grows faster than the other $\forall y$

Hence for $y \geq 0$, $\frac{d}{dy}\left(\frac{y}{y+1}\right) \leq \frac{d}{dy}\left(\log(1+y)\right)$, hence

$\frac{y}{y+1} \leq \log(1+y)$ for $y \in [0,\infty)$

Hence since $\frac{f(x)}{n} \geq 0$, we have that $\frac{\frac{f(x)}{n}}{1+\frac{f(x)}{n}} \leq \log\left(1+\frac{f(x)}{n}\right)$, hence $(*)$ is positive for all x, n, hence the sequence $f_n(x)$ is increasing as $n \to \infty$ for all x.

Therefore we may apply MCT:

$$\lim_{n\to\infty} \int n \log\left(1+\frac{f(x)}{n}\right) dx = \int \lim_{n\to\infty} n \log\left(1+\frac{f(x)}{n}\right) dx \quad (**)$$

Now see that, by L'Hôpital's rule:

$$\lim_{n\to\infty} n \log\left(1+\frac{f(x)}{n}\right) = \lim_{n\to\infty} \frac{\log\left(1+\frac{f(x)}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\left(\frac{1}{1+\frac{f(x)}{n}}\right)\left(\frac{-f(x)}{n^2}\right)}{\frac{-1}{n^2}}$$

$= \lim_{n\to\infty} \frac{f(x)}{1+\frac{f(x)}{n}} = f(x)$, hence:

$$(**) = \int f(x)\, dx$$

③ $f \in L^1(m)$. Let $f_k(x) = k\int_{j/k}^{j+1/k} f(t)\,dt$ for $x \in \left[\frac{j}{k}, \frac{j+1}{k}\right] = I_k^j$

Show that $f_k \to f$ in L^1.

See that $f_k(x) = \sum_j k \chi_{I_j^k}(x) \int_{I_j^k} f(t)\,dt$

→ Now suppose that $f(x) = \sum_{i=1}^n a_i \chi_{E_i}(t)$ (for E_i = finite union of intervals, an integral simple function)

Then $\|f - f_k\|_{L^1} = \int \left| \sum_{i=1}^n a_i \chi_{E_i}(x) - \sum_{j=\infty}^{\infty} k \chi_{I_j^k}(x) \left(\int_{I_j^k} \sum_{i=1}^n a_i \chi_{E_i}(t) \right) \right| dx$

Let j_x be s.t. $x \in I_{j_x}^k$ and let k be large enough s.t. $x \in I_{j_x}^k \subseteq E_{i_x}$ some i_x

$$= \int \left| a_{i_x} - k\left(\int_{I_{j_x}^k} a_{i_x}\,dt \right) \right| dx$$

$m(I_{j_x}^k) = \frac{1}{k}$

$$= \int \left| a_{i_x} - k a_{i_x} m(I_{j_x}^k) \right| dx = \int |a_{j_x} - a_{j_x}|\,dx = 0$$

→ See that:

$$\int_{\mathbb{R}} |f_k(x)| = \int_{\mathbb{R}} \left| \sum_j k \chi_{I_j^k}(x) \left(\int_{I_j^k} f(t)\,dt \right) \right| dx$$

$$\leq \int_{\mathbb{R}} k \sum_j \chi_{I_j^k}(x) \left(\int_{I_j^k} |f(t)|\,dt \right) dx$$

$$= k \sum_j \int_{\mathbb{R}} \int_{I_j^k} \chi_{I_j^k}(x) |f(t)|\,dt\,dx$$

$$= k \sum_j \int_{I_j^k} \int_{\mathbb{R}} \chi_{I_j^k}(x) |f(t)|\,dx\,dt$$

$$= k \sum_j \int_{I_j^k} \frac{1}{k} |f(t)|\,dt = \sum_j \int_{I_j^k} |f(t)|\,dt = \int_{\mathbb{R}} |f(t)|\,dt < \infty,$$

hence $f_k \in L^1(m)$ for all k.

→ Since f, f_k are L^1, we know that for $\varepsilon > 0$, $\exists$ int. simple fns ϕ, ϕ_k s.t. $\|f - \phi\| < \varepsilon$ and $\|f_k - \phi_k\| < \varepsilon$, hence for $\varepsilon > 0$,

$$\|f - f_k\|_{L^1} = \int |f - f_k| = \int |f - \phi + \phi + \phi_k - \phi_k - f_k| \leq \int (|f - \phi| + |\phi_k - f_k| + |\phi - \phi_k|)\,dx$$

$$< \varepsilon + \varepsilon + \int |\phi - \phi_k|\,dx = \varepsilon + \varepsilon + 0 = 2\varepsilon,$$ hence $f_k \to f$ in L^1

by above, for k large enough

(4) E Borel in $\mathbb{R}^n$. Let $D_E(x) = \lim_{r\to 0} \dfrac{m(E\cap B_r(x))}{m(B_r(x))}$

(a) Show $D_E(x) = 1$ for a.e. $x\in E$
$D_E(x) = 0$ for a.e. $x \notin E$.

(b) Find E, x s.t. $D_E(x) = \alpha \in (0,1)$

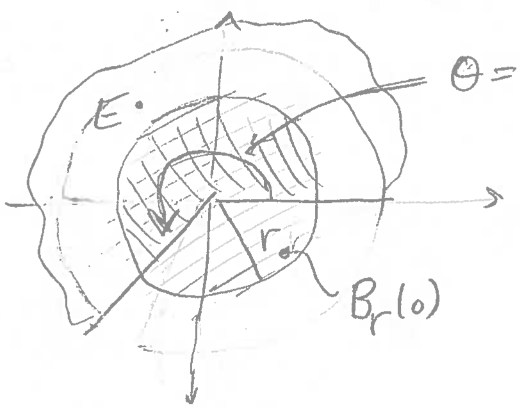
(c) Find E, x s.t. $D_E(x)$ DNE

(a) Define the measure $d\nu = \chi_E\, dm$

Then: $D_E(x) = \lim_{r\to 0} \dfrac{m(E\cap B_r(x))}{m(B_r(x))} = \lim_{r\to 0} \dfrac{\int_{B_r(x)} \chi_E}{m(B_r(x))} = \lim_{r\to 0} \dfrac{\nu(B_r(x))}{m(B_r(x))}$

$= \dfrac{d\nu}{dm} = \chi_E$ almost everywhere; hence $D_E(x) = 1$ for a.e. $x\in E$, and $= 0$ for a.e. $x \in E^c$

(b) Let E be the region pictured:



See that the area of $E\cap B_r(0)$ is $\frac{1}{2}\theta r^2 = \frac{1}{2} 2\pi\alpha r^2$
$= \alpha\pi r^2$

Therefore:

$D_E(0) = \lim_{r\to 0} \dfrac{m(E\cap B_r(x))}{m(B_r(x))} = \dfrac{\alpha\cancel{\pi r^2}}{\cancel{\pi r^2}} = \alpha$

(c) Consider $\mathbb{R}$ and $E = \bigcup_{n \text{ odd}} \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right)$

$\to$ <u>m odd</u>: Let $r = \frac{1}{2^{m-1}}$

Then $m(B_r(0)\cap E) = \sum_{\substack{n\geq m\\ n \text{ odd}}} \left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right) = \frac{4}{3}\cdot\frac{1}{2^m}$ (?)

and $m(B_r(0)) = \frac{2}{2^{m-1}}$, hence $\dfrac{m(B_r(0)\cap E)}{m(B_r(0))} = \dfrac{2^{m-1}\cdot 4}{2\cdot 2^m \cdot 3} = \dfrac{2^{m+1}}{2^{m+1}3} = \frac{1}{3}$

$\to$ <u>m even</u>: Let $r = \frac{1}{2^{m-1}}$ (E is empty from $r = \frac{1}{2^{m-1}}$ to $r = \frac{1}{2}$)

Then $m(B_r(0)\cap E) = m(B_{\frac{1}{2}r}(0)\cap E) = \sum_{\substack{n\geq m+1\\ n \text{ odd}}} \left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right) = \frac{4}{3}\cdot\frac{1}{2^{m+1}}$

and $m(B_r(0)) = \frac{2}{2^{m-1}}$, so $\dfrac{m(B_r(0)\cap E)}{m(B_r(0))} = \dfrac{4\cdot 2^{m-1}}{3\cdot 2^{m+1}\cdot 2} = \dfrac{2^{m+1}}{3\cdot 2^{m+1}\cdot 2} = \frac{1}{6}$

$\to$ Hence limit DNE since alt. for n even/odd

REAL ANALYSIS GRADUATE EXAM

Spring 2013

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Suppose that $\{f_n\}$ is a sequence of of real valued continuously differentiable functions on $[0,1]$ such that

- Definitions

$$\lim_{n\to\infty}\int_0^1 |f_n(x)|\,dx = 0 \text{ and } \lim_{n\to\infty}\int_0^1 |f_n'(x)|\,dx = 0.$$

Show that $\{f_n\}$ converges to 0 uniformly on $[0,1]$.

2. Investigate the convergence of $\sum_{n=0}^{\infty} a_n$, where

- geom sum
- $\lim_{x\to 0}\frac{\sin x}{x} = 1$

$$a_n = \int_0^1 \frac{x^n}{1-x}\sin(\pi x)\,dx.$$

W 3. Let $(X, \mathcal{M}, \mu)$ be a measure space, $f_n, f \in L^1(\mu)$. Show that $\int_X |f_n - f|\,d\mu \to 0$ as $n\to\infty$ if and only if

- split into $f_n - f \geq 0$ 3 $f_n - f < 0$

$$\sup_{A\in\mathcal{M}}\left|\int_A f_n d\mu - \int_A f d\mu\right| \to 0$$

as $n\to\infty$.

W 4. Let μ and ν be σ-finite positive measures, $\mu \geq \nu$ and assume that $\nu << \mu - \nu$ (ν is absolutely continuous with respect to $\mu - \nu$).

Radon-Nikodym

Prove that

$$\mu\left(\{\frac{d\nu}{d\mu} = 1\}\right) = 0.$$

Spring 13 :

① $(f_n) \subseteq C^1([0,1])$ s.t. $\overset{(1)}{\lim_{n\to\infty}\int_0^1 |f_n(x)|dx = 0}$ and $\overset{(2)}{\lim_{n\to\infty}\int_0^1 |f_n'(x)|dx = 0}$

Show $f_n \to 0$ uniformly :

Want to show that for $\varepsilon > 0$, $\exists N \geq 0$ s.t. $n \geq N \Rightarrow |f_n(x)| < \varepsilon \ \forall x$.

First see that for $\frac{\varepsilon}{2} > 0$, $\exists N \geq 0$ s.t. $n \geq N \Rightarrow \int_0^1 |f_n'(x)|dx < \frac{\varepsilon}{2}$, namely that, for $a, b \in [0,1]$:

$$\left.|f_n(b) - f_n(a)| = \left|\int_a^b f_n'(x)dx\right| \leq \int_a^b |f_n'(x)|dx \leq \int_0^1 |f_n'(x)|dx < \frac{\varepsilon}{2}\right] \text{use (2)}$$

Therefore, $|f_n(0) - f_n(x)| < \frac{\varepsilon}{2}$ for all $x \in [0,1]$, hence

$$f_n(0) - \frac{\varepsilon}{2} < |f_n(x)| < f_n(0) + \frac{\varepsilon}{2}$$

$$\Rightarrow f_n(0) < |f_n(x)| + \frac{\varepsilon}{2} \Rightarrow \int_0^1 f_n(0)\,dx < \int_0^1 \left(|f_n(x)| + \frac{\varepsilon}{2}\right) \ \text{use (1)}$$

$$\Rightarrow f_n(0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $f_n(0) \to 0$ as $n \to \infty$, i.e. for $\frac{\varepsilon}{2} > 0$, $\exists M$ s.t. $n \geq M$ implies

$$|f_n(x)| < f_n(0) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for all } x.$$

Thus $f_n \to 0$ unif.

② Determine the convergence of $\sum_{n=0}^{\infty} \int_0^1 \frac{x^n \sin(\pi x)}{(1-x)^2}dx$

First see that $\sum_{n=0}^{\infty} \int \frac{\sin \pi x}{(1-x)^2}dx = \int_0^1 \sum_{n=0}^{\infty} \frac{x^n \sin(\pi x)}{(1-x)^2}dx$ by Tonelli.

$$= \int_0^1 \frac{\sin(\pi x)}{(1-x)^2}dx \quad \text{by geometric sum}$$

Then, by substitution $u = 1-x$:

$$\int_0^1 \frac{\sin(\pi x)}{(1-x)^2}dx = \int_1^0 \frac{-\sin(\pi(1-u))du}{u^2} = \int_0^1 \frac{\sin(\pi(1-u))}{u^2}du = \int_0^1 \frac{\sin(\pi u)}{u^2}du$$

by symmetry of $\sin(\pi u)$ over $[0,1]$

$= \pi \int_0^1 \frac{\sin(\pi u)}{\pi u^2}du$; recall that $\lim_{u\to 0} \frac{\sin(\pi u)}{\pi u} = 1$,

hence we may choose δ s.t. $|u| < \delta \Rightarrow \left|\frac{\sin \pi u}{\pi u} - 1\right| < \frac{1}{2}$,

i.e. $-\frac{1}{2} < \frac{\sin \pi u}{\pi u} - 1 < \frac{1}{2} \Rightarrow \frac{1}{2} < \frac{\sin \pi u}{\pi u} < \frac{3}{2}$

$$\Rightarrow \pi \int_0^1 \frac{\sin(\pi u)}{\pi u} \cdot \frac{du}{u} > \pi \int_\delta^1 \frac{1}{2}\frac{du}{u} = \frac{\pi}{2}\int_\delta^1 \frac{du}{u} = \infty$$

as $\delta \to 0$.

hence divergent

(3) (X,μ) meas, $f_n, f \in L^1(\mu)$. Show

$$\int_X |f_n - f|\,d\mu \to 0 \text{ as } n\to\infty \iff \sup_{A\in\mathcal{M}_\mu} \left|\int_A f_n\,d\mu - \int_A f\,d\mu\right| \to 0 \text{ as } n\to\infty$$

($\Rightarrow$) $\varepsilon>0$, $\exists N$ s.t. $n\geq N \Rightarrow \int_X |f_n - f|\,d\mu < \varepsilon$

Hence: $\sup_{A\in\mathcal{M}_\mu} \left|\int_A f_n - \int_A f\right| \leq \sup_{A\in\mathcal{M}_\mu} \int_A |f_n - f| \leq \int_X |f_n - f| < \varepsilon$,

thus $\sup_{A\in\mathcal{M}_\mu} \left|\int_A f_n - \int_A f\right| \to 0$ as $n\to\infty$

($\Leftarrow$) $\varepsilon>0$, $\exists N$ st. $n\geq N \Rightarrow \sup_{A\in\mathcal{M}_\mu} \left|\int_A (f_n - f)\,d\mu\right| < \varepsilon$

Now:
$$\int_X |f_n - f|\,d\mu = \int_{\{f_n - f\geq 0\}} |f_n - f|\,d\mu + \int_{\{f_n - f<0\}} |f_n - f|\,d\mu$$
$$= \left|\int_{\{f_n - f\geq 0\}} (f_n - f)\,d\mu\right| + \left|\int_{\{f_n - f<0\}} (f_n - f)\,d\mu\right|$$
$$\leq \sup_{A\in\mathcal{M}_\mu} \left|\int_A (f_n - f)\,d\mu\right| + \sup_{A\in\mathcal{M}_\mu} \left|\int_A (f_n - f)\,d\mu\right|$$
$$= 2\sup_{A\in\mathcal{M}_\mu} \left|\int_A (f_n - f)\,d\mu\right| < 2\varepsilon$$

Hence $\int_X |f_n - f|\,d\mu \to 0$ as $n\to\infty$.

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(4) μ, ν σ-finite, $\mu \geq \nu$, $\nu \ll \mu - \nu$. Claim: $\mu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = 0$

First, see that $\mu(E) = 0 \Rightarrow \nu(E) \leq \mu(E) = 0 \Rightarrow \nu(E) = 0$ since ν is positive, hence the derivative $\frac{d\nu}{d\mu}$ is valid.

Now, $\nu \ll \mu - \nu$, hence $d\nu = g\,d(\mu-\nu)$ for some g. Hence:
$\nu(E) = \int_E g\,d\mu - g\,d\nu$. Then:

$$\nu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = \int_{\left\{\frac{d\nu}{d\mu}=1\right\}} d\nu = \int_{\left\{\frac{d\nu}{d\mu}=1\right\}} \frac{d\nu}{d\mu}\,d\mu = \int_{\left\{\frac{d\nu}{d\mu}=1\right\}} d\mu = \mu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right),$$

but:
$$\nu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = \int_{\left\{\frac{d\nu}{d\mu}=1\right\}} g\,d\mu - g\,d\nu = \int_{\left\{\frac{d\nu}{d\mu}=1\right\}} g\,d\mu - g\frac{d\nu}{d\mu}\,d\mu$$
$$= \int_{\left\{\frac{d\nu}{d\mu}=1\right\}} g\,d\mu - g\,d\mu = 0, \text{ i.e. } 0 = \nu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = \mu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right)$$